Some characterizations of $\beta$-paracompactness in ideal topological space

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Abstract. In this paper, we introduce $\beta$-paracompactness with respect to an ideal ($I\beta$-paracompactness) as a weak form of $\beta$-paracompactness and $I$-paracompactness. We give some relations between this concept and some other types of paracompactness, and also we study some of its fundamental properties.

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1. Introduction

Paracompactness is one of the important concepts of general topology. In literature, different kinds of generalized paracompactness such as $S$-paracompactness [5], $P_3$-paracompactness [6] and $\beta$-paracompactness [11] are studied.

The concept of $I$-paracompactness as generalization of paracompactness was given by Zahid [24]. Furthermore, this concept was studied by Hamlet et al. [13] and Sathiyasundari and Remikadevi [22]. Recently, $S$-paracompactness with respect to an ideal which is weaker form of $I$-paracompactness was studied by J. Sanabria et al. [21].

Here, we introduce $I\beta$-paracompactness and we compare this concept with the other types of paracompactness. Then, we give counterexamples showing that the opposite directions of Proposition 1 and 2 do not hold. Furthermore, adding some conditions, we find that the reverse directions may happen to be true. Besides, we investigate some of its essential properties. Finally, we examine $I\beta$-paracompactness under some functions.

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Throughout this work, \((X, \tau)\) denotes a topological space on which no separation axioms are assumed unless clearly indicated. If \(A\) is a subset of \((X, \tau)\), then the closure of \(A\) and the interior of \(A\) will be denoted by \(\text{cl}(A)\) and \(\text{int}(A)\), respectively. Also, the class of all subsets of \(X\) will be denoted by \(\mathcal{P}(X)\). A subset \(A\) of \((X, \tau)\) is said to be semi-open [16] if there exists \(U \in \tau\) such that \(U \subseteq A \subseteq \text{cl}(U)\). This is equivalent to say that \(A \subseteq \text{cl}((\text{int}(A)))\).

A is said to be \(\beta\)-open [1] if \(\text{int}(\text{cl}(A))) \subseteq A \subseteq \text{cl}(\text{int}(A)))\). The concept of \(\beta\)-open sets is equal to that of semi-preopen sets in [7]. The family of all semi-open (resp. \(\beta\)-open and preopen) sets of \((X, \tau)\) is denoted by \(SO(X, \tau)\) (resp. \(\beta SO(X, \tau)\) and \(PO(X, \tau)\)). The complement of a semi-open (resp. \(\beta\)-open and preopen) set is said to be semi-closed [10] (resp. \(\beta\)-closed) if \(\text{cl}(A)\) and \(\text{precl}(A)\), respectively. Also, the class of all semi-closure (resp. \(\beta\)-closed and preclosed) sets containing \(A\), denoted by \(\text{scl}(A)\) (resp. \(\beta cl(A)\) and \(pcl(A)\)), is the intersection of all semi-closed (resp. \(\beta\)-closed and preclosed) sets containing \(A\). Note that, \(\beta cl(A)\) is \(\beta\)-closed [3, 7].

**Lemma 1.** [3, 7] For a subset \(A\) of a topological space \((X, \tau)\), the following conditions hold:

(i) \(x \in \beta cl(A)\) if and only if \(A \cap U \neq \emptyset\) for every \(U \in \beta O(X, \tau)\) containing \(x\),

(ii) \(A\) is \(\beta\)-closed if and only if \(A = \beta cl(A)\).

**Theorem 1.** [19] Let \((X, \tau)\) be a space, \(A \subseteq Y \subseteq X\) and \(Y\) be \(\beta\)-open in \((X, \tau)\). Then \(A\) is \(\beta\)-open in \((X, \tau)\) if and only if \(A\) is \(\beta\)-open in the subspace \((Y, \tau Y)\).

A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be pre \(\beta\)-closed [17] (pre \(\beta\)-open [17]) if for every \(\beta\)-closed (\(\beta\)-open) set \(A\) of \((X, \tau)\), \(f(A)\) is \(\beta\)-closed (\(\beta\)-open) in \((Y, \sigma)\) and \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be \(\beta\)-irresolute [17] if for every \(\beta\)-open set \(B\) of \((Y, \sigma)\), \(f^{-1}(B)\) is \(\beta\)-open in \((X, \tau)\). If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is continuous and open, then \(f\) is \(\beta\)-irresolute and pre \(\beta\)-open.

**Lemma 2.** [11] Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a surjective function. Then \(f\) is pre \(\beta\)-closed if and only if for every \(y \in Y\) and every \(\beta\)-open set \(U\) in \((X, \tau)\) which contains \(f^{-1}(y)\), there exists a \(V \in \beta O(Y, \sigma)\) such that \(y \in V\) and \(f^{-1}(V) \subseteq U\).

A space \((X, \tau)\) is called extremally disconnected[23](briefly, e. d.) if the closure of every open set in \(X\) is open and called submaximal [8] if each dense subset of \(X\) is open in \(X\).

**Lemma 3.** [20] \((X, \tau)\) is submaximal if and only if every pre-open set is open.

**Lemma 4.** [9] \((X, \tau)\) is e. d. if and only if every \(\beta\)-open set is pre-open.

A collection \(\mathcal{V}\) of subsets of a space \((X, \tau)\) is said to be locally finite [23](resp. \(s\)-locally finite [4], \(\beta\)-locally finite [11] and \(p\)-locally finite[6]), if for each \(x \in X\) there exists \(U_x \in \tau\) (resp. \(U_x \in SO(X, \tau)\), \(U_x \in \beta O(X, \tau)\) and \(U_x \in PO(X, \tau)\)) containing \(x\) and \(U_x\) intersects at most finitely many members of \(\mathcal{V}\).
Proposition 1. If $A - X$ necessarily a cover such that $A$ space.

Definition 1. I respect to $V$ and $\beta$-open sets has a finite subcover. Also a space $(X, \tau)$ is said to be $\sigma$-locally finite if $A = \bigcup_{n=1}^{\infty} A_n$ where each $A_n$ is locally finite family.

Theorem 2. [11] Let $(X, \tau)$ be an e.d. submaximal space. Then every $\beta$-locally finite collection of subsets of $X$ is locally finite.

A space $(X, \tau)$ is said to be $\beta$-compact [2] if every cover of $X$ by $\beta$-open sets has a finite subcover. Also a space $(X, \tau)$ is said to be paracompact [23] (resp. $S$-paracompact [5], $\beta$-paracompact [11] and $P_3$-paracompact [6]), if every open cover of $X$ has a locally finite open (resp. locally finite semi-open, $\beta$-locally finite $\beta$-open and $p$-locally finite preopen) refinement which covers to $X$.

An ideal is defined as a nonempty collection $I$ of subsets of $X$ satisfying the following two conditions:

1. If $A \in I$ and $B \subseteq A$, then $B \in I$,
2. If $A \in I$ and $B \in I$, then $A \cup B \in I$.

Given a topological space $(X, \tau)$ with an ideal $I$ on $X$ and if $\mathcal{P}(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^*: \mathcal{P}(X) \to \mathcal{P}(X)$, called a local function [15] of $A$ with respect to $\tau$ and $I$ is defined as follows: for $A \subseteq X$, $A^*(I, \tau) = \{x \in X : V \cap A \notin I \text{ for every } V \in \tau(x)\}$ where $\tau(x) = \{V \in \tau : x \in V\}$. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(I, \tau)$, called the $^*$-topology, finer than $\tau$, is defined by $cl^*(A) = A \cup A^*(I, \tau)$ [14]. A basis $\beta(I, \tau)$ for $\tau^*(I, \tau)$ can be described as follows: $\beta(I, \tau) = \{V - J : V \in \tau$ and $J \in I\}$[14]. We will simply write $A^*$ for $A^*(I, \tau)$, $\tau^*$ or $\tau^*(I)$ for $\tau^*(I, \tau)$ and $\beta$ for $\beta(I, \tau)$. If $I$ is an ideal on $X$, then $(X, \tau, I)$ is called an ideal topological space.

A space $(X, \tau, I)$ is said to be $I$-$\beta$-paracompact [24] ($I$-$S$-paracompact [21]) if every open cover $\mathcal{U}$ of $X$ has a locally finite open (semi-open) refinement $\mathcal{V}$, not necessarily a cover, such that $X - \bigcup\{V : V \in \mathcal{V}\} \subseteq I$. A collection $\mathcal{V}$ of subsets of $X$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \subseteq I$ is called an $I$-cover [24] of $X$. A space $(X, \tau, I)$ is said to be $I$-regular[12] if for each closed set $F$ and a point $p \notin F$, there exist disjoint open sets $U$ and $V$ such that $p \in U$ and $F - V \subseteq I$.

3. $I$-$\beta$-paracompactness

Definition 1. A space $(X, \tau, I)$ is said to be $I$-$\beta$-paracompact or $\beta$-paracompact with respect to $I$ if every open cover $\mathcal{U}$ of $X$ has a $\beta$-locally finite $\beta$-open refinement $\mathcal{V}$ (not necessarily a cover) such that $X - \bigcup\{V : V \in \mathcal{V}\} \in I$.

A subset $A$ of a space $(X, \tau, I)$ is called an $I$-$\beta$-paracompact set in $(X, \tau, I)$ if every open cover $\mathcal{U}$ of $A$ has a $\beta$-locally finite (with respect to $\tau$) $\beta$-open refinement $\mathcal{V}$ such that $A - \bigcup\{V : V \in \mathcal{V}\} \subseteq I$.

Proposition 1. If $(X, \tau)$ is $\beta$-paracompact, then $(X, \tau, I)$ is $I$-$\beta$-paracompact.

Proof. It is obvious since $\emptyset \in I$. 

Obviously, every compact space is $I$-$\beta$-paracompact since every compact space is $\beta$-paracompact [11].

The following example shows that the converse of Proposition 1 may not be true, in general.

**Example 1.** Let $X = \mathbb{N}$ be the set of natural numbers with the topology $\tau = \{G \subseteq \mathbb{N} : 5 \in G\} \cup \{\emptyset\}$ and the ideal $I = \{U \subseteq \mathbb{N} : 5 \notin U\}$. Observe that $(X, \tau, I)$ is $I$-$\beta$-paracompact space but $(X, \tau)$ is not $\beta$-paracompact since the collection $\{\{5, x\} : x \in \mathbb{N}\}$ is an open cover of $X$ which admits no $\beta$-locally finite $\beta$-open refinement in $X$.

**Remark 1.**

1. If $I = \{\emptyset\}$, then $(X, \tau, I)$ is $I$-$\beta$-paracompact if and only if $(X, \tau)$ is $\beta$-paracompact.
2. If $I = \{\emptyset\}$ and $(X, \tau, I)$ is an e.d. space, then $(X, \tau, I)$ is $I$-$\beta$-paracompact if and only if $(X, \tau)$ is $P_3$-paracompact.

**Proposition 2.** If $(X, \tau, I)$ is $I$-$S$-paracompact then it is $I$-$\beta$-paracompact.

**Proof.** Since every locally finite collection of subsets of $X$ is $\beta$-locally finite and every semi-open set is $\beta$-open, it is clear.

Clearly, every $S$-paracompact space is $I$-$\beta$-paracompact since every $S$-paracompact space is $I$-$S$-paracompact[21]. Also, every $I$-paracompact space is $I$-$\beta$-paracompact since every $I$-paracompact space is $I$-$S$-paracompact[21].

The following example shows that the converse of Proposition 2 may not be true, in general.

**Example 2.** Let $X = [0, 2] \cup [3, 10]$ with the topology $\tau = \{U \subseteq [0, 2] \subseteq U\} \cup \{\emptyset\}$ and the ideal $I = \{A : A \subseteq [0, 2]\}$. Then $(X, \tau, I)$ is $I$-$\beta$-paracompact since every open cover of $X$ has $\beta$-locally finite $\beta$-open refinement $\mathcal{V} = \{\{x\} : x \in [0, 2]\} \cup \{\{y, z\} : y \in [0, 2], z \in [3, 10]\}$ such that $X - \bigcup\{\{V : V \in \mathcal{V}\} \in I$. But it is not $I$-$S$-paracompact since $\tau = \text{SO}(X)$.

**Theorem 3.** If $(X, \tau, I)$ is an e.d. submaximal $I$-$\beta$-paracompact space, then it is $I$-$S$-paracompact.

**Proof.** It is obvious from Lemma 3, Lemma 4 and Theorem 2.

**Theorem 4.** If $(X, \tau, I)$ is $I$-$\beta$-paracompact and $J$ is an ideal on $X$ with $I \subseteq J$, then $(X, \tau, J)$ is $J$-$\beta$-paracompact.

**Proof.** Let $(X, \tau, I)$ be $I$-$\beta$-paracompact and $I \subseteq J$. And let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $X$. Since $(X, \tau, I)$ is $I$-$\beta$-paracompact, $\mathcal{U}$ has a $\beta$-locally finite $\beta$-open refinement $\mathcal{V}$ such that $X - \bigcup\{\{V : V \in \mathcal{V}\} \in I$. Since $I \subseteq J$, $X - \bigcup\{\{V : V \in \mathcal{V}\} \in J$. Thus, $(X, \tau, J)$ is $J$-$\beta$-paracompact.

**Lemma 5.** [11] Let $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ be a collection of subsets of a space $(X, \tau)$. $\mathcal{V}$ is $\beta$-locally finite if and only if $\{\text{cl}(V_\lambda) : \lambda \in \Lambda\}$ is $\beta$-locally finite.
Lemma 6. If a cover $U = \{U_\lambda : \lambda \in \Lambda\}$ of a space $(X, \tau, I)$ has a $\beta$-locally finite $\beta$-open refinement $\mathcal{V}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in I$ then there exists a $\beta$-locally finite precise $\beta$-open refinement $\mathcal{H} = \{H_\lambda : \lambda \in \Lambda\}$ of $U$ such that $X - \bigcup\{H_\lambda : H_\lambda \in \mathcal{H}\} \in I$.

Proof. The proof is similar to that of Lemma 1.3 in [21].

Definition 2. A collection $\mathcal{A}$ of subsets of a space $(X, \tau)$ is said to be $\sigma$-$\beta$-locally finite if $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ where each collection $\mathcal{A}_n$ is a $\beta$-locally finite family.

Lemma 7. Every $\beta$-locally finite collection of subsets of a space $(X, \tau)$ is $\sigma$-$\beta$-locally finite.

Proof. It is obvious.

Theorem 5. Let $(X, \tau)$ be a regular space. If $(X, \tau, I)$ is $I$-$\beta$-paracompact, then every open cover of $X$ has a $\beta$-closed $\beta$-locally finite I-cover refinement.

Proof. Let $U$ be an open cover of $X$. By regularity of $X$, for each $x \in X$ and $U_x \in U$ containing $x$, there exists an open set $G_x$ of $x$ such that $cl(G_x) \subseteq U_x$. Then $U_1 = \{G_x : x \in X\}$ is an open cover of $X$. Since $X$ is $I$-$\beta$-paracompact, $U_1$ has $\beta$-locally finite $\beta$-open refinement $\mathcal{V}_1 = \{V_\lambda : \lambda \in \Lambda\}$ such that $X - \bigcup\{V_\lambda : \lambda \in \Lambda\} \in I$. Then $X - \bigcup\{\beta cl(V_\lambda) : \lambda \in \Lambda\} \in I$. By Lemma 5, $\mathcal{V} = \{\beta cl(V_\lambda) : V_\lambda \in \mathcal{V}_1\}$ is $\beta$-locally finite. Since $\mathcal{V}_1$ refines $U_1$, for every $\lambda \in \Lambda$, there is some $G_x \in U_1$ such that $V_\lambda \subseteq G_x$. Then $\beta cl(V_\lambda) \subseteq cl(V_\lambda) \subseteq cl(G_x)$ implies $\beta cl(V_\lambda) \subseteq U_x$. Hence $\mathcal{V}$ refines $U$. So, $\mathcal{V} = \{\beta cl(V_\lambda) : V_\lambda \in \mathcal{V}_1\}$ is $\beta$-closed $\beta$-locally finite I-cover refinement.

Remark 2. If $(X, \tau, I)$ is considered to be e.d. submaximal regular space, then the Theorem 5 becomes the Theorem 2.20 in [22].

Theorem 6. If $(X, \tau, I)$ is $I$-$\beta$-paracompact, then every open cover of $X$ has a $\beta$-open $\sigma$-$\beta$-locally finite I-cover refinement.

Proof. It is obvious by Lemma 7.

Theorem 7. Let $(X, \tau, I)$ be a regular space and $\beta O(X, \tau)$ be closed under finite intersection. Then, $(X, \tau, I)$ is $I$-$\beta$-paracompact if and only if every open cover of $X$ has a $\beta$-open $\sigma$-$\beta$-locally finite I-cover refinement.

Proof. To show sufficiency, let $U$ be an open cover of $X$. By hypothesis, there exists a $\sigma$-$\beta$-locally finite $\beta$-open refinement $\mathcal{V}$ of $U$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in I$. Also, $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ where each collection $\mathcal{V}_n$ is a $\beta$-locally finite. For each $n \in \mathbb{N}$, let $H_n = \bigcup\{V : V \in \mathcal{V}_n\}$ so that $X - \bigcup\{H_n : n \in \mathbb{N}\} \in I$. For each $n \in \mathbb{N}$, let $G_n = H_n - \bigcup_{i=1}^{n-1} H_i$. Then $\{G_n : n \in \mathbb{N}\}$ refines $\{H_n : n \in \mathbb{N}\}$. Let $x \in X$, and let $n$ be the smallest member of $\{n \in \mathbb{N} : x \in H_n\}$. Then $x \in G_n$ and $X - \bigcup\{G_n : n \in \mathbb{N}\} \in I$. Also, $G_n$ is a $\beta$-open set containing $x$ that intersects only finite family number of members of $G_n$ so
that \{G_n : n \in \mathbb{N}\} is \(\beta\)-locally finite. Let \(\mathcal{O} = \{V \cap G_n : V \in \mathcal{V}_n \text{ and } n \in \mathbb{N}\}\). Since \(\{G_n : n \in \mathbb{N}\}\) is \(\beta\)-locally finite, \(\mathcal{O}\) is \(\beta\)-locally finite. Also, since \(\beta\mathcal{O}(X, \tau)\) is closed under finite intersection and \(\mathcal{V}\) is \(\beta\)-open refinement of \(\mathcal{U}\), \(\mathcal{O}\) is \(\beta\)-open refinement of \(\mathcal{U}\). Then, \(X - \bigcup\{V \cap G_n : n \in \mathbb{N}\} \in I\) because \(X - \bigcup\{G_n : n \in \mathbb{N}\} \in I\). Thus, \((X, \tau, I)\) is \(I\)-\(\beta\)-paracompact.

**Remark 3.** If \((X, \tau, I)\) is considered to be e.d. submaximal regular space, then Theorem 7 becomes Theorem 2.22 in [22].

**Theorem 8.** For any ideal topological space \((X, \tau, I)\), the following are equivalent:

(i) For every closed subset \(A\) of \(X\) and every \(x \notin A\), there exist disjoint \(\beta\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(A - V \in I\).

(ii) For every open subset \(G\) of \(X\) and every \(x \in G\), there exists a \(\beta\)-open set \(U\) such that \(x \in U\) and \(\beta\text{cl}(U) - G \in I\).

**Proof.** (i) \(\Rightarrow\) (ii) Let \(G \subseteq X\) be open and \(x \in G\). Then \(X - G = A\) is closed and \(x \notin A\). From (i), there exist disjoint \(\beta\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(A - V \in I\). Since \(U\) and \(V\) are disjoint, we have \(\beta\text{cl}(U) \subseteq X - V\). Thus, \(A \cap \beta\text{cl}(U) \subseteq A - V\). Then, \(\beta\text{cl}(U) \cap (X - G) \in I\). Therefore, \(\beta\text{cl}(U) - G \in I\).

(ii) \(\Rightarrow\) (i) Let \(A \subseteq X\) be closed and \(x \notin A\). Then, \(X - A = G\) is open and \(x \in G\). From (ii), there exists a \(\beta\)-open set \(U\) such that \(x \in U\) and \(\beta\text{cl}(U) - G \in I\). Thus, \(X - \beta\text{cl}(U) = V \in \beta\mathcal{O}(X)\) and \(U \cap V = \emptyset\). Furthermore, \(A - V = (X - G) - (X - \beta\text{cl}(U)) = \beta\text{cl}(U) - G \in I\).

The following example reveals that for a locally finite collection of subsets of \(Y = \{V_\lambda : \lambda \in \Lambda\}\) of a space \((X, \tau)\), the equality \(\text{cl}(\bigcup\{V_\lambda : \lambda \in \Lambda\}) = \bigcup\{\text{cl}(V_\lambda) : \lambda \in \Lambda\}\) always holds whereas for \(\beta\)-locally finite collection of subsets \(U = \{U_\lambda : \lambda \in \Lambda\}\) of a space \((X, \tau)\), the equality \(\beta\text{cl}(\bigcup\{U_\lambda : \lambda \in \Lambda\}) = \bigcup\{\beta\text{cl}(U_\lambda) : \lambda \in \Lambda\}\) does not hold in general.

**Example 3.** Consider the real number \(\mathbb{R}\) with usual topology \(\tau\). Let \(\mathcal{V} = \{(0, 1), (1, 2)\}\). Then \(\mathcal{V}\) is \(\beta\)-locally finite in \((\mathbb{R}, \tau)\) since it is finite. But \(\beta\text{cl}(\{(0, 1) \cup (1, 2)\}) \neq \beta\text{cl}(\{(0, 1)\}) \cup \beta\text{cl}(\{(1, 2)\})\).

**Theorem 9.** Suppose that for a \(\beta\)-locally finite collection of subsets \(Y = \{V_\lambda : \lambda \in \Lambda\}\) of a space \((X, \tau, I)\), the equality \(\beta\text{cl}(\bigcup\{V_\lambda : \lambda \in \Lambda\}) = \bigcup\{\beta\text{cl}(V_\lambda) : \lambda \in \Lambda\}\) holds. If \((X, \tau, I)\) is Hausdorff \(I\)-\(\beta\)-paracompact, then for every closed subset \(A\) of \(X\) and every \(x \notin A\), there exist disjoint \(\beta\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(A - V \in I\).

**Proof.** Let \(A \subseteq X\) closed and \(x \notin A\). Since \(X\) is Hausdorff space, there exists an open set \(H_y\) containing \(y\) for each \(y \in A\) such that \(x \notin \text{cl}(H_y)\). Thus, \(\mathcal{H} = \{H_y : y \in A\} \cup \{X - A\}\) is an open cover of \(X\). By hypothesis and Lemma 6, \(\mathcal{H}\) has a \(\beta\)-locally finite precise \(\beta\)-open refinement \(\mathcal{W} = \{W_y : y \in A\} \cup \{G\}\) such that \(W_y \subseteq H_y\) for each \(y \in A\), \(G \subseteq X - A\) and \(X - (\bigcup\{W_y : y \in A\} \cup \{G\}) \in I\). Since \(A - (\bigcup\{W_y : y \in A\}) = A - (\bigcup\{W_y : y \in A\}) \in I\), we have \(A - (\bigcup\{W_y : y \in A\}) \in I\). Let
Corollary 1. If \((X, \tau, I)\) is an e.d. submaximal Hausdorff \(I\)-\(\beta\)-paracompact space, then \((X, \tau, I)\) is \(I\)-regular.

Theorem 10. Let \(A\) and \(B\) be subsets in ideal topological space \((X, \tau, I)\). If \(A\) is \(I\)-\(\beta\)-paracompact set in \(X\) and \(B\) is closed in \(X\), then \(A \cap B\) is \(I\)-\(\beta\)-paracompact set in \(X\).

Proof. Let \(U = \{U_\lambda : \lambda \in \Lambda\}\) be an open cover of \(A \cap B\). Since \(X - B\) is open in \(X\), \(U' = \{U_\lambda : \lambda \in \Lambda\} \cup \{X - B\}\) is an open cover of \(A\). By hypothesis and Lemma 6, \(U'\) has a \(\beta\)-locally finite precise \(\beta\)-open refinement \(\{V_\lambda : \lambda \in \Lambda\} \cup \{V\}\) such that \(V_\lambda \subseteq U_\lambda\) for each \(\lambda \in \Lambda\), \(V \subseteq X - B\) and \(A - (\bigcup \{V_\lambda : \lambda \in \Lambda\} \cup \{V\}) \in I\). Since \((A \cap B) - (\bigcup \{V_\lambda : \lambda \in \Lambda\}) = (A \cap B) - (\bigcup \{V_\lambda : \lambda \in \Lambda\} \cup \{V\}) \subseteq A - (\bigcup \{V_\lambda : \lambda \in \Lambda\} \cup \{V\})\), we have \((A \cap B) - (\bigcup \{V_\lambda : \lambda \in \Lambda\}) \in I\). Hence, \(A \cap B\) is \(I\)-\(\beta\)-paracompact set in \(X\).

Corollary 2. Let \((X, \tau, I)\) be an \(I\)-\(\beta\)-paracompact space and \(A \subseteq X\). If \(A\) is closed in \(X\), then \(A\) is an \(I\)-\(\beta\)-paracompact set in \(X\).

Lemma 8. [13] If \(I \neq \emptyset\) is an ideal on \(X\) and \(Y\) is a subset of \(X\), then \(I_Y = \{Y \cap G : G \in I\} = \{G \in I : G \subseteq Y\}\) is an ideal on \(Y\).

Theorem 11. Let \(A\) and \(B\) be subsets in ideal topological space \((X, \tau, I)\) such that \(B \subseteq A\). If \(A\) is \(\beta\)-open in \(X\) and \(B\) is an \(I_A\)-\(\beta\)-paracompact set in \(A\) then \(B\) is an \(I\)-\(\beta\)-paracompact set in \(X\).

Proof. Let \(U = \{U_\lambda : \lambda \in \Lambda\}\) be an open cover of \(B\) in \(X\). Then, \(U_B = \{U_\lambda \cap A : \lambda \in \Lambda\}\) is an open cover of \(B\) in \(A\). Since \(B\) is an \(I_A\)-\(\beta\)-paracompact set in \(A\), \(U_B\) has a \(\beta\)-locally finite precise \(\beta\)-open refinement \(V_B\) in \(A\) such that \(B - \bigcup \{V_\lambda : V_\lambda \in V_B\} \in I_A\). Thus, \(V_B\) is a \(\beta\)-locally finite precise \(\beta\)-open refinement in \(X\) by Theorem 1. Also, \(B - \bigcup \{V_\lambda : V_\lambda \in V_B\} \in I\). Hence, \(B\) is an \(I\)-\(\beta\)-paracompact set in \(X\).

Theorem 12. Let \(f : (X, \tau, I) \to (Y, \sigma, J)\) be a continuous, open and pre \(\beta\)-closed surjection with \(f^{-1}(y)\) \(\beta\)-compact for every \(y \in Y\) and \(f(I) \subseteq J\). If \((X, \tau, I)\) is \(I\)-\(\beta\)-paracompact, then \((Y, \sigma, J)\) is \(J\)-\(\beta\)-paracompact.

Proof. Let \(U = \{U_\lambda : \lambda \in \Lambda\}\) be an open cover of \(Y\). Then, \(\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}\) is an open cover of \(X\). Since \((X, \tau, I)\) is \(I\)-\(\beta\)-paracompact, this open cover has a \(\beta\)-locally finite precise \(\beta\)-open refinement \(V = \{V_\lambda : \lambda \in \Lambda\}\) such that \(X - \bigcup \{V_\lambda : V_\lambda \in V\} \in I\). Since \(f\) is pre \(\beta\)-open, \(f(V) = \{f(V_\lambda) : \lambda \in \Lambda\}\) is a precise \(\beta\)-open refinement of \(U\). Also, \(Y - \bigcup \{f(V_\lambda) : \lambda \in \Lambda\} \in J\). Now, let we prove that \(f(V)\) is \(\beta\)-locally finite. Let \(y \in Y\). Since \(V\) is \(\beta\)-locally finite, for \(x \in f^{-1}(y)\), there exists a \(\beta\)-open set \(G_x\) containing \(x\) such that \(G_x\) intersects at most finitely members of \(V\). Since \(f^{-1}(y)\) is \(\beta\)-compact, \(\{G_x : x \in \)
\( f^{-1}(y) \) has a finite subcollection \( H_y \) such that \( f^{-1}(y) \subseteq \bigcup H_y \) and \( \bigcup H_y \) intersects at most finitely members of \( \mathcal{V} \). By Lemma 2, there exists a \( \beta \)-open set \( W_y \) containing \( y \) such that \( f^{-1}(W_y) \subseteq \bigcup H_y \). Then, \( f^{-1}(W_y) \) intersects at most finitely members of \( \mathcal{V} \). This implies that \( W_y \) intersects at most finitely members of \( f(\mathcal{V}) \). Hence, \( f(\mathcal{V}) \) is \( \beta \)-locally finite in \( Y \). So, \( (Y, \sigma, J) \) is \( \beta \)-\( J \)-paracompact.

**Theorem 13.** Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) be an open, \( \beta \)-irresolute bijective mapping and \( I = f^{-1}(J) \). If \( A \) is \( \beta \)-\( J \)-paracompact in \( Y \), then \( f^{-1}(A) \) is \( \beta \)-\( I \)-paracompact in \( X \).

**Proof.** Let \( \mathcal{U} = \{U_\lambda : \lambda \in \Lambda\} \) be an open cover of \( f^{-1}(A) \). Since \( f \) is open, \( \mathcal{U}_1 = \{f(U_\lambda) : \lambda \in \Lambda\} \) is an open cover of \( A \). By hypothesis, this open cover has a \( \beta \)-locally finite precise \( \beta \)-open refinement \( \mathcal{V}_1 = \{V_\lambda : \lambda \in \Lambda\} \) such that \( A - \bigcup \{V_\lambda : \lambda \in \Lambda\} \in J \). Then, \( f^{-1}(A) - \bigcup \{f^{-1}(V_\lambda) : \lambda \in \Lambda\} \in f^{-1}(J) = I \). Since \( f \) is \( \beta \)-irresolute, \( \mathcal{V} = \{f^{-1}(V_\lambda) : \lambda \in \Lambda\} \) is \( \beta \)-locally finite \( \beta \)-open. Let \( f^{-1}(V_\lambda) \in \mathcal{V} \). Since \( \mathcal{V}_1 \) refines \( \mathcal{U}_1 \), there exists \( f(U_\lambda) \in \mathcal{U}_1 \) such that \( V_\lambda \subseteq f(U_\lambda) \). Then \( f^{-1}(V_\lambda) \subseteq f^{-1}(f(U_\lambda)) = U_\lambda \). Hence \( \mathcal{V} \) refines \( \mathcal{U} \). Therefore \( f^{-1}(A) \) is \( \beta \)-\( I \)-paracompact in \( X \).

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**References**


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