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Weakly Prime and Weakly primary ideals in gamma seminearrings

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Abstract. We introduce and discuss the weakly prime and weakly primary ideals of a gamma seminearrings with illustrative examples. We also present few of characterizations of these ideals.

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1. Introduction and Preliminaries

The concept of seminearring was introduced by W. G. van Hoorn et al. in [1]. Seminearfields have been introduced in [5]. As a generalization of seminearrings that is Γ-seminear-rings were introduced in [2]. Subsequently, prime and semiprime ideals in gamma seminearrings have been explored in [3]. In a sequel, we introduce the notion of weakly prime and weakly primary ideals Γ-seminearring and few of their characterizations. We recall some useful concepts for the sake of completeness. A nonempty set R with two binary operations " + "(addition) and "." (multiplication) is called a seminearring if it satisfies (i) $(R, +)$ and $(R, .)$ are semigroups; (ii) $(x+y).z = x.z + y.z$ for all $x, y, z \in R$. In 2005, Krishna & Chatterjee [4], introduced the condition of minimality of generalized linear sequential machines using the theory of near-semirings. Near-semirings have proven to be useful in studying automata and formal languages. Following [3], Γ-seminearring is a triple $(R, +, \Gamma)$ where, (i) Γ is a non-empty set of binary operators on R such that for each $\alpha \in \Gamma$, $(R, +, .)$ is a seminearring, (ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Similarly, let R be a Γ-seminearring, a subsemigroup A of $(R, +)$ is called a left (resp., right) ideal of R if $R\Gamma A \subseteq A$ (resp., $A\Gamma R \subseteq A$). A left and right ideal is called an ideal. Let

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R be a Γ-seminearring and $I, J \subseteq R$. We denote it by $I\Gamma J = \{a\alpha b \mid a, b \in R \text{ and } \alpha \in \Gamma\}$. A mapping $f : R \to R'$ between two gamma seminearrings is called a Γ-seminearring homomorphism (Γ-homomorphism), if $f(x+y) = f(x) + f(y)$ and $f(x\gamma y) = f(x)\gamma f(y)$ for all $x, y \in R$ and $\gamma \in \Gamma$. Let R and R' be a Γ-seminearrings and f : $R \to R'$ be a Γ-seminearring homomorphism. Then, (i) $f(I_1\Gamma I_2) = f(I_1)\Gamma f(I_2)$ for all $I_1, I_2 \in R$, (ii) $f^{-1}(J_1)\Gamma f^{-1}(J_2) = f^{-1}(J_1\Gamma J_2)$ for all J_1, $J_2 \in R'$.

2. Weakly prime and weakly primary ideals in Γ-seminearrings

In this section we introduce the notion of weakly prime and weakly primary ideals in Γ-seminearrings. By an ideal we mean two-sided ideal unless otherwise stated. We begin with the following definition.

Definition 1. Let R be a Γ-*seminearring.* A proper ideal P of R is called weakly prime if for ideals I and J, $0 \neq I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

Proposition 1. *Let P be a proper ideal of a Γ-seminearring R. The following statements are equivalent.*
(i) *P is weakly prime.*
(ii) *For ideals I and J of R, $0 \neq (I\Gamma J) \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.*
(iii) *For elements i and j in R, $i \notin P$ and $j \notin P$ implies $0 \neq (i)\Gamma(j) \nsubseteq P$.*

Proof. Following definition1, clearly (i) and (ii) are equivalent. Now, (i) $\Longrightarrow$ (iii) Let P be a weakly prime, $i \notin P$ and $j \notin P$. Assume $0 \neq (i)\ \Gamma(j) \subseteq P \Rightarrow (i) \subseteq P$ or $(j) \subseteq P$. Hence, $i \in P$ or $j \in P$, a contradiction. Thus, $0 \neq (i)\Gamma(j) \nsubseteq P$. ($iii$) $\Longrightarrow$ (i) Assume that $I \nsubseteq P$ and $J \nsubseteq P$. Then there exists $i \in I\backslash P$ and $j \in J\backslash P$. Hence, $0 \neq (i)\Gamma(j) \subseteq 0 \neq I\Gamma J$ but $0 \neq (i)\Gamma(j) \nsubseteq P$ by (iii). Thus, $0 \neq I\Gamma J \nsubseteq P$.

Example 1. Let $R = \{0, 1, e, a, b, c\}$ be a Γ-seminearring with $\Gamma = \{\alpha, 1\}$.

$+$	0	1	e	a	b	c
0	0	1	e	a	a	c
1	1	1	1	1	1	1
e	e	1	e	1	1	e
a	a	1	1	a	a	a
b	a	1	1	a	a	a
c	c	1	e	a	a	c

α	0	1	e	a	b	c
0	0	0	0	0	0	0
1	0	1	e	a	b	c
e	0	e	e	0	c	c
a	0	a	0	a	a	0
b	0	b	0	a	a	0
c	0	c	0	0	0	0

$P = \{0, a, b\}$ of a seminearring R is a weakly prime ideal but not a prime ideal, since $c\alpha c = 0$ and $c \notin P$. On the other hand, consider a prime ideal $Q = \{0, e, c\}$ of R. It is easy to show that Q is a weakly prime ideal. For this, let I and J are ideals of R, where $I = \{0, e, c\}$ and $J = \{0, a, b\}$. Then, $0 \neq I\Gamma J \subseteq Q \Longrightarrow I \subseteq Q \Longrightarrow Q$ is a weakly prime ideal. Hence every prime ideal of a gamma seminearring is a weakly prime ideal.

Example 2. Let $R = \{0, 1, e, a, b, c\}$ be a Γ-seminearring with $\Gamma = \{\alpha, 1\}$ as defined in example*1*. Let $S = \{0, 1, e, a, c\} \subseteq R$ be a Γ-sub-seminearring of R with $\Gamma = \{1, \alpha\}$.Clearly, in S the ideals $I = \{0, a\}$ and $J = \{0, c\}$ are weakly prime ideals but not prime.

Proposition 2. *Let P be a proper ideal of a Γ-seminearring R and $\{0 \neq a\alpha r\beta b : r \in R, \alpha, \beta \in \Gamma\} \subseteq P$ if and only if $a \in P$ or $b \in P$, then P is a weakly prime ideal.*

Proof. Let I and J are ideals of R with $0 \neq I\Gamma J \subseteq P$. Let $I \nsubseteq P$, and for $a \in I \backslash P$, $b \in J$, we have $\{0 \neq a\alpha r\beta b : r \in R, \alpha, \beta \in \Gamma\} \subseteq I\Gamma J \neq 0 \subseteq P$. Since $a \notin P$ and $b \in P \Rightarrow J \subseteq P$. Hence, P is a weakly prime ideal.

Proposition 3. *Intersection of finite numbers of weakly prime ideals of a Γ-seminearring R which are totally ordered by inclusion is a weakly prime ideal.*

Proof. Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be the family of weakly prime ideals which are totally ordered by inclusion. Suppose I and J be ideals of R. If $0 \neq I\Gamma J \subseteq \cap_{\alpha\in\Lambda} P_\alpha$, then $0 \neq I\Gamma J \subseteq P_\alpha$, for all $\alpha \in \Lambda$. Suppose that there exists $\alpha \in \Lambda$ such that $I \nsubseteq P_\alpha$. Then, $J \subseteq P_\alpha$ and hence $J \subseteq P_\beta$ for all $\beta \geq \alpha$. We assume that there exist $\gamma < \alpha$ such that $J \subseteq P_\gamma$.Then, $I \subseteq P_\gamma$ and hence $I \subseteq P_\alpha$, which is impossible. Hence, $J \subseteq P_\beta$ for any $\beta \in \Lambda$. Thus, $\cap_{\alpha\in\Lambda} P_\alpha$ is a weakly prime ideal of a Γ-seminearring R.

Below we provide an illustrative example.

Example 3. Let $S = \{0, 1, e, a, c\}$ with $\Gamma = \{1, \alpha\}$ be a Γ-seminearring defined in the tables given below.

+	0	1	e	a	c
0	0	1	e	a	c
1	1	1	1	1	1
e	e	1	e	1	e
a	a	1	1	a	a
c	c	1	e	a	c

α	0	1	e	a	c
0	0	0	0	0	0
1	0	1	e	a	c
e	0	e	e	0	c
a	0	a	0	a	0
c	0	c	0	0	0

Consider $P_1 = \{0, c\}$ and $P_2 = \{0, e, c\}$ are the weakly prime ideals of R and are totally ordered by inclusion as well. Since, $P_1 \cap P_2 = P_1$, which is a weakly prime ideal of R. Hence, $\cap_{\alpha \in A} P_\alpha$ is a weakly prime ideal.

Proposition 4. *Let I be an ideal of a Γ-seminearring R with $R + I \subseteq I$ and $I + R \subseteq I$. Let P be a proper ideal of R containing I and $\psi : R \to R/I$ be the canonical epimorphism. Then, P is a weakly prime ideal if and only if $\psi(P)$ is a weakly prime.*

Proof. Let P be a weakly prime ideal of R. Suppose J_1 and J_2 are ideals in R/I such that $0 \neq J_1\Gamma J_2 \subseteq \psi(P)$. Assume that $\psi^{-1}(J_1) = I_1$ and $\psi^{-1}(J_2) = I_2$. Then, $0 \neq I_1\Gamma I_2 = 0 \neq \psi^{-1}(J_1)\Gamma\psi^{-1}(J_2) \subseteq 0 \neq \psi^{-1}(J_1\Gamma J_2) \subseteq 0 \neq \psi^{-1}(\pi(P)) = P$. Since P is a weakly prime ideal, it implies $I_1 \subseteq P$ or $I_2 \subseteq P$. Hence, $J_1 = \psi(\psi^{-1}(J_1)) = \psi(I_1) \subseteq \psi(P)$ or $J_2 = \psi(\psi^{-1}(J_2)) = \psi(J_2) \subseteq \psi(P)$. Hence, $\psi(P)$ is a weakly prime. Conversely, suppose $\psi(P)$ be a weakly prime ideal and let I_1, I_2 are ideals of R such that $0 \neq I_1\Gamma I_2 \subseteq P$. Then, $0 \neq \psi(I_1)\Gamma\psi(I_2) = 0 \neq \psi(I_1\Gamma I_2) \subseteq \psi(P)$.Since $\psi(P)$ is a weakly prime ideal, it implies that $\psi(I_1) \subseteq \psi(P)$ or $\psi(I_2) \subseteq \psi(P)$. Thus, $I_1 \subseteq P$ or $I_2 \subseteq P$, and hence P is a weakly prime ideal of a Γ-seminearring R.

Definition 2. Let R be a Γ-seminearring and M be a non-empty subset of R. We call M an m-system if for a, $b \in M$, there exist $a_1 \in (a)$, $b_1 \in (b)$ and $\alpha \in \Gamma$ such that $0 \neq a_1\alpha b_1 \in M$.

Proposition 5. *Let P be a proper ideal of a Γ-seminearring R. Then, P is a weakly prime ideal if and only if $R\backslash P$ is m-system.*

Proof. Let P be a weakly prime ideal of a Γ-seminearring R. Consider a, $b \in R\backslash P$ and $0 \neq (a)\Gamma(b) \nsubseteq P$. Let $a_1 \in (a)$, $b_1 \in (b)$ and $\alpha \in \Gamma$ such that $0 \neq a_1\alpha b_1 \notin P$, i.e., $a_1\alpha b_1 \in R\backslash P$. Thus, $R\backslash P$ is an m-system. Conversely, suppose $R\backslash P$ is an m-system and let a, $b \in R\backslash P$. Then, there exist $a_1 \in (a)$, $b_1 \in (b)$ and $\alpha \in \Gamma$ such that $a_1\alpha b_1 \in R\backslash P$. Thus, $0 \neq (a)\Gamma(b) \nsubseteq P$ and hence P is a weakly prime ideal of a Γ-seminearring R.

Definition 3. *A subset A of a Γ-seminearring R is a subtractive, if $a \in A$ and $a + b \in A$ implies $b \in A$.*

Proposition 6. *Let R be a Γ-seminearring whose all ideals are subtractive, and let P be a proper ideal of R. Then, P is a weakly prime if and only if for any ideals I, J of R, $P \subset I$ and $P \subset J$ implies $0 \neq I\Gamma J \nsubseteq P$.*

Proof. Suppose for any ideals I, J of R, $P \subset I$ and $P \subset J$ implies $0 \neq I\Gamma J \nsubseteq P$. Let us suppose that $I \nsubseteq P$ and $J \nsubseteq P$. Then there exist $i \in I\backslash P$ and $j \in J\backslash P$ and hence $P \subset P + (i)$. By hypothesis, $0 \neq (P + (i))\Gamma(P + (j)) \nsubseteq P$ and so there exist $i' \in (i)$, $j' \in (j)$, $p, p' \in P$ and $\alpha \in \Gamma$ such that $0 \neq (p+i')\alpha(p'+j') \notin P$. Since, $0 \neq p\alpha(p'+j') \in P$, $0 \neq i'\alpha(p' + j') \notin P$ and P is an ideal, then $i' \notin P$ and $p' + j' \notin P$. Thus, $i' \notin P$ and $j' \notin P$ because P is subtractive. It implies $0 \neq (i')\Gamma(j') \nsubseteq P$. But $0 \neq (i')\Gamma(j') \subseteq 0 \neq I\Gamma J \Rightarrow 0 \neq I\Gamma J \nsubseteq P$. Hence, P is a weakly prime ideal. The converse is obvious by the definition of a weakly prime ideal of a Γ-seminearring.

Theorem 1. *Let M be an m-system of a Γ-seminearring R whose each ideal is a subtractive. Let I be an ideal with $I \cap M = \emptyset$. Then, there exists a weakly prime ideal P such that $I \subseteq P$ and $P \cap M = \emptyset$.*

Proof. Let $\Im = \{J : J \text{ is an ideal of } R,\ I \subseteq J \text{ and } J \cap M \neq \emptyset\}$. Then, $\Im \neq \emptyset$ and let $\{J_\alpha\}_{\alpha \in A}$ be a chain in I which is ordered under set inclusion. Then, $I \subseteq \cap_{\alpha\in\Lambda} J_\alpha$ and $(\cup_{\alpha\in\Lambda} J_\alpha) \cap M = \cup_{\alpha\in\Lambda}(J_\alpha \cap M) \neq \emptyset$. Thus, $\cup_{\alpha\in\Lambda} J_\alpha \in I$. By Zorn's Lemma, $\Im$ has a maximal element say P. We also claim that P is a weakly prime ideal. If $P \subset K_1$ and $P \subset K_2$, then there exist $k_1 \in K_1 \cap M$, $k_2 \in K_2 \cap M$ and $\alpha \in \Gamma$ such that $0 \neq (k_1)\alpha(k_2) \subseteq 0 \neq K_1\Gamma K_2$ and there exist $k_1' \in (k_1)$ and $k_2' \in (k_2)$ such that $0 \neq k_1'\alpha k_2' \in M$. Thus, $0 \neq k_1'\alpha k_2' \in 0 \neq K_1\Gamma K_2 \cap M$. Since $P \cap M = \emptyset, (K_1\Gamma K_2) \nsubseteq P$. Hence, P is a weakly prime ideal.

Now we present few results about such a Γ-seminearring R in which each ideal is weakly prime.

Proposition 7. *Every ideal of a Γ-seminearring R is a weakly prime if and only if for any ideals I, J, K of R, $I\Gamma J = I$, $I\Gamma J = J$, $I\Gamma J = K$ where K is the ideal contained in both I and J, or $I\Gamma J = 0$.*

Proof. Suppose that every ideal of R is a weakly prime. Let I, J are ideals of a Γ-seminearring R. If $I\Gamma J \neq R$, then $I\Gamma J$ is a weakly prime. If $0 \neq I\Gamma J \subseteq I\Gamma J$, then we have $I \subseteq I\Gamma J$ or $J \subseteq I\Gamma J$ i.e., $I = I\Gamma J$ or $J = I\Gamma J$. If $I\Gamma J = K$ then clearly $K = I \cap J$ is a weakly prime ideal then by proposition3, $K \subset I$ and $K \subset J$. Finally, if $I\Gamma J = R$, then we have $I = J = R$ and hence $R\Gamma R = R$.

Conversely, let L be any proper ideal of R and suppose that $0 \neq I\Gamma J \subseteq L$ for ideals I and J of R. Then, we have either $I = I\Gamma J \subseteq L$ or $J = I\Gamma J \subseteq L$. And if $K = I\Gamma J \subseteq L$, where $K \subset I \cap J$ and hence $K \cap I \subseteq I$ and $K \cap J \subseteq L$.

Example 4. Refer to the Γ-seminearring S defined by tables in example*3*. Clearly, S has four ideals, $I = \{0, a\}$, $J = \{0, c\}$, $K = \{0, e, c\}$ and $L = \{0, a, c\}$. Now, $I\Gamma J = \{0\}$, $I\Gamma K = \{0\}$, $I\Gamma L = I$, $J\Gamma I = \{0\}$, $J\Gamma K = \{0\}$, $J\Gamma L = \{0\}$, $K\Gamma I = \{0\}$, $K\Gamma J = J$, $K\Gamma L = J$ where $J \subset K$ and $J \subset L$, $L\Gamma I = I$, $L\Gamma J = \{0\}$, $L\Gamma K = \{0\}$. Hence, we can check easily that every ideal of S is a weakly prime ideal.

Corollary 1. *Let R be a Γ-seminearring in which every ideal of R is a weakly prime. Then for any ideal I of R, either $I\Gamma I = I^2 = I$ or $I\Gamma I = I^2 = 0$.*

Example 5. Refer to example*4*, since $I = \{0, a\}$ be the weakly prime ideal of S and hence $I\Gamma I = I^2 = I$. Also, for another weakly prime ideal $J = \{0, c\}$ of S we have $J\Gamma J = J^2 = 0$.

In the above example the ideal $K = \{0, e, c\}$ we have $K\Gamma K = K^2 = \{0, e\}$ which is a subset of Γ-seminearring but not an ideal. And for the ideal $L = \{0, a, c\}$ we have $L\Gamma L = L^2 = \{0, a\}$which is a weakly prime ideal of S.)

Proposition 8. *Suppose that every ideal of a Γ-seminearring R is a weakly prime. If M_1 and M_2 are two maximal ideals of R then $M_1\Gamma M_2 = 0$ or $M_1\Gamma M_2 = N = M_1 \cap M_2$.*

Proof. Suppose every ideal of a Γ-seminearring R is a weakly prime ideal. Let M_1 and M_2 be the two distinct maximal ideals. Since, $M_1 \cap M_2$ is a weakly prime and hence $M_1 \Gamma M_2 \subseteq M_1 \cap M_2$, we must have $M_1 \Gamma M_2 = 0$ and similarly $M_2 \Gamma M_1 = 0$, or $M_2 \Gamma M_1 = N$, being every ideal a weakly prime ideal of R, the result follows from proposition3.

Example 6. Refer to the Γ-seminearring S defined in tables of an example*3*. Let $I = \{0,$ $a,$ $c\}$ and $J = \{0,\ e,\ c\}$ be the two maximal ideals of S. Clearly $I\Gamma J = 0$ and $J\Gamma I = \{0,$ $c\} = I \cap J = \{0,\ c\}$.

Corollary 2. *Let every ideal of a* Γ*-seminear-ring* R *is a weakly prime. Then, every nonzero ideal of* $R/N(R)$ *is prime.*

Corollary 3. Suppose that every ideal of a Γ-seminear-ring R is a weakly prime. Then $(N(R))\Gamma(N(R)) = 0$ and every prime ideal $P(R)$ contains $N(R)$. There are three possibilities.

(a) $N(R) = R$.

(b) $N(R) = P(R)$ is the smallest prime ideal and all other prime ideals are idempotent and are linearly ordered. If $N(R) \neq 0$, then it is the only non-idempotent prime ideal.
(c) $N(R) = P(R)$ is not a prime ideal. And in such case there exist two nonzero minimal prime ideals J_1 and J_2 with $N(R) = J_1 \cap J_2$ and $J_1 \Gamma J_2 = \{0\}$ or $(d), J_2 \Gamma J_1 = \{0\}$ or (c). All other ideals containing $N(R)$ also contain $J_1 + J_2$ and they are linearly ordered.

We elaborate the above proposition in the below example.

Example 7. Refer to the Γ-seminearring S defined in tables of an example*3*. In S, $N(S) =$ $\{0,\ c\}$ and we have $(N(S))^2 = (N(R))\ \Gamma(N(R)) = \{0,\ c\}^2 = \{0\}$. As, $N(S) = P(S)$ is not a prime ideal and possibility (c) of the above corollary*3* is valid for this i.e., there exist two nonzero minimal prime ideals J_1 and J_2 with $N(R) = J_1 \cap J_2$ and $J_1 \Gamma J_2 = \{0\}$ *and* $J_2 \Gamma J_1 = (c) = \{0, c\}$. All other ideals containing $N(R)$ also contain $J_1 + J_2$ and are linearly ordered. Let $J_1 = \{0,\ a,\ c\}$ and $J_2 = \{0,\ e,\ c\}$ be the minimal prime ideals of S. We have $N(S) = \{0,\ c\} = J_1 \cap J_2$ and $J_1 J_2 = J_2 J_1 = 0$. Beside these two ideals another ideal of S is S itself and clearly it contains $N(S)$ and also $J_1 + J_2$ where $J_1 + J_2 = S$.

Example 8. *Let* $T = \{0,\ a,\ b\}$ *be a right seminearring under the operations defined in given below tables.*

+	0	a	b
0	0	a	b
a	a	a	a
b	b	b	b

.	0	a	b
0	0	0	0
a	0	a	a
b	0	a	b

Here $N(T) = P(T) = \{0\}$ and it is the smallest prime ideal. Possibility (b) of above corollary3 is valid for this seminearring.

Definition 4. Let R be a Γ-seminearring under the mapping from $R \times \Gamma \times R$ into R, say f, and D be the set of all destributive elements of R, i.e., $D = \{d \in R \mid d\alpha(a + b) =$

$d\alpha a + d\alpha b$ for all a, b $\in R$ and $\alpha \in \Gamma\}$. Then R is called distributively generated (in short, d.g.) if the set D is non empty subset of R which $f_{D\times\Gamma\times D} : D \times \Gamma \times D \to D$ and $(< D, + >) = (R, +)$ where $< D >= \{\sum_{i=1}^{m} \alpha_i d_i \mid m, \alpha_i \in N$ and $d_i \in D$ for all i $\}$. In fact, $< D >= \{\sum_{i=1}^{n} d_i \mid n \in N$ and $d_i \in D\}$ where all $d_i^{'}s$ in $\sum d_i$ may not be distinct. In addition, $(< D, + >) = (R, +)$ means that every element in R can be written as a finite sum of desrtributive elements.

Example 9. Refer to the Γ-seminearring S defined in tables of an example*3*. Let $D = \{0, 1\}$, where all elements of D are distributive elements of R i.e., $D = \{d \in R \mid d\alpha(a + b) = d\alpha a + d\alpha b$ for all a, $b \in R$ and $\alpha \in \Gamma\}$. S is called distributively generated because the set $D = \{0, 1\}$ is a nonempty subset of R which satisfies $f_{D\times\Gamma\times D} : D \times \Gamma \times D \to D$ and $(< D >, +) = (R, +)$.

Theorem 2. *Let R be a distributively generated Γ-seminearring.*

(1) *If A is weakly prime ideal of R and B is a nonempty subset of R. Then, $A\Gamma B$ is a weakly prime ideal of R.*

(2) *If A and B are weakly prime ideals of R, then $A\Gamma B$ is an ideal of R.*

Example 10. Refer to the Γ-seminearring S defined in tables of an example*3. Let $A = \{0, a\}$ be a weakly prime ideal of R and $B = \{1, e\}$ be a nonempty subset of R. Clearly $A\Gamma B = \{0, a\}$ is a weakly prime ideal of S. Let $C = \{0, c\}$ be another weakly prime ideal. Also $A\Gamma C = \{0\}$ and it is a minimal prime ideal of S.*

Weakly primary ideals

Definition 5. Let R be a Γ-seminearring. A proper ideal P of R is said to be a weakly primary ideal if $0 \neq p\gamma q \in P$ implies $p \in P$ or $q^n \in P$.

+	0	1	e	a	b	c
0	0	1	e	a	a	c
1	1	1	1	1	1	1
e	e	1	e	1	1	e
a	a	1	1	a	a	a
b	a	1	1	a	a	a
c	c	1	e	a	a	c

α	0	1	e	a	b	c
0	0	0	0	0	0	0
1	0	1	e	a	b	c
e	0	e	e	0	c	c
a	0	a	0	a	a	0
b	0	b	0	a	a	0
c	0	c	0	0	0	0

Example 11. Let $R = \{0, 1, e, a, b, c\}$ be a Γ-seminearring with $\Gamma = \{\alpha, 1\}$ defined in example*1*. Here $I = \{0, a\}$, $J = \{0, a, c\}$ are weakly primary ideals but not a weakly prime. Clearly, $b\alpha b = a \in I$ but $b^2 = a \in I$. Similarly, J is also a weakly primary but not a weakly prime ideal. neither prime because in J, as $e.b = c \in J$. Clearly, $e, b \notin J$ but $b^2 = a \in J$.

Proposition 9. *Every weakly prime ideal is a weakly primary ideal but converse is not true.*

Example 12. Let $R = \{0, 1, e, a, b, c\}$ be a Γ-seminearring with $\Gamma = \{\alpha, 1\}$. In R the ideal $I = \{0, a, b\}$ is weakly prime and also by above proposition it is weakly primary but it is not prime b/c $c\alpha c = 0$ and $c \notin I$. Another ideal $J = \{0, a\}$ is weakly primary but not weakly prime neither prime b/c $b\alpha b = a \in I$. Clearly, $b \notin I$ but $b^2 = a \in I$.

Proposition 10. *Intersection of finite numbers of weakly primary ideals of a Γ-seminearring R which are totally ordered by inclusion is a weakly primary ideal.*

Proof. Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be the family of weakly primary ideals which are totally ordered by inclusion. Suppose I and J be ideals of R. If $0 \neq I\Gamma J \subseteq \cap_{\alpha\in\Lambda} P_\alpha$, then $0 \neq I\Gamma J \subseteq P_\alpha$, for all $\alpha \in \Lambda$. Suppose that there exists $\alpha \in \Lambda$ such that $I \nsubseteq P_{_\alpha}$. Then, $J^n \subseteq P_\alpha$ and hence $J^n \subseteq P_\beta$ for all $\beta \geq \alpha$. We assume that there exist $\gamma < \alpha$ such that $J^n \subseteq P_\gamma$.Then, $I \subseteq P_\gamma$ and hence $I \subseteq P_\alpha$, which is impossible. Hence, $J^n \subseteq P_\beta$ for any $\beta \in \Lambda$. Thus, $\cap_{\alpha\in\Lambda} P_\alpha$ is a weakly primary ideal of a Γ-seminearring R.

Example 13. Let $R = \{0, 1, e, a, b, c\}$ be a Γ-seminearring with $\Gamma = \{\alpha, 1\}$. *Here $I = \{0, a\}$, $J = \{0, a, c\}$ are weakly primary ideals but not weakly prime and $K = \{0, a, b, c\}$ is prime and hence weakly primary because every prime ideal is weakly primary. Clearly, these ideals are totally ordered by inclusion i.e. $I \subseteq J \subseteq K$. Since, $I \cap J \cap K = I = \{0, a\}$, which is also a primary ideal b/c $b.b = a \in I$. Clearly, $b \notin I$ but $b^2 = a \in I$.*

Proposition 11. *Every ideal of a Γ-seminearring R is a weakly primary if and only if for any ideals I, J, K of R, $I\Gamma J = I$, $I\Gamma J = J$, $I\Gamma J = K$ where K is the ideal contained in both I and J or either in I or in J i.e.$K \subseteq I, J$ or $K \subseteq I$ or $K \subseteq J$, or $I\Gamma J = 0$.*

Proof. Suppose that every ideal of R is a weakly prime. Let I, J are ideals of a Γ-seminearring R. If $I\Gamma J \neq R$, then $I\Gamma J$ is a weakly prime. If $0 \neq I\Gamma J \subseteq I\Gamma J$, then we have $I \subseteq I\Gamma J$ or $J^n \subseteq I\Gamma J$ i.e., $I = I\Gamma J$ or $J^n = I\Gamma J$. If $I\Gamma J = K$ then clearly $K = I \cap J$ is a weakly primary ideal then, $K \subset I$ and $K \subset J^n$. Finally, if $I\Gamma J = R$, then we have $I = J = R$ and hence $R\Gamma R = R$.

Conversely, let L be any proper ideal of R and suppose that $0 \neq I\Gamma J \subseteq L$ for ideals I and J of R. Then, we have either $I = I\Gamma J \subseteq L$ or $J^n = I\Gamma J \subseteq L$. And if $K = I\Gamma J \subseteq L$, where $K \subset I \cap J$ and hence $K \cap I \subseteq I$ and $K \cap J \subseteq L$.

Example 14. Let $R = \{0, 1, e, a, b, c\}$ be a Γ-seminearring with $\Gamma = \{1, \alpha\}$. As R has six different ideals i.e. $I = \{0, a\}$, $J = \{0, c\}$, $K = \{0, e, c\}$, $L = \{0, a, c\}$, $M = \{0, a, b\}$, and

$N = \{0, a, b, c\}$. Now, $I\Gamma I = I$, $I\Gamma J = \{0\}$, $I\Gamma K = \{0\}$, $I\Gamma L = I$, $I\Gamma M = I, I\Gamma N = I$, $J\Gamma I = \{0\}, J\Gamma J = \{0\}$, $J\Gamma K = \{0\}$, $J\Gamma L = \{0\}$, $J\Gamma M = \{0\}, J\Gamma N = \{0\}$, $K\Gamma I = \{0\}$, $K\Gamma J = J$, $K\Gamma K = K$, $K\Gamma L = J$, $K\Gamma M = J$ where $J \subseteq K$, $K\Gamma N = J$, where $J \subseteq K$ and $J \subseteq N$, $L\Gamma I = I$, $L\Gamma J = \{0\}$, $L\Gamma K = \{0\}$, $L\Gamma L = I$, where $I \subseteq L$, $L\Gamma M = I$ where $I \subseteq L$ and $I \subseteq M$, $L\Gamma N = I$, where $I \subseteq L$ and $I \subseteq N$, $M\Gamma I = I$, $M\Gamma J = \{0\}$, $M\Gamma K = \{0\}$, $M\Gamma L = I$ where $I \subseteq M$ and $I \subseteq L$, $M\Gamma M = I$, where $I \subseteq M$, $M\Gamma N = I$, where $I \subseteq M$ and $I \subseteq N$. Hence, we can easily check that every ideal of R is weakly primary.

Proposition 12. *Suppose that every ideal of a Γ-seminearring R is a weakly primary. If M_1 and M_2 are two maximal ideals of R then either $M_1\Gamma M_2 = 0$ or $M_1\Gamma M_2 = N = M_1 \cap M_2$.*

Example 15. Let $R = \{0, 1, e, a, b, c\}$ be a Γ- seminearring with $\Gamma = \{1, \alpha\}$. Let $I = \{0, a, b, c\}$ and $J = \{0, e, c\}$ be the two maximal ideals of R. Clearly $I\Gamma J = \{0\}$ and $J\Gamma I = \{0, c\} = I \cap J$.

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