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Some characterizations of β-paracompactness in ideal topological space

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Abstract. In this paper, we introduce β-paracompactness with respect to an ideal (I-β-paracompactness) as a weak form of β-paracompactness and I-paracompactness. We give some relations between this concept and some other types of paracompactness, and also we study some of its fundamental properties.

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1. Introduction

Paracompactness is one of the important concepts of general topology. In literature, different kinds of generalized paracompactness such as S-paracompactness [5], P_3-paracompactness [6] and β-paracompactness [11] are studied.

The concept of I-paracompactness as generalization of paracompactness was given by Zahid [24]. Furthermore, this concept was studied by Hamlet et al. [13] and Sathiyasundari and Renukadevi [22]. Recently, S-paracompactness with respect to an ideal which is weaker form of I-paracompactness was studied by J. Sanabria et al. [21].

Here, we introduce I-β-paracompactness and we compare this concept with the other types of paracompactness. Then, we give counterexamples showing that the opposite directions of Proposition 1 and 2 do not hold. Furthermore, adding some conditions, we find that the reverse directions may happen to be true. Besides, we investigate some of its essential properties. Finally, we examine I-β-paracompactness under some functions.

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2. Preliminaries

Throughout this work, (X, τ) denotes a topological space on which no separation axioms are assumed unless clearly indicated. If A is a subset of (X, τ), then the closure of A and the interior of A will be denoted by $cl(A)$ and $int(A)$, respectively. Also, the class of all subsets of X will be denoted by $\mathcal{P}(X)$. A subset A of (X, τ) is said to be semi-open [16] if there exists $U \in \tau$ such that $U \subseteq A \subseteq cl(U)$. This is equivalent to say that $A \subseteq cl(int(A))$. Also, A is said to be β-open [1] (preopen [18]) if $A \subseteq cl(int(cl(A)))(A \subseteq int(cl(A)))$. The concept of β-open sets is equal to that of semi-preopen sets in [7]. The family of all semi-open (resp. β-open and preopen) sets of (X, τ) is denoted by $SO(X, \tau)$ (resp. $\beta O(X, \tau)$ and $PO(X, \tau)$). The complement of a semi-open (resp. β-open and preopen) set is said to be semi-closed [10] (resp. β-closed [1, 7] and preclosed [18]). The semi-closure [10] (resp. β-closure [3, 7] and preclosure[18]) of A, denoted by $scl(A)$ (resp. $\beta cl(A)$ and $pcl(A)$), is the intersection of all semi-closed (resp. β-closed and preclosed) sets containing A. Note that, $\beta cl(A)$ is β-closed [3, 7].

Lemma 1. *[3, 7] For a subset A of a topological space (X, τ), the following conditions hold:*

(i) $x \in \beta cl(A)$ if and only if $A \cap U \neq \emptyset$ for every $U \in \beta O(X, \tau)$ containing x,

(ii) A is β-closed if and only if $A = \beta cl(A)$.

Theorem 1. *[19] Let (X, τ) be a space, $A \subseteq Y \subseteq X$ and Y be β-open in (X, τ). Then A is β-open in (X, τ) if and only if A is β-open in the subspace (Y, τ_Y).*

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be pre β-closed [17] (pre β-open [17]) if for every β-closed (β-open)set A of (X, τ), $f(A)$ is β-closed (β-open) in (Y, σ) and $f : (X, \tau) \to (Y, \sigma)$ is said to be β-irresolute [17] if for every β-open set B of (Y, σ), $f^{-1}(B)$ is β-open in (X, τ). If $f : (X, \tau) \to (Y, \sigma)$ is continuous and open, then f is β-irresolute and pre β-open.

Lemma 2. *[11] Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective function. Then f is pre β-closed if and only if for every $y \in Y$ and every β-open set U in (X, τ) which contains $f^{-1}(y)$, there exists a $V \in \beta O(Y, \sigma)$ such that $y \in V$ and $f^{-1}(V) \subseteq U$.*

A space (X, τ) is called extremally disconnected[23](briefly, e. d.) if the closure of every open set in X is open and called submaximal [8] if each dense subset of X is open in X.

Lemma 3. *[20] (X, τ) is submaximal if and only if every pre-open set is open.*

Lemma 4. *[9] (X, τ) is e.d. if and only if every β-open set is pre-open.*

A collection $\mathcal{V}$ of subsets of a space (X, τ) is said to be locally finite [23](resp. s-locally finite [4], β-locally finite [11] and p-locally finite[6]), if for each $x \in X$ there exists $U_x \in \tau$ (resp. $U_x \in SO(X, \tau)$, $U_x \in \beta O(X, \tau)$ and $U_x \in PO(X, \tau)$) containing x and U_x intersects at most finitely many members of $\mathcal{V}$. Every locally finite collection of subsets of a space

(X,τ) is β-locally finite[11] and p-locally finite[6]. Also, a collection $\mathcal{A}$ of subsets of a space (X,τ) is said to be σ-locally finite if $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ where each $\mathcal{A}_n$ is locally finite family [13].

Theorem 2. *[11] Let (X,τ) be an e.d. submaximal space. Then every β-locally finite collection of subsets of X is locally finite.*

A space (X,τ) is said to be β-compact [2] if every cover of X by β-open sets has a finite subcover. Also a space (X,τ) is said to be paracompact [23] (resp. S-paracompact [5], β-paracompact [11] and P_3-paracompact [6]), if every open cover of X has a locally finite open (resp. locally finite semi-open, β-locally finite β-open and p-locally finite preopen) refinement which covers to X.

An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions:

(1) If $A \in I$ and $B \subseteq A$, then $B \in I$,

(2) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

Given a topological space (X,τ) with an ideal I on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^* : \mathcal{P}(X) \to \mathcal{P}(X)$, called a local function [15] of A with respect to τ and I is defined as follows: for $A \subseteq X$, $A^*(I,\tau) = \{x \in X : V \cap A \notin I$ for every $V \in \tau(x)\}$ where $\tau(x) = \{V \in \tau : x \in V\}$. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(I,\tau)$, called the *-topology, finer than τ, is defined by $cl^*(A) = A \cup A^*(I,\tau)$ [14]. A basis $\beta(I,\tau)$ for $\tau^*(I,\tau)$ can be described as follows: $\beta(I,\tau) = \{V - J : V \in \tau$ and $J \in I\}$[14]. We will simply write A^* for $A^*(I,\tau)$, τ^* or $\tau^*(I)$ for $\tau^*(I,\tau)$ and β for $\beta(I,\tau)$. If I is an ideal on X, then (X,τ,I) is called an ideal topological space.

A space (X,τ,I) is said to be I-paracompact [24] (I-S-paracompact [21]) if every open cover $\mathcal{U}$ of X has a locally finite open (semi-open) refinement $\mathcal{V}$, not necessarily a cover, such that $X - \bigcup\{V : V \in \mathcal{V}\} \in I$. A collection $\mathcal{V}$ of subsets of X such that $X - \bigcup\{V : V \in \mathcal{V}\} \in I$ is called an I-cover [24] of X. A space (X,τ,I) is said to be I-regular[12] if for each closed set F and a point $p \notin F$, there exist disjoint open sets U and V such that $p \in U$ and $F - V \in I$.

3. I-β-paracompactness

Definition 1. *A space (X,τ,I) is said to be I-β-paracompact or β-paracompact with respect to I if every open cover $\mathcal{U}$ of X has a β-locally finite β-open refinement $\mathcal{V}$ (not necessarily a cover) such that $X - \bigcup\{V : V \in \mathcal{V}\} \in I$.*

A subset A of a space (X,τ,I) is called an I-β-paracompact set in (X,τ,I) if every open cover $\mathcal{U}$ of A has a β-locally finite (with respect to τ) β-open refinement $\mathcal{V}$ such that $A - \bigcup\{V : V \in \mathcal{V}\} \in I$.

Proposition 1. *If (X,τ) is β-paracompact, then (X,τ,I) is I-β-paracompact.*

Proof. It is obvious since $\emptyset \in I$.

Obviously, every compact space is I-β-paracompact since every compact space is β-paracompact [11].

The following example shows that the converse of Proposition 1 may not be true, in general.

Example 1. *Let $X = \mathbb{N}$ be the set of natural numbers with the topology $\tau = \{G \subseteq \mathbb{N} : 5 \in G\} \cup \{\emptyset\}$ and the ideal $I = \{U \subseteq \mathbb{N} : 5 \notin U\}$. Observe that (X, τ, I) is I-β-paracompact space but (X, τ) is not β-paracompact since the collection $\{\{5, x\} : x \in \mathbb{N}\}$ is an open cover of X which admits no β-locally finite β-open refinement in X.*

Remark 1.
(1) If $I = \{\emptyset\}$, then (X, τ, I) is I-β-paracompact if and only if (X, τ) is β-paracompact.
(2) If $I = \{\emptyset\}$ and (X, τ, I) is an e.d. space, then (X, τ, I) is I-β-paracompact if and only if (X, τ) is P_3-paracompact.

Proposition 2. *If (X, τ, I) is I-S-paracompact then it is I-β-paracompact.*
Proof. Since every locally finite collection of subsets of X is β-locally finite and every semi-open set is β-open, it is clear.

Clearly, every S-paracompact space is I-β-paracompact since every S-paracompact space is I-S-paracompact[21]. Also, every I-paracompact space is I-β-paracompact since every I-paracompact space is I-S-paracompact[21].

The following example shows that the converse of Proposition 2 may not be true, in general.

Example 2. *Let $X = [0, 2] \cup [3, 10]$ with the topology $\tau = \{U \subseteq X : [0, 2] \subseteq U\} \cup \{\emptyset\}$ and the ideal $I = \{A : A \subseteq [0, 2]\}$. Then (X, τ, I) is I-β-paracompact since every open cover of X has β-locally finite β-open refinement $\mathcal{V} = \{\{x\} : x \in [0, 2]\} \cup \{\{y, z\} : y \in [0, 2], z \in [3, 10]\}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in I$. But it is not I-S-paracompact since $\tau = SO(X)$.*

Theorem 3. *If (X, τ, I) is an e.d. submaximal I-β-paracompact space, then it is I-S-paracompact.*
Proof. It is obvious from Lemma 3, Lemma 4 and Theorem 2.

Theorem 4. *If (X, τ, I) is I-β-paracompact and J is an ideal on X with $I \subseteq J$, then (X, τ, J) is J-β-paracompact.*
Proof. Let (X, τ, I) be I-β-paracompact and $I \subseteq J$. And let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X. Since (X, τ, I) is I-β-paracompact, $\mathcal{U}$ has a β-locally finite β-open refinement $\mathcal{V}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in I$. Since $I \subseteq J$, $X - \bigcup\{V : V \in \mathcal{V}\} \in J$. Thus, (X, τ, J) is J-β-paracompact.

Lemma 5. *[11]Let $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ be a collection of subsets of a space (X, τ). $\mathcal{V}$ is β-locally finite if and only if $\{\beta cl(V_\lambda) : \lambda \in \Lambda\}$ is β-locally finite.*

Lemma 6. *If a cover $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of a space (X, τ, I) has a β- locally finite β-open refinement $\mathcal{V}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in I$ then there exists a β-locally finite precise β- open refinement $\mathcal{H} = \{H_\lambda : \lambda \in \Lambda\}$ of $\mathcal{U}$ such that $X - \bigcup\{H_\lambda : H_\lambda \in \mathcal{H}\} \in I$.*

Proof. The proof is similar to that of Lemma 1.3 in [21].

Definition 2. *A collection $\mathcal{A}$ of subsets of a space (X, τ) is said to be σ-β-locally finite if $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ where each collection $\mathcal{A}_n$ is a β- locally finite family.*

Lemma 7. *Every β-locally finite collection of subsets of a space (X, τ) is σ-β-locally finite.*

Proof. It is obvious.

Theorem 5. *Let (X, τ) be a regular space. If (X, τ, I) is I-β-paracompact, then every open cover of X has a β-closed β-locally finite I-cover refinement.*

Proof. Let $\mathcal{U}$ be an open cover of X. By regularity of X , for each $x \in X$ and $U_x \in \mathcal{U}$ containing x, there exists an open set G_x of x such that $cl(G_x) \subseteq U_x$. Then $\mathcal{U}_1 = \{G_x : x \in X\}$ is an open cover of X. Since X is I-β-paracompact, $\mathcal{U}_1$ has β-locally finite β-open refinement $\mathcal{V}_1 = \{V_\lambda : \lambda \in \Lambda\}$ such that $X - \bigcup\{V_\lambda : \lambda \in \Lambda\} \in I$. Then $X - \bigcup\{\beta cl(V_\lambda) : \lambda \in \Lambda\} \in I$. By Lemma 5, $\mathcal{V} = \{\beta cl(V_\lambda) : V_\lambda \in \mathcal{V}_1\}$ is β-locally finite. Since $\mathcal{V}_1$ refines $\mathcal{U}_1$, for every $\lambda \in \Lambda$, there is some $G_x \in \mathcal{U}_1$ such that $V_\lambda \subseteq G_x$. Then $\beta cl(V_\lambda) \subseteq cl(V_\lambda) \subseteq \mathrm{cl}(G_x)$ implies $\beta cl(V_\lambda) \subset U_x$. Hence $\mathcal{V}$ refines $\mathcal{U}$. So, $\mathcal{V} = \{\beta cl(V_\lambda) : V_\lambda \in \mathcal{V}_1\}$ is β-closed β-locally finite I-cover refinement.

Remark 2. *If (X, τ, I) is considered to be e.d. submaximal regular space, then the Theorem 5 becomes the Theorem 2.20 in [22].*

Theorem 6. *If (X, τ, I) is I-β-paracompact, then every open cover of X has a β-open σ-β-locally finite I-cover refinement.*

Proof. It is obvious by Lemma 7.

Theorem 7. *Let (X, τ, I) be a regular space and $\beta O(X, \tau)$ be closed under finite intersection. Then, (X, τ, I) is I-β-paracompact if and only if every open cover of X has a β-open σ-β-locally finite I-cover refinement.*

Proof. To show sufficiency, let $\mathcal{U}$ be an open cover of X. By hypothesis, there exists a σ-β-locally finite β-open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in I$. Also, $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ where each collection $\mathcal{V}_n$ is a β- locally finite. For each $n \in \mathbb{N}$, let H_n $=\bigcup\{V : V \in \mathcal{V}_n\}$ so that $X - \bigcup\{H_n : n \in \mathbb{N}\} \in I$. For each $n \in \mathbb{N}$, let $G_n = H_n - \bigcup_{i=1}^{n-1} H_i$. Then $\{G_n : n \in \mathbb{N}\}$ refines $\{H_n : n \in \mathbb{N}\}$. Let $x \in X$, and let n be the smallest member of $\{n \in \mathbb{N} : x \in H_n\}$. Then $x \in G_n$ and $X - \bigcup\{G_n : n \in \mathbb{N}\} \in I$. Also, G_{nx} is a β-open set containing x that intersects only finite family number of members of G_n so that $\{G_n : n \in \mathbb{N}\}$ is β-locally finite. Let $\mathcal{O}= \{V \cap G_n : V \in \mathcal{V}_n \text{ and } n \in \mathbb{N}\}$. Since $\{G_n : n \in \mathbb{N}\}$ is β-locally finite, $\mathcal{O}$ is β-locally finite. Also, since $\beta O(X, \tau)$ is closed under finite intersection and $\mathcal{V}$ is β-open refinement of $\mathcal{U}$, $\mathcal{O}$ is β-open refinement of $\mathcal{U}$. Then, $X - \bigcup\{V \cap G_n : n \in \mathbb{N}\} \in I$ because $X - \bigcup\{G_n : n \in \mathbb{N}\} \in I$. Thus, (X, τ, I) is I-β-paracompact.

Remark 3. *If (X, τ, I) is considered to be e.d. submaximal regular space, then Theorem 7 becomes Theorem 2.22 in [22].*

Theorem 8. *For any ideal topological space (X, τ, I), the following are equivalent:*

(i) For every closed subset A of X and every $x \notin A$, there exist disjoint β-open sets U and V such that $x \in U$ and $A - V \in I$.

(ii) For every open subset G of X and every $x \in G$, there exists a β-open set U such that $x \in U$ and $\beta cl(U) - G \in I$.

Proof. *(i) ⇒ (ii) Let $G \subseteq X$ be open and $x \in G$. Then $X - G = A$ is closed and $x \notin A$. From (i), there exist disjoint β-open sets U and V such that $x \in U$ and $A - V \in I$. Since U and V are disjoint, we have $\beta cl(U) \subseteq X - V$. Thus, $A \cap \beta cl(U) \subseteq A - V$. Then, $\beta cl(U) \cap (X - G) \in I$. Therefore, $\beta cl(U) - G \in I$.*

(ii) ⇒ (i) Let $A \subseteq X$ be closed and $x \notin A$. Then, $X - A = G$ is open and $x \in G$. From (ii), there exists a β-open set U such that $x \in U$ and $\beta cl(U) - G \in I$. Thus, $X - \beta cl(U) = V \in \beta O(X)$ and $U \cap V = \emptyset$. Furthermore, $A - V = (X - G) - (X - \beta cl(U)) = \beta cl(U) - G \in I$.

The following example reveals that for a locally finite collection of subsets of $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of a space (X, τ), the equality $cl(\bigcup\{V_\lambda : \lambda \in \Lambda\}) = \bigcup\{cl(V_\lambda) : \lambda \in \Lambda\}$ always holds whereas for β-locally finite collection of subsets $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of a space (X, τ), the equality $\beta cl(\bigcup\{U_\lambda : \lambda \in \Lambda\}) = \bigcup\{\beta cl(U_\lambda) : \lambda \in \Lambda\}$ does not hold in general.

Example 3. *Consider the real number $\mathbb{R}$ with usual topology τ. Let $\mathcal{V} = \{[0, 1), (1, 2]\}$. Then $\mathcal{V}$ is β-locally finite in $(\mathbb{R}, \tau)$ since it is finite. But $\beta cl([0, 1) \cup (1, 2]) \neq \beta cl([0, 1)) \cup \beta cl((1, 2])$.*

Theorem 9. *Suppose that for a β-locally finite collection of subsets $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of a space (X, τ, I), the equality $\beta cl(\bigcup\{V_\lambda : \lambda \in \Lambda\}) = \bigcup\{\beta cl(V_\lambda) : \lambda \in \Lambda\}$ holds. If (X, τ, I) is Hausdorff I-β-paracompact, then for every closed subset A of X and every $x \notin A$, there exist disjoint β-open sets U and V such that $x \in U$ and $A - V \in I$.*

Proof. Let $A \subseteq X$ closed and $x \notin A$. Since X is Hausdorff space, there exists an open set H_y containing y for each $y \in A$ such that $x \notin cl(H_y)$. Thus, $\mathcal{H} = \{H_y : y \in A\} \cup \{X - A\}$ is an open cover of X. By hypothesis and Lemma 6, $\mathcal{H}$ has a β-locally finite precise β-open refinement $\mathcal{W} = \{W_y : y \in A\} \cup \{G\}$ such that $W_y \subseteq H_y$ for each $y \in A$, $G \subseteq X - A$ and $X - (\bigcup\{W_y : y \in A\} \cup \{G\}) \in I$. Since $A - (\bigcup\{W_y : y \in A\}) = A - (\bigcup\{W_y : y \in A\} \cup \{G\}) \subseteq X - (\bigcup\{W_y : y \in A\} \cup \{G\})$, we have $A - (\bigcup\{W_y : y \in A\}) \in I$. Let *we say* $V = \bigcup\{W_y : y \in A\}$*. Then,* V *is* β*-open set in* X *and* $A - V \in I$*. Since* $x \notin cl(H_y)$*, we have* $x \notin cl(W_y)$*. This implies that* $x \notin \beta cl(W_y)$*. Since* $\mathcal{W}$ *is* β*-locally finite,* $\beta cl(V) = \beta cl(\bigcup\{W_y : y \in A\}) = \bigcup\{\beta cl(W_y) : y \in A\}$ *by hypothesis. Thus, for a* β*-open set* $U = X - \beta cl(V)$*, we have* $U \cap V = \emptyset$ *such that* $x \in U$*.*

From Theorem 8 and Theorem 9, we have the following Corollary.

Corollary 1. *If* (X, τ, I) *is an e.d. submaximal Hausdorff* I*-*β*-paracompact space, then* (X, τ, I) *is* I*-regular.*

Theorem 10. *Let* A *and* B *be subsets in ideal topological space* (X, τ, I)*. If* A *is* I*-*β*-paracompact set in* X *and* B *is closed in* X*, then* $A \cap B$ *is* I*-*β*-paracompact set in* X*.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ *be an open cover of* $A \cap B$*. Since* $X - B$ *is open in* X*,* $\mathcal{U}' = \{U_\lambda : \lambda \in \Lambda\} \cup \{X - B\}$ *is open cover of* A*. By hypothesis and Lemma 6,* $\mathcal{U}'$ *has a* β*- locally finite precise* β*-open refinement* $\{V_\lambda : \lambda \in \Lambda\} \cup \{V\}$ *such that* $V_\lambda \subseteq U_\lambda$ *for each* $\lambda \in \Lambda$*,* $V \subseteq X - B$ *and* $A - (\bigcup\{V_\lambda : \lambda \in \Lambda\} \cup \{V\}) \in I$*. Since* $(A\cap B)-(\bigcup\{V_\lambda : \lambda \in \Lambda\}) = (A\cap B)-(\bigcup\{V_\lambda : \lambda \in \Lambda\}\cup\{V\}) \subseteq A-(\bigcup\{V_\lambda : \lambda \in \Lambda\}\cup\{V\})$*, we have* $(A \cap B) - (\bigcup\{V_\lambda : \lambda \in \Lambda\}) \in I$*. Hence,* $A \cap B$ *is* I*-*β*-paracompact set in* X*.*

Corollary 2. *Let* (X, τ, I) *be an* I*-*β*-paracompact space and* $A \subseteq X$*. If* A *is closed in* X*, then* A *is an* I*-*β*-paracompact set in* X*.*

Lemma 8. *[13] If* $I \neq \emptyset$ *is an ideal on X and Y is a subset of X, then* $I_Y = \{Y \cap G | G \in I\} = \{G \in I | G \subseteq Y\}$ *is an ideal on Y.*

Theorem 11. *Let* A *and* B *be subsets in ideal topological space* (X, τ, I) *such that* $B \subseteq A$*. If* A *is* β*-open in* X *and* B *is an* I_A*-*β*-paracompact set in* A *then* B *is an* I*-*β*-paracompact set in* X*.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ *be an open cover of* B *in* X*. Then,* $\mathcal{U}_\mathcal{B} = \{U_\lambda \cap A : \lambda \in \Lambda\}$ *is an open cover of* B *in* A*. Since* B *is an* I_A*-*β*-paracompact set in* A*,* $\mathcal{U}_\mathcal{B}$ *has a* β*-locally finite precise* β*-open refinement* $\mathcal{V}_\mathcal{B}$ *in* A *such that* $B - \bigcup\{V_\lambda : V_\lambda \in \mathcal{V}_\mathcal{B}\} \in I_A$*. Thus,* $\mathcal{V}_\mathcal{B}$ *is a* β*-locally finite precise* β*-open refinement in* X *by Theorem 1. Also,* $B - \bigcup\{V_\lambda : V_\lambda \in \mathcal{V}_\mathcal{B}\} \in I$*. Hence,* B *is an* I*-*β*-paracompact set in* X*.*

Theorem 12. *Let* $f : (X, \tau, I) \to (Y, \sigma, J)$ *be a continuous, open and pre* β*-closed surjection with* $f^{-1}(y)$ β*-compact for every* $y \in Y$ *and* $f(I) \subseteq J$*. If* (X, τ, I) *is* I*-*β*-paracompact, then* (Y, σ, J) *is* J*-*β*-paracompact.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ *be an open cover of* Y*. Then,* $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ *is an open cover of* X*. Since* (X, τ, I) *is* I*-*β*-paracompact, this open cover has a* β*-locally finite precise* β*-open refinement* $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ *such that* $X - \bigcup\{V_\lambda : V_\lambda \in \mathcal{V}\} \in I$*. Since* f *is pre* β*-open,* $f(\mathcal{V}) = \{f(V_\lambda) : \lambda \in \lambda\}$ *is a precise* β*-open refinement of* $\mathcal{U}$*. Also,* $Y - \bigcup\{f(V_\lambda) : \lambda \in \Lambda\} \in J$*. Now, let we prove that* $f(\mathcal{V})$ *is* β*-locally finite. Let* $y \in Y$*. Since* $\mathcal{V}$ *is* β*-locally finite, for* $x \in f^{-1}(y)$*, there exists a* β*-open set* G_x *containing* x *such that* G_x *intersects at most finitely members of* $\mathcal{V}$*. Since* $f^{-1}(y)$ *is* β*-compact,* $\{G_x : x \in$

$f^{-1}(y)\}$ *has a finite subcollection* H_y *such that* $f^{-1}(y) \subseteq \bigcup H_y$ *and* $\bigcup H_y$ *intersects at most finitely members of* $\mathcal{V}$. *By Lemma 2, there exists a* β*-open set* W_y *containing* y *such that* $f^{-1}(W_y) \subseteq \bigcup H_y$. *Then,* $f^{-1}(W_y)$ *intersects at most finitely members of* $\mathcal{V}$. *This implies that* W_y *intersects at most finitely members of* $f(\mathcal{V})$. *Hence,* $f(\mathcal{V})$ *is* β*-locally finite in* Y. *So,* (Y, σ, J) *is* J-β*-paracompact.*

Theorem 13. *Let* $f : (X, \tau, I) \to (Y, \sigma, J)$ *be an open,* β*- irresolute bijective mapping and* $I = f^{-1}(J)$. *If* A *is* J-β*-paracompact in* Y*, then* $f^{-1}(A)$ *is* I-β*-paracompact in* X.

Proof. *Let* $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ *be an open cover of* $f^{-1}(A)$. *Since* f *is open,* $\mathcal{U}_1 = \{f(U_\lambda) : \lambda \in \Lambda\}$ *is an open cover of* A. *By hypothesis, this open cover has a* β*-locally finite precise* β*-open refinement* $\mathcal{V}_1 = \{V_\lambda : \lambda \in \Lambda\}$ *such that* $A - \bigcup\{V_\lambda : \lambda \in \Lambda\} \in J$. *Then,* $f^{-1}(A) - \bigcup\{f^{-1}(V_\lambda) : \lambda \in \Lambda\} \in f^{-1}(J) = I$. *Since* f *is* β*-irresolute,* $\mathcal{V} = \{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ *is* β*-locally finite* β*-open. Let* $f^{-1}(V_\lambda) \in \mathcal{V}$. *Since* $\mathcal{V}_1$ *refines* $\mathcal{U}_1$*, there exists* $f(U_\lambda) \in \mathcal{U}_1$ *such that* $V_\lambda \subseteq f(U_\lambda)$. *Then* $f^{-1}(V_\lambda) \subseteq f^{-1}(f(U_\lambda)) = U_\lambda$. *Hence* $\mathcal{V}$ *refines* $\mathcal{U}$. *Therefore* $f^{-1}(A)$ *is* I-β*-paracompact in* X.

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