



Existence and Uniqueness Solution under Non-lipschitz condition of the mixed fractional Heston's model

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Abstract. This paper focuses on a mixed fractional version of Heston model in which the volatility Brownian and price Brownian are replaced by mixed fractional Brownian motion with the Hurst parameter $H \in (\frac{3}{4}, 1)$ so that the model exhibits the long range dependence. The existence and uniqueness of solution of mixed fractional Heston model is established under various non-Lipschitz condition and a related Euler discretization method is discussed. An example on the American put option price using Least Squares Monte Carlo Algorithm to produce acceptable results under the mixed fractional Heston model is presented to illustrate the applicability of the theory. The numerical result obtained proves the performance of our results.

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1. Introduction

The fractional Brownian motion (fBm) $B^H = \{B_t^H, t \in [0, T]\}$ with Hurst parameter $H \in (0, 1)$ is a Gaussian self-similar process with stationary increments. This process was introduced by [18] and studied by [23], where a stochastic integral representation in terms of a standard Brownian motion (Bm for short) was established. The parameter H is called Hurst index from the statistical analysis, developed by the climatologist [17]. The self-similarity and stationary increments properties make the fBm an appropriate model for many applications in diverse fields from biology to finance [19], [22]. If $H \neq \frac{1}{2}$, the process B_t^H is neither a semimartingale, nor Markovian process. Therefore, the classical Ito calculus can not be used to analyze the fBm process. In order to overcome this problem, we use a mixed fractional Brownian motion (mfBm for short) with a, b and H parameters

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which is a linear combination of Brownian motion and fractional Brownian motion with parameter H , defined for any $t \in [0, T]$ by:

$$M_t^H = \{aB_t + bB_t^H, \forall a, b > 0\}, \quad (1)$$

where B^H is the fBm with Hurst parameter $H \in (0, 1)$. [10] proved that the mfBm process with Hurst parameter $H \in]\frac{3}{4}, 1[$ is equivalent to a martingale aB_t and hence it is arbitrage-free.

Let $T > 0$ be a fixed time and $(\Omega, \mathcal{F}, (F_t)_{t \in [0, T]}, \mathbb{P})$ be a given filtered complete probability space with $(F_t)_{t \in [0, T]}$ being a filtration that satisfies the usual hypothesis.

The aim of this paper is to study the following stochastic differential equation (SDE for short) on \mathbb{R}^n

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dM_t^H. \quad (2)$$

On most occasions, the coefficients of SDEs driven by mfBm are assumed to satisfy the Lipschitz condition. The existence and uniqueness of solutions of SDEs driven by fBm with Lipschitz condition have been studied in [13], [11]. Fortunately, many researchers have investigated the SDEs under non-Lipschitz condition and they presented many meaningful results [31], [5], [29, 30]. But, to the best of our knowledge, the existence and uniqueness of solutions of SDEs driven by mfBm with a non-Lipschitz condition have not been considered. This point motivates us to carry out the present study. In the present paper, we discuss the SDEs with mfBm under the non-Lipschitz condition. Using the successive approximation method, the existence and uniqueness theorems of solutions to the following non-Lipschitz SDEs driven by mfBm are proved:

$$X(t) = x_0 + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma_1(s, X(s))dB_s + \int_0^t \sigma_2(s, X(s))dB_s^H, \quad (3)$$

where $t \in [0, T]$, $x_0 = \xi \in \mathbb{R}^n$ is a random variable, $0 \leq T < \infty$, the process B represent a m -dimensional standard F_t -Brownian motion and the process B^H represent a d -dimensional F_t -adapted fractional Brownian motion with the Hurst index $H \in (\frac{3}{4}, 1)$ defined in a same complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mu(t, X(t)) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\sigma_1(t, X(t)) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma_2(t, X(t)) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are all mesurable functions.

The main difficulty when considering Equation (3) lies in the fact that both stochastic integrals are dealt in different ways. However, the integral with respect to the Bm is an It integral, while the integral with respect to the fBm has to be understood in the pathwise sense. Finally, using the LSM Algorithm, we will calculate the value of the American put option price under the Heston model governed by M^H (mixed fractional Heston model (in short MFH)) for differents values of the Hurst parameter H and we compare the result with the value of American option price under Heston Model (HM). We remind that in (3), $\int_0^t \cdot dB_s$ stand for the stochastic integral w.r.t Bm, $\int_0^t \cdot dB_s^H$ stand for the stochastic integral w.r.t fBm.

The article is organized as follows. In section 2, we recall briefly the malliavin calculus in order to define the integral with respect to fBm and introduce proper normed spaces and we also state our assumptions on the coefficients μ , σ_1 and σ_2 of Equation (3).

In section 3, we give a version of Heston's model which the volatility Brownian and stock price Brownian are replaced by the mixed fractional Brownian motion. The main existence and uniqueness results are discussed under the non-Lipschitz condition and simulation result using the

discretization of Euler's method for 80 simulation paths of the stock price is provided. Finally, in section 4, we give the applicability of the general theory to calculate the value of American put option price under the MFH model and we compare the result with the value of American put option price under the Heston Model (HM).

2. Preliminaries

The (elementary) background needed here about the theory of integral with respect fBm [26], [20]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete filtered probability space satisfying the usual assumptions. We fix some notation throughout the paper $|\cdot|$ will denote the absolute value of a real number, the Euclidean norm of a vector, or the operator norm of a matrix. The symbol K will denote a generic constant, whose value may change from one value to another. \diamond denote the Wick product and is defined in [7]. If A is a vector or matrix, its transpose is denoted by A^T .

Stochastic differential equation with respect to fBm have been interpreted via various stochastic integral, such as the Wick-integral, the Wiener integral, the Skorohod integral, and path-wise integral [26], [14],[2, 3], [9], [25]. In this paper, we consider the path-wise integral with respect to fBm [26].

2.1. Stochastic integral with respect to fBm.

We begin by a brief review of the malliavin calculus. We start with the definition of the integral with respect to fBm as a path-wise stochastic integrals (symmetric, forward and backward integral) for fBm on the slow-fast systems, following the work of [7].

Definition 1. Let $u(t)$ be a stochastic process with integrable trajectories.

(i) The symmetric integral of $u(t)$ with respect to B_t^H is defined as

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^T u(s) [B_{s+\varepsilon}^H - B_{s-\varepsilon}^H] ds, \quad (4)$$

provided that the limit exists in probability, and is denoted by $\int_0^T u(s) d^\circ B_s^H$.

(ii) The forward integral of $u(t)$ with respect to B_t^H is defined as

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^T u(s) \left[\frac{B_{s+\varepsilon}^H - B_s^H}{\varepsilon} \right] ds, \quad (5)$$

provided that the limit exists in probability, and is denoted by $\int_0^T u(s) d^- B_s^H$.

(iii) The backward integral of $u(t)$ with respect to B_t^H is defined as

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^T u(s) \left[\frac{B_{s-\varepsilon}^H - B_s^H}{\varepsilon} \right] ds, \quad (6)$$

provided that the limit exists in probability, and is denoted by $\int_0^T u(s) d^+ B_s^H$.

For the convenience of readers, some basic properties of the path-wise stochastic integrals are provided as follows:

Let $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by

$$\varphi(t, s) = H(2H - 1)|t - s|^{2H-2}, t, s \in \mathbb{R}_+, \quad (7)$$

where H is a constant with $\frac{1}{2} < H < 1$.

Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ be borel measurable, define

$$L^2_\varphi(\mathbb{R}_+) = \left\{ g : \|g\|_\varphi^2 = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} g(t)g(s)\varphi(t, s)dsdt < \infty \right\}, \tag{8}$$

if we equip $L^2_\varphi(\mathbb{R}_+)$ and with the inner product

$$\langle g_1, g_2 \rangle_\varphi = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} g_1(t)g_2(s)\varphi(t, s)dsdt, \quad g_1, g_2 \in \mathbb{R}_+ \tag{9}$$

then $L^2_\varphi(\mathbb{R}_+)$ become respectively the separable Hilbert space.

Let \mathcal{S} be the set of smooth and cyndrical random variables of the form

$$F(\omega) = f \left(\int_0^T \psi_1(t)dB_t^H, \dots, \int_0^T \psi_n(t)dB_t^H \right) \tag{10}$$

where $n \geq 1, f \in C_b^\infty(\mathbb{R}_+^n)$ (f and all its partial derivatives are bounded) and $\psi_i \in \mathcal{H}$.

\mathcal{H} is the completion of the measurable functions such that $\|\psi_i\|^2 < \infty$ and ψ_n , is the sequence in \mathcal{H} such that $\langle \psi_i, \psi_j \rangle_\varphi = \delta_{ij}$. We denote by \mathcal{H} the space of measurable functions h on $[0, T]$ satisfying

$$\|h\|_{\mathcal{H}}^2 = \int_0^T \int_0^T |h(t)||h(s)|\varphi(t, s)dsdt < \infty, \tag{11}$$

where \mathcal{H} is a Banach space with the norm $\|\cdot\|_{\mathcal{H}}$.

The malliavin derivative D_t^H of a smooth and cylindrical random variable $F \in \mathcal{S}$ is defined as the \mathcal{H} -value random variable:

$$D_t^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \left(\int_0^T \psi_1(t)dB_t^H, \dots, \int_0^T \psi_n(t)dB_t^H \right) \psi_i(t), \tag{12}$$

then for any $p \geq 1$, the derivative operator D_t^H is a closable operator from $L^p(\Omega)$ into $L^p(\Omega, \mathcal{H})$. We define the φ -derivative of F :

$$D_t^\varphi F = \int_{\mathbb{R}_+} \varphi(t, v)D_v^H F dv. \tag{13}$$

Definition 2. The space $L_\varphi[0, T]$ of integrands is defined as the family of stochastic process $u(t)$ on $[0, T]$, such that $\mathbb{E} \int_0^T \|u(s)\|_\varphi^2 < \infty$ $u(t)$ is φ -differentiable, the trace of $D_t^\varphi u(t)$ exists; $0 \leq s \leq T, 0 \leq t \leq T, \mathbb{E} \int_0^T \int_0^T [D_s^\varphi u(s)]^2 dsdt < \infty$ and for each sequence of partition $(\pi_n, n \in \mathbb{N})$ such that $|\pi_n| \rightarrow 0$ as $n \rightarrow 0$

$$\sum_{i=0}^{n-1} \mathbb{E} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \left| D_s^\varphi u^\pi(t_i^{(n)})u^\pi(t_j^{(n)}) - D_s^\varphi u(t)D_t^\varphi u(s) \right| dsdt$$

and $\mathbb{E}[\|u^\pi - u\|_\varphi^2]^2$ tend to 0 as $n \rightarrow 0$, where $\pi_n = t_0^{(n)} < t_1^{(n)} < \dots < t_{n-1}^{(n)} < t_n^{(n)} = T$.

According to the remark 1 in [3] and Proposition 6.2.3 in [7]. Let $u(t)$ be a stochastic process in the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$, and satisfies $\int_0^T \int_0^T |D_s^H u(s)|^{2H-2} dsdt < \infty$ then we can see that symetric integral $\int_0^T u(s)d^\circ B_s^H$ coincides with the forward and backward integrals.

If $u(s) \in L_\varphi[0, T]$, then one of the pathwise integrals exist and the following relation holds :

$$\int_0^T u(s) d^\circ B_s^H = \int_0^T u(s) \diamond dB_s^H + \int_0^T D_t^\varphi u(s) ds. \tag{14}$$

Lemma 1. Let B_t^H be the fBm with $\frac{1}{2} < H < 1$ and $u(t)$ be a stochastic process in $D^{1,2}(|\mathcal{H}|) \cap (L_\varphi[0, T])$, then for every $T < \infty$,

$$\mathbb{E} \left[\int_0^T u(s) d^\circ B_s^H \right]^2 \leq 2HT^{2H-1} \mathbb{E} \left[\int_0^T |u(s)|^2 ds \right] + 4T \mathbb{E} \left[\int_0^T D_s^\varphi u(s) \right]^2 ds. \tag{15}$$

Proof. We have:

$$\begin{aligned} \mathbb{E} \left[\int_0^T u(s) d^\circ B_s^H \right]^2 &= \mathbb{E} \left[\int_0^T u(s) \diamond dB_s^H + \int_0^T D_t^\varphi u(s) ds \right]^2 \\ &\leq 2 \mathbb{E} \left[\int_0^T u(s) \diamond dB_s^H \right]^2 + 2 \mathbb{E} \left[\int_0^T D_s^\varphi u(s) \right]^2 ds \\ &\leq 2HT^{2H-1} \mathbb{E} \left[\int_0^T |u(s)|^2 ds \right] + 4T \mathbb{E} \left[\int_0^T D_s^\varphi u(s) \right]^2 ds. \end{aligned}$$

2.2. Hypothesis of non-Lipschitz condition

Throughout this paper we assume that the coefficients μ , σ_1 and σ_2 , which are continuous, satisfy, for all $x, y \in \mathbb{R}^n$ and $t \in [0, T]$, the assumptions **(A.1)** and **(A.2)**:

A.1 The functions μ and σ_1 have a linear growth and satisfy suitable modulus of continuity with respect to variable x uniformly in t . Assumption **A.1** means that μ and σ_1 satisfy:

- A.1.1 $|\mu(t, x)| \leq K(1 + |x|)$
- A.1.2 $|\mu(t, x) - \mu(t, y)|^2 \leq \varrho(|x - y|^2)$
- A.1.3 $|\sigma_1(t, x)| \leq K(1 + |x|)$
- A.1.4 $|\sigma_1(t, x) - \sigma_1(t, y)| \leq \varrho(|x - y|^2)$,

where ϱ is a concave increasing function from \mathbb{R}_+ to \mathbb{R}_+ such that $\varrho(0) = 0$, $\varrho(u) > 0$ for $u > 0$ and

$$\int_{0^+} \frac{du}{\varrho(u)} = +\infty. \tag{16}$$

A.2 The functions $\mu(t, 0)$ and $\sigma_2(t, 0)$ are locally integral with respect to t , and the function σ_2 is continuously differentiable in the first variable t . Assumption **A.2** means that μ and σ_2 satisfy $\forall t \in [0, T]$, $\mu(t, \cdot), \sigma_2(t, \cdot) \in L_\varphi([0, T]) \cap \mathbb{D}^{1,2}(|\mathcal{H}|)$:

$$\mathbb{E} |\tilde{\mu}(t, x, y)|^2 + \mathbb{E} |\tilde{\sigma}_2(t, x, y)|^2 + \mathbb{E} |D_t^\varphi(\tilde{\sigma}_2(t, x, y))|^2 \leq \varrho(\mathbb{E}|x - y|^2), \tag{17}$$

with φ is given by (7), and

$$\begin{cases} \tilde{\sigma}_2(t, x, y) = \sigma_2(t, x) - \sigma_2(t, y) \\ \tilde{\mu}(t, x, y) = \mu(t, x) - \mu(t, y). \end{cases} \tag{18}$$

The non-Lipschitz condition has a variety of forms [4], [27], [1] and [28].

Now, let define two sequences of functions $\{\chi_n(t)\}_{n=1,2,\dots}$ and $\{\tilde{\chi}_{n,p}(t)\}_{n=1,2,\dots}$ such that $\chi_1(t) = Ct$, $\chi_{n+1}(t) = \int_0^t \varrho_1(\chi_n(s))ds$ and $\tilde{\chi}_{n,p}(t) = \sup_{0 \leq s \leq t} \mathbb{E} |X_{n+p}(s) - X_n(s)|^2$, $n = 1, 2, \dots$ where $p \geq 1$ is fixed arbitrarily.

Lemma 2. (Liu, [20]) Under the non-Lipschitz condition,

$$0 \leq \tilde{\chi}_{n,p}(t) \leq \chi_n(t) \leq \chi_{n-1}(t) \leq \dots \leq \chi_1(t), \tag{19}$$

for all positive integer n .

Lemma 3. There exists a positive number G ,

$$\begin{aligned} \forall \mu(t, \cdot), \sigma_2(t, \cdot) \in L_\varphi([0, T]) \cap \mathbb{D}^{1,2}(|\mathcal{H}|), \\ \mathbb{E} |\mu(t, x)|^2 + \mathbb{E} |\sigma_2(t, x)|^2 + \mathbb{E} |D_t^\varphi(\sigma_2(t, x))|^2 \leq G (1 + \mathbb{E}|x|^2). \end{aligned} \tag{20}$$

Proof. Since $\varrho(u)$ is a concave and non-negative function, we can choose two positive constants $a > 0$ and $b > 0$, so that $\kappa(u) \leq a + bu$

$$\begin{aligned} & \mathbb{E} |\mu(t, x)|^2 + \mathbb{E} |\sigma_2(t, x)|^2 + \mathbb{E} |D_t^\varphi(\sigma_2(t, x))|^2 \\ & \leq 2\mathbb{E} (|\mu(t, 0)|^2 + |\sigma_2(t, 0)|^2 + |D_t^\varphi(\sigma_2(t, 0))|^2) \\ & + 2\mathbb{E} |\mu(t, x) - \mu(t, 0)|^2 + 2\mathbb{E} |\sigma_2(t, x) - \sigma_2(t, 0)|^2 + 2\mathbb{E} |D_t^\varphi(\sigma_2(t, x)) - D_t^\varphi(\sigma_2(t, 0))|^2 \\ & \leq 2 \sup_{0 \leq t \leq T} \mathbb{E} (|\mu(t, 0)|^2 + |\sigma_2(t, 0)|^2 + |D_t^\varphi(\sigma_2(t, 0))|^2) + 2\varrho(\mathbb{E}(x)^2) \\ & \leq G(1 + \mathbb{E}(x)^2), \end{aligned}$$

where

$$G = 2 \sup_{0 \leq t \leq T} \{ \mathbb{E} (|\mu(t, 0)|^2 + |\sigma_2(t, 0)|^2 + |D_t^\varphi(\sigma_2(t, 0))|^2) + 2a, 2b \} < \infty.$$

3. The main results

3.1. The MFH model framework

Mixed Fractional Heston model is the Heston model in which the volatility Brownian and the price Brownian are replaced by the MFBM. So we first consider the Heston model which will be described in Definition 3.

Definition 3. (Heston Model, [15]) The model given by [15] as one of the most the important stochastic volatility models. In this model the volatility is a stochastic process and it is determined by the stochastic differential equation (SDE) as follows, see [21]

$$dS_t = S_t \mu dt + \sqrt{V_t} S_t dB_{1,t} \tag{21}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma \sqrt{V_t} dB_{2,t} \tag{22}$$

$$dB_{1,t} \times dB_{2,t} = \rho dt, \tag{23}$$

where $B_{1,t}$ and $B_{2,t}$ are two Brownian motion process with correlation $\rho \in (-1, 1)$ and S represent the current stock price, V is the volatility, κ is the rate which V reverts to θ , θ is the long variance and σ is the volatility of the volatility.

Definition 4. (MFH model) Let us consider a probability space (Ω, F, \mathbb{P}) on some Brownian motion $B_i = B_{i,t}$, fractional Brownian motion $B_i^H = B_{i,t}^H$, for $i = 1, 2$. Let $(F_t)_{t \geq 0}$ be a filtration generated by these three above process and \mathbb{P} a risk neutral probability under the asset price process S_t at time $t \geq 0$. Let V_t be stochastic volatility process at time $t \geq 0$. In the Heston model, if we substitute $B_{i,t}$ by $M_i^H = M_{i,t}^H$ then, we obtain a Mixed Fractional Heston model and its SDE's is given by

$$\begin{cases} dS_t = S_t \mu dt + \sqrt{V_t} S_t dM_{1,t}^H \\ dV_t = \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dM_{2,t}^H \end{cases} \tag{24}$$

with $dM_{1,t}^H \times dM_{2,t}^H = \rho(a^2 dt + b^2 dt^{2H})$, $\rho \in (0, 1)$. i.e.

$$dS_t = S_t \mu dt + a \sqrt{V_t} S_t dB_{t,1} + b \sqrt{V_t} S_t dB_{1,t}^H \tag{25}$$

and

$$dV_t = \kappa(\theta - V_t) dt + a \sigma \sqrt{V_t} S_t dB_{t,2} + b \sigma \sqrt{V_t} S_t dB_{2,t}^H, \tag{26}$$

where κ control the speed of mean reversion of the volatility and θ is the long-run mean of the volatility, σ is the volatility of V_t process. S_0 and V_0 are spot asset price and spot variance respectively.

V_t is strictly positive when $2\kappa\theta \geq \sigma^2$ and non-negative when $0 \leq 2\kappa\theta < \sigma^2$ (Feller condition). ρ is the coefficient of correlation between B_i , fractional Brownian motion B_i^H .

3.2. Simulation of MFH model

Euler's scheme is the simplest way to discretize the stochastic differential equations [16]. We perform Euler discretisation on the MFH model. The Euler discretization can be used to approximate the asset path of the stock price on a discrete time grid [16]. Let S_t be an asset price which implies in (25) and V_t satisfying (26). Let $\Lambda = \{t_0, t_1, \dots, t_N\}$ be a partition of the interval $[0, T]$. i.e $0 < t_0 < t_1 < \dots < t_N = T$ then, we have for all $0 \leq j \leq N - 1$ and $i = 1, 2$,

$$S_{j+1} = S_j + \mu S_j \Delta t + \sqrt{V_j} S_j \Delta M_{ij}^H \tag{27}$$

$$V_{j+1} = V_j + \kappa(\theta - V_j) \Delta t + \sigma \sqrt{V_j} \Delta M_{ij}^H. \tag{28}$$

We have the following formula, for all $0 \leq j \leq N - 1$, $t_j = j \Delta t$ and $i = 1, 2, 3$.

$$\Delta M_{ij}^H = M_i^H(t_{j+1}) - M_i^H(t_j); \tag{29}$$

and

$$\Delta M_{ij}^H \sim \mathcal{N}(0, a^2 \Delta t + b^2 \Delta t^{2H}). \tag{30}$$

According to the central limit theorem, we have

$$\Delta M_{ij}^H = \mathcal{N}\left(0, a Z_i \sqrt{\Delta t} + b Z_i \sqrt{\Delta t^{2H}}\right) \tag{31}$$

and

$$Z_i \sim \mathcal{N}(0, 1).$$

Therefore, we have

$$S_{j+1} = S_j + \mu S_j \Delta t + a \sqrt{V_j} \Delta t S_j Z_1 + b S_j Z_1 \sqrt{V_j \Delta t^{2H}}, \tag{32}$$

and

$$V_{j+1} = V_j + \kappa(\theta - V_j)\Delta t + a\sigma\sqrt{V_j\Delta t}\phi_1 + b\sigma\phi_1\sqrt{V_j\Delta t^{2H}}, \tag{33}$$

where the correlated normal variables, $\phi_1 = \rho Z_1 + \sqrt{1 - \rho^2} Z_2$ generated by the Cholesky's method.

The parameters of the MFH model are taken from [24] and are presented in the following table 1. The asset price has been estimated under the MFH model, where the parameter of the option model is given by the table 1 defined by:

T	ρ	S_0	V_0	μ	σ	κ	θ	Δt	E
2	0.26	0.04	0.04	0.07	0.04	2	3	0.001	100

Table 1: **Parameter of MFH model**

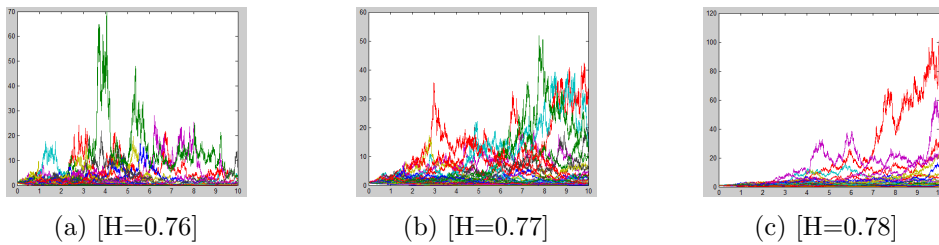


Figure 1: Simulated asset paths of MFH model.

In Figure 1 above, we see 80 simulated paths for the asset price with different Hurst parameter H such that $H > \frac{3}{4}$. Figure 1 below shows that increasing or decreasing the hurst parameters affects the future price of the asset so that, by increasing the Hurst parameter, the difference between expected lowest price and the highest price will be increased.

The simulation of the MFH model is given by the following algorithm

Algorithm 1. MFH model simulation process.

- (i) Set $\Delta = \frac{t}{N}$.
 - (ii) For $i = 1$ to number of simulation.
 - (iii) Generate independent standard normal variables, $Z_j \sim \mathcal{N}(0, 1), j = 1, \dots, N$.
 - (iv) Set $S_{j+1} \leftarrow S_j + \mu S_j \Delta t + a(V_j \Delta t)^{\frac{1}{2}} S_j Z_1 + b S_j Z_1 (V_j \Delta t^{2H})^{\frac{1}{2}}$.
 - (v) For $V_{j+1} \leftarrow V_j + \kappa(\theta - V_j)\Delta t + a\sigma(V_j \Delta t)^{\frac{1}{2}}\phi_1 + b\sigma\phi_1(V_j \Delta t^{2H})^{\frac{1}{2}}$.
 - (vi) End For.
-

3.3. Existence and Uniqueness

Now we will discuss the solutions for non-Lipschitz SDE's with Brownian motion, fBm for each equation defined in (58) by using an iteration of Picard [12]. Let $X_0(t) = \zeta$ be a random variable with $\mathbb{E}|\zeta|^2 < +\infty$. In the general case, we construct an approximative sequence of stochastic process $\{X_p(t)\}_{p \geq 1}$ as follows

$$\begin{aligned}
 X_p(t) &= \zeta + \int_0^t \mu(s, X_{p-1}(s))ds + \int_0^t \sigma_1(s, X_{p-1}(s))dB_s \\
 &+ \int_0^t \sigma_2(s, X_{p-1}(s))dB_s^H.
 \end{aligned}
 \tag{34}$$

Theorem 1. *Under the assumptions A.1 and A.2, the path-wise uniqueness holds for (3), $t \in [0, T]$.*

Proof. Let $Y(t)$ and $Z(t)$ be two solutions of (34) and $Y(0) = Z(0)$, we have

$$\begin{aligned}
 Y(t) - Z(t) &= \int_0^t \tilde{\mu}(s)ds + \int_0^t \tilde{\sigma}_1(s)dB_s + \int_0^t \tilde{\sigma}_2(s)d^\circ B_s^H \\
 &= I_\mu(t) + I_{\sigma_1}(t) + I_{\sigma_2}(t)
 \end{aligned}$$

with

$$\begin{cases}
 I_\mu(t) = \int_0^t \tilde{\mu}(s)ds \\
 I_{\sigma_1}(t) = \int_0^t \tilde{\sigma}_1(s)dB_s \\
 I_{\sigma_2}(t) = \int_0^t \tilde{\sigma}_2(s)d^\circ B_s^H.
 \end{cases}
 \tag{35}$$

where

$$\begin{cases}
 \tilde{\mu}(s) = \mu(s, Y(s)) - \mu(s, Z(s)) \\
 \tilde{\sigma}_1(s) = \sigma_1(s, Y(s)) - \sigma_1(s, Z(s)) \\
 \tilde{\sigma}_2(s) = \sigma_2(s, Y(s)) - \sigma_2(s, Z(s)),
 \end{cases}
 \tag{36}$$

By employing the following inequality

$$\forall a_1, a_2, a_3 \in \mathbb{R}, \quad |a_1 + a_2 + a_3|^2 \leq 3|a_1|^2 + 3|a_2|^2 + 3|a_3|^2.
 \tag{37}$$

It follows that

$$|Y(t) - Z(t)|^2 \leq 3 \left| \int_0^t \tilde{\mu}(s)ds \right|^2 + 3 \left| \int_0^t \tilde{\sigma}_1(s)dB_s \right|^2 + 3 \left| \int_0^t \tilde{\sigma}_2(s)d^\circ B_s^H \right|^2.
 \tag{38}$$

We observe

$$\mathbb{E}|Y(t) - Z(t)|^2 \leq 3\mathbb{E}|I_\mu(t)|^2 + 3\mathbb{E}|I_{\sigma_1}(t)|^2 + 3\mathbb{E}|I_{\sigma_2}(t)|^2.
 \tag{39}$$

We have to estimate $\mathbb{E}|I_\mu(t)|^2$, $\mathbb{E}|I_{\sigma_1}(t)|^2$ and $\mathbb{E}|I_{\sigma_2}(t)|^2$.

According to the Ito's Isometry we have

$$\mathbb{E} \left| \int_0^t \tilde{\sigma}_1(s)dB_s \right|^2 = \mathbb{E} \int_0^t |\tilde{\sigma}_1(s)|^2 ds.
 \tag{40}$$

By using Fubini's Theorem, we have

$$\mathbb{E} \int_0^t |\tilde{\sigma}_1(s)|^2 ds = \int_0^t \mathbb{E} |\tilde{\sigma}_1(s)|^2 ds.
 \tag{41}$$

Using the linear growth assumption in **(A.1.4)**, it is easy to see that

$$|\tilde{\sigma}_1(s)|^2 \leq \varrho(|Y(s) - Z(s)|^2) \tag{42}$$

and then

$$\mathbb{E} |\tilde{\sigma}_1(s)|^2 \leq \mathbb{E}[\varrho(|Y(s) - Z(s)|^2)]. \tag{43}$$

According to the Jensen's Inequality, we have

$$\mathbb{E} \left[\varrho \left(|Y(s) - Z(s)|^2 \right) \right] \leq \varrho \left(\mathbb{E} |Y(s) - Z(s)|^2 \right), \tag{44}$$

it follows that

$$\int_0^t \mathbb{E} |\tilde{\sigma}_1(s)|^2 ds \leq \int_0^t \varrho \left(\mathbb{E} |Y(s) - Z(s)|^2 \right) ds. \tag{45}$$

Therefore,

$$\mathbb{E} |I_{\sigma_1}(t)|^2 \leq \int_0^t \varrho \left(\mathbb{E} |Y(s) - Z(s)|^2 \right) ds. \tag{46}$$

Using the simple estimation, we have

$$\mathbb{E} |I_\mu(t)|^2 = \mathbb{E} \left| \int_0^t \tilde{\mu}(s) ds \right|^2 \leq T \mathbb{E} \int_0^t |\tilde{\mu}(s)|^2 ds, \tag{47}$$

and by Fubini's Theorem, we have

$$\mathbb{E} |I_\mu(t)|^2 \leq T \int_0^t \mathbb{E} |\tilde{\mu}(s)|^2 ds. \tag{48}$$

Consequently

$$\mathbb{E} |I_\mu(t)|^2 \leq 8T \int_0^t \mathbb{E} |\tilde{\mu}(s)|^2 ds. \tag{49}$$

We know that B_t^H is the fBm with $\frac{1}{2} < H < 1$ and $\tilde{\sigma}_2(t)$ is a stochastic process in $D^{1,2}(|\mathcal{H}|) \cap (L_\varphi[0, T])$, for every $t \in [0, T]$, we have from lemma 1 that

$$\mathbb{E} \left[\int_0^t \tilde{\sigma}_2(s) d^\circ B_s^H \right]^2 \leq 2Ht^{2H-1} \mathbb{E} \left[\int_0^t |\tilde{\sigma}_2(s)|^2 ds \right] + 4t \mathbb{E} \left[\int_0^t D_s^\varphi \tilde{\sigma}_2(s) \right]^2 ds. \tag{50}$$

The inequality (50) implies that

$$\mathbb{E} |I_{\sigma_2}(t)|^2 \leq 8T \int_0^t [\mathbb{E} |\tilde{\sigma}_2(s)|^2 + \mathbb{E} |D_s^\varphi \tilde{\sigma}_2(s)|^2] ds. \tag{51}$$

By combining (51) and (49), we obtain

$$\mathbb{E} |I_\mu(t)|^2 + \mathbb{E} |I_{\sigma_2}(t)|^2 \leq 8T \int_0^t [\mathbb{E} |\tilde{\mu}(s)|^2 ds + \mathbb{E} |\tilde{\sigma}_2(s)|^2 + \mathbb{E} |D_s^\varphi \tilde{\sigma}_2(s)|^2] ds. \tag{52}$$

Using the inequality (17) of the assumption **(A.2)**, we obtain

$$\mathbb{E} |I_\mu(t)|^2 + \mathbb{E} |I_{\sigma_2}(t)|^2 \leq 8T \int_0^t \varrho \left(\mathbb{E} |Y(t) - Z(t)|^2 \right) ds. \tag{53}$$

The inequality (46) and (53) give

$$\mathbb{E}|I_{\sigma_1}(t)|^2 + \mathbb{E}|I_{\mu}(t)|^2 + \mathbb{E}|I_{\sigma_2}(t)|^2 \leq (1 + 8T) \int_0^t \varrho(\mathbb{E}|Y(s) - Z(s)|^2)ds. \tag{54}$$

The inequality (39) give

$$\mathbb{E}|Y(t) - Z(t)|^2 \leq (3 + 24T) \int_0^t \varrho(\mathbb{E}|Y(s) - Z(s)|^2)ds. \tag{55}$$

Noticing that from (16), the inequality (55) implies that $\mathbb{E}|Y(t) - Z(t)|^2 = 0, \forall t \in [0, T]$. Since $T > 0$ is an arbitrary, $Y(t) \equiv Z(t), \forall t \in [0, T]$. Thus the path-wise uniqueness holds for (3).

To prove the existence of Theorem 1, we show that under the non-Lipschitz condition,

$$\lim_{n,i \rightarrow \infty} \sup_{0 \leq t \leq T} \mathbb{E}|X_p(t) - X_i(t)|^2 = 0. \tag{56}$$

We call $\{X_p(\cdot)\}_{p \geq 1}$ a Cauchy sequence which $X(\cdot)$ is its limit. By letting $p \rightarrow \infty$ in (34), we deduce that the solution to (1) exist. We fix $p \geq 1$ arbitrary and define two sequences of functions $\{\chi_n(t)\}_{n \geq 1}$ and $\{\tilde{\chi}_{n,p}(t)\}_{n \geq 1}$ where

$$\begin{cases} \chi_1(t) = Ct \\ \chi_{n+1}(t) = \int_0^t \varrho_1(\chi_n(s))ds \\ \tilde{\chi}_{n,p}(t) = \sup_{0 \leq t \leq T} \mathbb{E}|X_p(t) - X_i(t)|^2. \end{cases} \tag{57}$$

By Lemma 2, we observe that $(\chi_n(t))$ decreases when $n \rightarrow \infty$ and is nonnegative function on $t \in [0, T]$, therefore, we define $\chi(t)$ as limit of $(\chi_n(t))$, we have $\chi(0) = 0$ and $\chi(t)$ is a continuous function on $t \in [0, T]$. Or $\chi(t) = \lim_{n \rightarrow \infty} \chi_n(t)$, we have

$$\lim_{n \rightarrow \infty} \chi_{n+1}(t) = \lim_{n \rightarrow \infty} \int_0^t \varrho_1(\chi_n(s))ds = \int_0^t \varrho_1(\chi(s))ds. \tag{58}$$

Since $\chi(0) = 0$ and $\int_{0^+} \frac{du}{\varrho(u)} = +\infty$, we say that (58) implies $\chi(t) = 0$, therefore we get from the inequality (19)

$$\begin{aligned} 0 \leq \lim_{n,p \rightarrow \infty} \chi_{n+1}(t) &= \lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} \mathbb{E}|X_p(t) - X_i(t)|^2 \\ &= \lim_{n,p \rightarrow \infty} \tilde{\chi}_{n,p}(T) \\ &\leq \lim_{n \rightarrow \infty} \tilde{\chi}_n(T) = 0, \end{aligned} \tag{59}$$

namely, $\lim_{n,i \rightarrow \infty} \sup_{0 \leq t \leq T} \mathbb{E}|X_n(t) - X_i(t)|^2 = 0$.

Remark 1. The asset price S_t satisfy

$$dS_t = S_t \mu dt + a \sqrt{V_t} S_t dB_{t,1} + b \sqrt{V_t} S_t dB_{1,t}^H. \tag{60}$$

With respect to the above equation defined by (60), we can say that the stock price equation of the MFH model has a unique solution.

Let $V_0(t) = \zeta$ be a random variable with $\mathbb{E}|\zeta|^2 < +\infty$. We also construct an approximative sequence of stochastic process $\{V_t^n\}_{n \geq 1}$ as follows

$$\begin{aligned}
 V_t^n &= \zeta + \int_0^t \kappa(\theta - V_s^{n-1})ds + \int_0^t \sigma a \sqrt{V_s^{n-1}}dB_s \\
 &+ \int_0^t \sigma b \sqrt{V_s^{n-1}}dB_s^H.
 \end{aligned}
 \tag{61}$$

Theorem 2. *Under the assumptions A.1 and A.2, the volatility of the mixed Heston model has a unique positive solution V_t , where $t \in [0, T]$ and $T = \inf\{t > 0 | X_t = 0\}$.*

Proof. We have

$$V_t = \zeta + \int_0^t \kappa(\theta - V_s)ds + \int_0^t \sigma \sqrt{V_s}dM_s^H.
 \tag{62}$$

Suppose that for some initial value ζ there are two continuous solutions V_t and \tilde{V}_t satisfy (62), then the difference satisfies

$$\begin{aligned}
 V_t - \tilde{V}_t &= \int_0^t \psi_1(s)ds + \sigma a \int_0^t \psi_2(s)dB_s + \sigma b \int_0^t \psi_3(s)dB_s^H \\
 &= J_1(t) + J_2(t) + J_3(t)
 \end{aligned}
 \tag{63}$$

where

$$\begin{cases}
 J_1(t) = \int_0^t \psi_1(s)ds \\
 J_2(t) = \sigma a \int_0^t \psi_2(s)dB_s \\
 J_3(t) = \sigma b \int_0^t \psi_3(s)dB_s^H
 \end{cases}
 \tag{64}$$

with

$$\begin{cases}
 \psi_1(t) = -\kappa(V_s - \tilde{V}_s)ds \\
 \psi_2(t) = \sigma a(\sqrt{V_s} - \sqrt{\tilde{V}_s}) \\
 \psi_3(t) = \sigma b(\sqrt{V_s} - \sqrt{\tilde{V}_s}).
 \end{cases}
 \tag{65}$$

We observe that

$$\mathbb{E}|V_t - \tilde{V}_t|^2 \leq 3\mathbb{E}|J_1(t)|^2 + 3\mathbb{E}|J_2(t)|^2 + 3\mathbb{E}|J_3(t)|^2.
 \tag{66}$$

Now we have to estimated $\mathbb{E}|J_i(t)|^2, i = 1, 2, 3$.

By letting

$$\epsilon = \min \left\{ \sqrt{V_t} + \sqrt{\tilde{V}_t} > 0 | t \in [0, T] \right\},
 \tag{67}$$

according to the Ito isometry, we get

$$\mathbb{E}|J_2(t)|^2 = \mathbb{E} \int_0^t |\psi_2(s)|^2 ds.
 \tag{68}$$

By Fubini's Theorem

$$\mathbb{E}|J_2(t)|^2 = \int_0^t \mathbb{E} |\psi_2(s)|^2 ds.
 \tag{69}$$

Using the assumption in (A.1.2), we have

$$\mathbb{E}|\psi_2(s)|^2 \leq \frac{\sigma^2 a^2}{\epsilon^2} \mathbb{E} \left(\varrho \left| (V_s - \tilde{V}_s) \right|^2 \right),
 \tag{70}$$

and the Jensen's inequality give

$$\mathbb{E} |\psi_2(s)|^2 \leq \frac{\sigma^2 a^2}{\epsilon^2} \varrho \left(\mathbb{E} \left| V_s - \tilde{V}_s \right|^2 \right). \tag{71}$$

Therefore

$$\mathbb{E} |J_2(t)|^2 \leq \frac{\sigma^2 a^2}{\epsilon^2} \int_0^t \varrho \left(\mathbb{E} \left| V_s - \tilde{V}_s \right|^2 \right) ds, \tag{72}$$

using the simple estimation and Fubini's Theorem, we have

$$\mathbb{E} |J_1(t)|^2 \leq T \int_0^t \mathbb{E} |\psi_1(s)|^2 ds. \tag{73}$$

Let $M = \min\{\frac{1}{\epsilon^2}, 1\}$, we have

$$\mathbb{E} |J_1(t)|^2 \leq 8TM \int_0^t \mathbb{E} |\psi_1(s)|^2 ds. \tag{74}$$

We know that B_t^H is the fBm with $\frac{1}{2} < H < 1$ and $\psi_3(t)$ is a stochastic process in $D^{1,2}(|\mathcal{H}|) \cap (L_\varphi[0, T])$, for every $t \in [0, T]$, we have from lemma 1 that

$$\mathbb{E} \left[\int_0^t \psi_3(s) d^\circ B_s^H \right]^2 \leq \frac{1}{\epsilon^2} \left[2Ht^{2H-1} \mathbb{E} \left[\int_0^t |\tilde{\psi}_3(s)|^2 ds \right] + 4t \mathbb{E} \left[\int_0^t D_s^\varphi \tilde{\psi}_3(s) \right]^2 ds \right] \tag{75}$$

with $\tilde{\psi}_3(s) = \sigma c(V_s - \tilde{V}_s)$.

The inequality (75) implies that

$$\mathbb{E} |J_3(t)|^2 \leq 8TM \int_0^t \left[\mathbb{E} |\tilde{\psi}_3(s)|^2 + \mathbb{E} |D_s^\varphi \tilde{\psi}_3(s)|^2 \right] ds. \tag{76}$$

By combining (76) and (74), we obtain

$$\mathbb{E} |J_1(t)|^2 + \mathbb{E} |J_3(t)|^2 \leq 8TM \int_0^t \left[\mathbb{E} |\psi_1(s)|^2 ds + \mathbb{E} |\psi_3(s)|^2 + \mathbb{E} |D_s^\varphi \psi_3(s)|^2 \right] ds. \tag{77}$$

Using the inequality (17) of the assumption **(A.2)**, we obtain

$$\mathbb{E} |J_1(t)|^2 + \mathbb{E} |J_2(t)|^2 \leq 8TM \int_0^t \varrho \left(\mathbb{E} |V_s - \tilde{V}_s|^2 \right) ds. \tag{78}$$

The inequality (72) and (78) give

$$\mathbb{E} |J_1(t)|^2 + \mathbb{E} |J_2(t)|^2 + \mathbb{E} |J_3(t)|^2 \leq M(\sigma^2 a^2 + 8T) \int_0^t \varrho \left(\mathbb{E} |V_s - \tilde{V}_s|^2 \right) ds. \tag{79}$$

The inequality (66) give

$$\mathbb{E} |V_t - \tilde{V}_t|^2 \leq 3M(\sigma^2 a^2 + 8T) \int_0^t \varrho \left(\mathbb{E} |V_s - \tilde{V}_s|^2 \right) ds. \tag{80}$$

Noticing that from (16), the inequality (80) implies that $\mathbb{E} |V_t - \tilde{V}_t|^2 = 0, \forall t \in [0, T]$. Since $T > 0$ is an arbitrary, $V_t \equiv \tilde{V}_t, \forall t \in [0, T]$. Thus the path-wise uniqueness holds for (3). To show the existence, we proceed in the same way as Theorem 1.

Theorem 3. *If V_t is the solution of (62), then $\mathbb{E}(|V_t|^2) < \infty$.*

Proof. To prove the existence of Theorem 3, we show that under the assumptions **A.1** and **A.2**,

$$\mathbb{E}|V_t^n|^2 < \infty, \tag{81}$$

where for all $n \geq 1$, V_t^n verify (61). By induction, we have for $n = 1$, from Lemma 1, and Lemma 15,

$$\begin{aligned} \mathbb{E}|V_t^1|^2 &\leq 3\mathbb{E}|\zeta|^2 + 3\mathbb{E}\left|\int_0^t \kappa(\theta - V_s^0)ds\right|^2 \\ &+ 3\mathbb{E}\left|\int_0^t \bar{\sigma}_1(s, \sqrt{V_s^0})dB_s\right|^2 + 3\mathbb{E}\left|\int_0^t \bar{\sigma}_2(s, \sqrt{V_s^0})dB_s^H\right|^2 \\ &\leq 3\mathbb{E}|\zeta|^2 + 3\kappa^2\theta^2t^2 + 12T \int_0^t \left[\mathbb{E}|\kappa(s, V_s^0)|^2 + \xi_1\right] ds \end{aligned}$$

with

$$\xi_1 = \mathbb{E}|\bar{\sigma}_2(s, V_s^0)|^2 + \mathbb{E}|D_s^\varphi \bar{\sigma}_2(s, V_s^0)|^2. \tag{82}$$

Let $\bar{\epsilon} = \min_{0 \leq t \leq T} \left\{ 1; \frac{1}{\sqrt{V_t^0}} \right\}$, we have

$$\mathbb{E}|V_t^1|^2 \leq 3\mathbb{E}|\zeta|^2 + 3\kappa^2\theta^2t^2 + 12T\bar{\epsilon}^2 \int_0^t \left[\mathbb{E}|\kappa(s, V_s^0)|^2 + \xi_1\right] ds.$$

Moreover

$$\mathbb{E}|\kappa(s, V_s^0)|^2 + \xi_1 \leq G(1 + \mathbb{E}|V_s^0|^2), \tag{83}$$

with $G = \max \left[2 \sup_{0 \leq t \leq T} \mathbb{E}(|\kappa V_s^0|^2 + \bar{\xi}_1) + 2a_1, 2b_1 \right]$, where

$$\bar{\xi}_1 = |\bar{\sigma}_2(s, 0)|^2 + |D_s^\varphi \bar{\sigma}_2(s, 0)|^2. \tag{84}$$

Therefore, we get

$$\mathbb{E}|V_t^0|^2 \leq 3\mathbb{E}|\zeta|^2 + 3\kappa^2\theta^2t^2 + 12TG\bar{\epsilon}^2 \int_0^t (1 + \mathbb{E}|V_s^0|^2) ds. \tag{85}$$

As we know that $V_s^0 = \zeta$, then we have

$$\mathbb{E}|V_t^1|^2 \leq 3\mathbb{E}|\zeta|^2 + 3\kappa^2\theta^2t^2 + 12TG\bar{\epsilon}^2t(1 + \mathbb{E}|\zeta|^2) \tag{86}$$

i.e

$$\mathbb{E}|V_t^1|^2 \leq 3\mathbb{E}|\zeta|^2 + 3\kappa^2\theta^2T^2 + 12GT^2\bar{\epsilon}^2(1 + \mathbb{E}|\zeta|^2) < \infty. \tag{87}$$

The relation (87) is true. Now assume that for all n ,

$$\mathbb{E}|V_t^n|^2 \leq \bar{A} + 3\mathbb{E}|\zeta|^2 \left[\sum_{i=0}^n \frac{(12GT)^i}{i!} t^i + \sum_{i=1}^p \frac{(12GT)^i}{i!} t^i \right],$$

with $\bar{A} = 3\mathbb{E}|\zeta|^2 + 3\kappa^2\theta^2T^2 + 12GT^2$. Then we have , for $n + 1$

$$\begin{aligned} \mathbb{E}|V_t^{n+1}|^2 &\leq 3\mathbb{E}|\zeta|^2 + 3\kappa^2\theta^2t^2 + 12GT \int_0^t (1 + \mathbb{E}|V_t^n|^2) ds \\ &+ 12GT \int_0^t (1 + \mathbb{E}|V_t^n|^2) ds \\ &\leq 3\mathbb{E}|\zeta|^2 + 3\kappa^2\theta^2t^2 + 12GT \int_0^t (1 + \mathbb{E}|V_t^n|^2) ds \\ &\leq \bar{A} + 12GT \int_0^t (\mathbb{E}|V_s^n|^2) ds \\ &\leq \bar{\bar{A}} + 3\mathbb{E}|\zeta|^2 \int_0^t \left[\sum_{i=0}^n \frac{(12GT)^{i+1}}{i!} s^i + \sum_{i=1}^p \frac{(12GT)^{i+1}}{i!} s^i \right] ds \\ &\leq \bar{\bar{A}} + 3\mathbb{E}|\zeta|^2 \int_0^t \left[\sum_{i=1}^{n+1} \frac{(12GT)^i}{i!} s^i + \sum_{i=2}^{p+1} \frac{(12GT)^i}{i!} s^i \right] ds \\ &\leq \bar{\bar{A}} + 3\mathbb{E}|\zeta|^2 \int_0^t \left[\sum_{i=1}^{n+1} \frac{(12GT)^i}{i!} s^i + \sum_{i=2}^{p+1} \frac{(12GT)^i}{i!} s^i \right] ds \\ &\leq \bar{\bar{A}} + 3\mathbb{E}|\zeta|^2 \left[\sum_{i=1}^{n+1} \frac{(12GT)^i}{i!} t^i + \sum_{i=2}^{p+1} \frac{(12GT)^i}{i!} t^i \right] \end{aligned}$$

with $\bar{\bar{A}} = \bar{A}(1 + 12GT^2)$.

Hence

$$\mathbb{E}|V_t^{n+1}|^2 \leq \bar{A}(1 + 12GT^2) + 3\mathbb{E}|\zeta|^2 \left[\sum_{i=1}^{n+1} \frac{(12GT)^i}{i!} t^i + \sum_{i=2}^{p+1} \frac{(12GT)^i}{i!} t^i \right].$$

Or $e^x \simeq \sum_{i=1}^n \exp(x)$, we have

$$\begin{aligned} \mathbb{E}|V_t^{n+1}|^2 &\leq 3\mathbb{E}|\zeta|^2(1 + 12GT^2) + 3\kappa^2\theta^2T^2(1 + 12GT^2) \\ &+ 12GT^2(1 + 12GT^2) + 3\mathbb{E}|\zeta|^2 \exp(12GT^2) < \infty. \end{aligned}$$

We can conclude that $\mathbb{E}|V_t^n|^2 < \infty$ and therefore $\mathbb{E}(|V_t|^2) < \infty$.

Remark 2. If S_t satisfy the stock price of the MFH model, then $\mathbb{E}|S_t|^2 < \infty$.

4. American put option under MFH model

In mathematical finance, Monte Carlo simulation methods are best ways for pricing the American option. The benefit of the Monte Carlo simulation method is to trade with dependent options. This method can simulate the underlying asset price path by path, then obtain the payoff associated with the data for each simulated path and using the average discounted payoff to approach the expected discounted payoff which is the value of path dependent option. Least Squares Monte Carlos method (in short LSM) is more suitable for problems in higher dimensions than oder comparable Monte Carlos Method [6], [8].

4.1. LSM Algorithm

In the LSM approach, in order to have a better price, we only recognize the in-the-money paths. The LSM algorithm is given as follows,

Algoriththeorem 2. LSM algorithm

- (i) Set S_j MFH model *Asset Path*.
 - (ii) If $S_j < E$.
 - (iii) Set $cashflow(j) = (E - S_j)e^{-r\delta t}, j = 1, \dots, number\ of\ paths$.
 - (iv) Else $cashflow(j) = 0$.
 - (v) End if.
 - (vi) For $j = N - 1 : -1 : 1$.
 - (vii) Set $index = find(E - S_j > 0)$.
 - (viii) Set $X = [ons(size(index))S(index)(S(index))^2]$.
 - (ix) Set $B = (X^T X)^{\frac{1}{2}} X cashflow(index)$.
 - (x) Set $conditional\ exp = XB$.
 - (xi) If $conditional\ exp < -E - S_j, j = 1 : size(index, 1)$.
 - (xii) Set $cashflow(index(j)) = (E - S_j), j = 1 : size(index, 1)$.
 - (xiii) End if.
 - (xiv) Set $cashflow = cashflow e^{(-r\delta t)}$.
 - (xv) End if.
 - (xvi) Set $American\ put\ option = mean(cashflow)$.
-

we apply the above LSM algorithm for pricing American put option when the underlying stock price follows the MFH model. The details of LSM Algorithm can be found in [21]. The concept of the put option is related to stopping time process. Indeed, it can be expired at any time until expiration date. Let Θ be a set of stopping times and S_t be a stock price. The price of the American put option is defined as follows

$$P(\tau, S(\tau)) = \sup \left\{ \mathbb{E} \left[e^{-r\tau} (E - S(\tau))^+ \right] \right\}, \tau \in \Theta, \quad (88)$$

where $S(0)$ is initial stock price.

If $\tau = +\infty$, then the value of the American put option is zero. Here, we investigate the value of the American put option of the MFH model by considering different values of the expiration date and a, b parameter.

4.2. Experimental results

In this section, we present some numerical results for the institution the price of the American put option of the MFH model by using the above LSM algorithm. We first simulate the price of the action, the paths based on the Euler scheme described in subsection 3.2. We use the parameters given in the table 1 for the price of the American put option under the MFH model. By applying the algorithm 1 and 2, we get the following results for differents value of Hurst parameters H such that $H \in (\frac{3}{4}, 1]$:

a	b	2	3
0.1	MFH	49.7453	49.6720
0.2	HM	49.5860	49.6969

Table 2: **Comparison of American put option using MFH models as function of S_0 and V_0 for $H=0.76$.**

a	b	2	3
0.1	MFH	49.7970	49.6887
0.2	HM	49.6450	49.7773

Table 3: **Comparison of American put option using MFH models and HM as function of S_0 and V_0 for $H=0.77$.**

a	b	2	3
0.1	MFH	49.7305	49.7286
0.2	HM	49.5938	49.6614

Table 4: **Comparison of American put option using MFH models and HM as function of S_0 and V_0 for $H=0.78$.**

a	b	2	3
0.1	MFH	49.7298	49.6620
0.2	HM	49.9172	49.7155

Table 5: **Comparison of American put option using MFH model and HM as function of S_0 and V_0 for $H=0.79$**

In tables 2, 3, 4 and 5 the results are shown for some quantities of Hurst parameters $H=0.76, 0.77, 0.78, 0.79$. We see that increase in the value of the Hurst parameter leads to a significant increase in the value of the American put option price under MFH.

a	b	2	3
0.1	MFH	49.5432	49.7045
0.2	HM	49.5249	49.7619

Table 6: **Comparison of American put option using MFH model and HM as function of S_0 and V_0 for $H=0.80$.**

a	b	2	3
0.1	MFH	49.9421	49.7051
0.2	HM	49.6671	49.6763

Table 7: **Comparison of American put option using MFH model and HM as function of S_0 and V_0 for $H=0.85$.**

a	b	2	3
0.1	MFH	49.8725	49.7520
0.2	HM	49.8213	49.7047

Table 8: **Comparison of American put option using MFH model and HM as function of S_0 and V_0 for $H=0.87$.**

a	b	2	3
0.1	MFH	49.8149	49.7072
0.2	HM	49.6484	49.7865

Table 9: **Comparison of American put option using MFH model and HM as function of S_0 and V_0 for $H=0.89$.**

In tables 6, 7, 8 and 9, the results are shown for some quantities of Hurst parameters $H=0.80, H=0.85, 0.87, 0.89$. We see that increase in the value of the Hurst parameter leads to a significant increase in the value of the American put option price under MFH.

a	b	2	3
0.1	MFH	49.7141	49.6956
0.2	HM	49.6721	49.6781

Table 10: **Comparison of American put option using MFH model and HM as function of S_0 and V_0 for $H=0.90$.**

a	b	2	3
0.1	MFH	49.8439	49.7099
0.2	HM	49.9487	49.6653

Table 11: **Comparison of American put option using MFH model and HM as function of S_0 and V_0 for $H=0.95$.**

a	b	2	3
0.1	MFH	49.7129	49.6898
0.2	HM	49.8712	49.7194

Table 12: **Comparison of American put option using MFH model and HM as function of S_0 and V_0 for $H=0.97$.**

a	b	2	3
0.1	MFH	49.7509	49.6964
0.2	HM	49.7540	49.7373

Table 13: **Comparison of American put option using MFH model and HM as function of S_0 and V_0 for $H=0.99$.**

In tables 10, 11, 12 and **13**, the results are shown for some quantities of Hurst parameters **$H=0.90, H=0.95, 0.97, 0.99$** . We see that increase in the value of the Hurst parameter leads to a significant increase in the value of the American put option price under MFH.

5. Conclusion

In this paper, we have studied the application of LSM algorithm to estimate the value of American put option price of the MFH model that both the stock price and volatility in the model are governed by distinct processes. For this reason, this version of the Heston model that we have proposed in this paper is intuitive and computational efficient. We have proved that our model has a unique solution. Moreover, we have used Euler discretization method which performed the MFH model. Numerical examples showed that the LSM algorithm produces acceptable results which generalized those of the Heston model.

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