Finite Groups With Certain Permutability Criteria

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Abstract. Let $G$ be a finite group. A subgroup $H$ of $G$ is said to be S-permutable in $G$ if it permutes with all Sylow subgroups of $G$. In this note we prove that if $P$, the Sylow $p$-subgroup of $G$ ($p > 2$), has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are S-permutable in $G$, then $G'$ is $p$-nilpotent.

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1. Introduction

Throughout this note, $G$ denotes a finite group. The relationship between the properties of the Sylow subgroups of a group $G$ and its structure has been investigated by many authors. Starting from Gaschütz and Itô ([10], Satz 5.7, p.436) who proved that a group $G$ is solvable if all its minimal subgroups are normal. In 1970, Buckely [4] proved that a group of odd order is supersolvable if all its minimal subgroups are normal (a subgroup of prime order is called a minimal subgroup). Recall that a subgroup is said to be S-permutable in $G$ if it permutes with all Sylow subgroup of $G$. This concept, as a generalization of normality, was introduced by Kegel [11] in 1962 and has been studied extensively in many notes. For example, Srinivasan [15] in 1980 obtained the supersolvability of $G$ under the assumption that the maximal subgroups of all Sylow subgroups are S-permutable in $G$. In 2000, Ballester-Bolinches et al. [3] introduced the $c$-supplementation concept of a finite group: A subgroup $H$ of a group $G$ is said to be $c$-supplemented in $G$ if there exists a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_G$, where $H_G = \text{Core}_G(H)$ is the largest normal subgroup of $G$ contained in $H$. By using this concept they were able to prove that a group $G$ is solvable if and only if every Sylow subgroup of $G$ is $c$-supplemented in $G$. Moreover, as an application, they got the supersolvability of a group $G$ if all its minimal subgroups and the cyclic subgroups of order 4 are $c$-supplemented in $G$.

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In 2014, Heliel [8] proved that $G$ is solvable if each subgroup of prime odd order of $G$ is $c$-supplemented in $G$. Also he proved that $G$ is solvable if and only if every Sylow subgroup of odd order of $G$ is $c$-supplemented in $G$. This improved and generalized the results of Hall [6, 7], Ballester-Bolinches and Guo [2], and Ballester-Bolinches et al. [3].

Heliel also posted the following conjecture:

Let $G$ be a finite group such that every non-cyclic Sylow subgroup $P$ of odd order of $G$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are $c$-supplemented in $G$. Is $G$ solvable?

In the same year, Li et al. [12] presented a counterexample to show that the answer of this conjecture is negative in general and then gave a generalization of Heliel’s theorems.

Example 1. Let $G = A_5 \times H$, where $A_5$ is the alternating group of degree 5 and $H$ is an elementary group of order $p^n$ with $p > 5$ and $n \geq 2$. Then $G$ satisfies the condition of the preceding conjecture, but $G$ is not solvable.

In 2015, Hijazi [9] continued the above mentioned investigations and proved the following: Suppose that each Sylow subgroup $P$ of $G$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are $S$-permutable in $G$. Then $G$ is solvable.

The main goal of this note is to prove the following main theorem:

Main Theorem 1. Let $P$ be a Sylow $p$-subgroup of $G$ ($p > 2$). Suppose that $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are $S$-permutable in $G$. Then $G'$ is $p$-nilpotent.

As immediate consequences of the main theorem we have:

Corollary 1. Let $P$ be a Sylow $p$-subgroup of $G$ ($p > 2$). Suppose that $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are permutable in $G$. Then $G'$ is $p$-nilpotent.

Corollary 2 ([9], Theorem 3.1). Suppose that each Sylow subgroup $P$ of $G$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are $S$-permutable in $G$. Then $G$ is solvable.

Corollary 3 (Gaschütz and Itô [10], Satz 5.7, p.436 ). A group $G$ is solvable if all its minimal subgroups are normal.

2. Proofs

We first prove the following theorems:

Theorem 2. Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is an odd prime. If each subgroup of $P$ of order $p$ is $S$-permutable in $G$, then $G'$ is $p$-nilpotent.

Proof. We prove the theorem by induction on $|G|$. Hence if each subgroup of $P$ of order $p$ is normal in $G$, then each subgroup of $G'$ of order $p$ is normal in $G'$. Let $L$ be a
In particular, hence by Schur-Zassenhaus Theorem, \( G' \) is \( p \)-nilpotent. Thus we may assume that there exists a subgroup \( H \) of order \( p \) such that \( H \) is not normal in \( G \). By the hypothesis, \( H \) is \( S \)-permutable in \( G \) and hence by ([13], Lemma A), \( O^p(G) \leq N_G(H) < G \). Let \( M \) be a maximal subgroup of \( G \) such that \( N_G(H) \leq M < G \). Then \( M \triangleleft G \) and \( |G/M| = p \). By induction on \(|G|\), \( M' \) is \( p \)-nilpotent. Hence if \( O_{p'}(G) \neq 1 \), \( G/O_{p'}(G) \) satisfies the hypothesis of the theorem and \( G/O_{p'}(G) \) is \( p \)-nilpotent which implies that \( G' \) is \( p \)-nilpotent. Thus assume that \( O_{p'}(G) = 1 \). Since \( M' \) char \( M \) and \( M < G \), we have \( M' \lhd G \). As \( M' \) is \( p \)-nilpotent and \( O_{p'}(G) = 1 \), we have \( M' \) is a \( p \)-group. Then \( P_1 \triangleleft M \) where \( P_1 \) is a Sylow \( p \)-subgroup of \( M \). By Schur-Zassenhaus Theorem ([5, Theorem 6.2.1, p. 221]), \( M = P_1 K \), where \( K \) is a \( p' \)-Hall subgroup of \( M \). Hence if \( C_G(P_1) \leq P_1 \), \( K \) is a \( p' \)-group of automorphisms of \( P_1 \), and since \( K \) leaves each subgroup of \( P_1 \) invariant because every subgroup of \( P \) of prime order is \( S \)-permutable, then by ([14], Lemma 2.20), \( K \) is cyclic. Let \( Q \) be a Sylow \( q \)-subgroup of \( K \), where \( q \) is a prime divisor of the order of \( K \). Hence if \( p < q \), then \( P_1 Q = P_1 \times Q \) and this means that \( Q \leq C_G(P_1) \), a contradiction. Thus \( p \) is the largest prime dividing \(|G|\) and since \( K \) is cyclic, it follows, by Burnside's \( p \)-Nilpotent Theorem ([10], Satz 2.8, p.420), that \( P < G \). But \( G/P \cong K \), therefore \( G/P \) is cyclic and so abelian, then \( G' \leq P \). This completes the proof of the theorem.

As a corollary of Theorem 2.1:

**Corollary 4.** If each subgroup of prime order of \( G \) is \( S \)-permutable in \( G \), then \( G \) is solvable, \( S \triangleleft G' \) and \( G'/S \) is nilpotent, where \( S \) is a Sylow 2-subgroup of \( G' \).

**Proof.** By Theorem 2.1, \( G' \) is \( p \)-nilpotent for each odd prime \( p \) dividing \(|G|\). So \( G'/S \) is nilpotent, \( S \) is a Sylow 2-subgroup of \( G' \) and hence \( G \) is solvable.

**Theorem 3.** Let \( p \) be an odd prime and let \( P \) be a Sylow \( p \)-subgroup of \( G \). Suppose that \( P \) has a subgroup \( D \) such that \( 1 < |D| < |P| \) and all subgroups \( H \) of \( P \) with \(|H| = |D| \) are normal in \( G \). Then \( G' \) is \( p \)-nilpotent.

**Proof.** We prove the theorem by induction on \(|G|\). Clearly, \( P \cap G' \) is a Sylow \( p \)-subgroup of \( G' \). Set \( P_1 = P \cap G' \). We deal with the following two cases:

**Case 1.** \(|P_1| \leq |D|\).

Hence if \(|D| = p\), \(|P_1| = p\), and \( P_1 \triangleleft G \). Then \( G' \leq C_G(P_1) \) and so \( P_1 \leq Z(G') \).

Hence, by Schur-Zassenhaus Theorem, \( G' = P_1 \times K \), where \( K \) is a \( p' \)-Hall subgroup of \( G' \).

In particular, \( G' \) is \( p \)-nilpotent.

Thus we may assume that \(|D| = p^n \) (\( n \geq 2 \)). Let \( H \) be a subgroup of \( P \) with \(|H| = |D| \) such that \( P_1 \leq H < P \). By the hypothesis, \( H \triangleleft G \). Assume that \( \Phi(H) \neq 1 \) and consider the factor group \( G/\Phi(H) \). Obviously, \( G/\Phi(H) \) satisfies the theorem hypothesis and so \( (G/\Phi(H))' = G'/\Phi(H) \). By induction on \(|G|\), \( G'/\Phi(H) \) is \( p \)-nilpotent.

But \( G'/\Phi(H) \) is \( p \)-nilpotent by the induction on \(|G|\). Hence, \( G'/\Phi(H) \leq G' \cap \Phi(H) \) and \( \Phi(H) \leq \Phi(G) \), then we have \( G' \cap \Phi(H) \leq G' \cap \Phi(G) \) and therefore...
$G'/G' \cap \Phi(G)$ is $p$-nilpotent. Now $G'\Phi(G)/\Phi(G) \cong G'/G' \cap \Phi(G)$ is $p$-nilpotent implies that $G'\Phi(G)$ is $p$-nilpotent and consequently $G'$ is $p$-nilpotent.

Thus we may assume that $\Phi(H) = 1$ and so $H$ is elementary abelian $p$-group of order $p^n$ ($n \geq 2$). Let $L$ be a subgroup of $P$ contains such that $H$ is maximal in $L$. Clearly, $L$ is not cyclic because $H$ is elementary abelian group of order $p^n$ ($n \geq 2$). Then $L$ contains a subgroup $H_1$ such that $|H_1| = |L|$ and $H_1 \neq H$. By the hypothesis, $H_1 < G$ and since $H < G$, we have $L = H_1 \triangleleft G$ and so $\Phi(L) \leq \Phi(G)$. Hence if $\Phi(L) \neq 1$, $\Phi(L) \leq H_1 < L \leq P$. Since $L$ is not cyclic, we have $\Phi(L)$ is contained properly in $H_1$. Now it is easy to notice that the factor group $G/\Phi(L)$ satisfies the hypothesis of the theorem, so by induction on $|G|$, $G'$ is $p$-nilpotent. Thus we may assume that $\Phi(L) = 1$ and so $P_1$ is elementary abelian $p$-group. Since $P_1 \leq H < L \leq P$ and $H$ is maximal in $L$, it follows that $|L| = p^{n+1}$. Let $L_1 = \langle x_1 \rangle$ be a subgroup of $P_1$ of order $p$. Then $L = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_{n+1} \rangle$.

By the hypothesis, each maximal subgroup of $L$ is normal in $G$. Applying ([1], Lemma 2.9) implies that each subgroup of $L$ of order $p$ is normal in $G$; in particular each subgroup $L_1$ of $P_1$ of order $p$ is normal in $G$. So, $G^p \leq C_G(L_1)$ and consequently $P_1 \leq Z(G')$.

By Schur-Zassenhaus Theorem, $G' = P_1 \times K_1$, where $K_1$ is a $p'$-Hall subgroup of $G$; in particular $G'$ is $p$-nilpotent.

**Case 2.** $|P_1| > |D|$. Hence if $|D| = p$, then every subgroup of $P_1$ of order $p$ is normal in $G$, so $\Omega_1(P_1) \leq Z(G')$ which implies that $G'$ is $p$-nilpotent by ([10], Satz 5.5(a), p 435). Thus assume that $|D| = p^n$ ($n \geq 2$). Hence if $\Phi(D) \neq 1$, $G/\Phi(D)$ satisfies the hypothesis of the theorem and so $(G/\Phi(D))'' = G'/\Phi(D)/\Phi(D)$ is $p$-nilpotent by induction on $|G|$ which implies that $G'/G' \cap \Phi(G)$ is $p$-nilpotent; in particular $G'$ is $p$-nilpotent. Thus we may assume that $\Phi(D) = 1$. Let $L \leq P_1$ such that $D$ is maximal in $L$. Then $|L| = p^{n+1}$ ($n \geq 2$).

Clearly $L$ is not cyclic. Then there exists a maximal subgroup $L_1 \neq D$ in $L$. By the hypothesis $L_1 \triangleleft G$ and $D \triangleleft G$ which implies that $L = L_1 D \triangleleft G$. Hence if $\Phi(L) \neq 1$, $\Phi(L) \leq D < L \leq P_1$ and since $L$ is not cyclic, it follows that $\Phi(L) < D$. By induction on $|G|$, $G'\Phi(L)/\Phi(L) \cong G'/G' \cap \Phi(L)$ is $p$-nilpotent. In particular, $G'\Phi(G)/\Phi(G)$ is $p$-nilpotent and it follows easily that $G'$ is $p$-nilpotent. So we may assume that $\Phi(D) = 1$ and so $L$ is elementary abelian. Let $L_1 < P$ such that $|L_1| = p$. Then $L_1 < L \leq P_1$ and so $L_1 \triangleleft G$ by ([1], Lemma 2.9). In particular, $\Omega_1(P_1) \leq Z(G')$. Again by ([10], Satz 5.5(a), p 435), $G'$ is $p$-nilpotent. This completes the proof of the theorem.

Now we can move forward to prove our main theorem:

**Proof.** We prove the theorem by induction on $|G|$. Hence if $O_{p'}(G) \neq 1$, $G/O_{p'}(G)$ satisfies the hypothesis of the theorem and so $(G/O_{p'}(G))''$ is $p$-nilpotent by induction on $|G|$; in particular, $G'$ is $p$-nilpotent. Thus we may assume that $O_{p'}(G) = 1$. If each subgroup $H$ of $P$ with $|H| = |D|$ is normal in $G$, then $G'$ is $p$-nilpotent by Theorem 2.2. So we may assume that there exists a subgroup $H$ of $P$ with $|H| = |D|$ and $H$ is not normal in $G$. By hypothesis, $H$ is $S$-permutable in $G$. Since $H \triangleleft G$ and $H \triangleleft G$ is $S$-permutable in $G$, we have by ([13], Lemma A) that $O_p(G) \leq N_G(H) < G$. Let $M$ be a maximal subgroup of $G$ contains $N_G(H)$ properly. Then $M \triangleleft G$ and $|G/M| = p$. Let $P = P \cap M$ be a Sylow $p$-subgroup of $M$. By the hypothesis, $|D| \leq |P_1|$. If $|D| = |P_1|$, then $|H| = |P_1|$. So
Proof of the theorem.

Let $P \leq N_G(H)$, and since $O^p(G) \leq N_G(H)$, we have $PO^p(G) = G \leq N_G(H) < M$ which is impossible. Thus we may assume that $|D| < |P_1|$. Now $M'$ is $p$-nilpotent, by the inductive hypothesis, implies that $M'$ is a $p$-group because $O_p'(G) = 1$. Then $P_1$ is characteristic in $M$ and since $M < G$, we have $P_1 < G$. If $P < G$, then $G/P$ is abelian and since all subgroups $H$ of $P$ with $|H| = |D|$ are $S$-permutable in $G$, we have that $G$ is supersolvable by ([14], Theorem 1.3) and so $G'$ is nilpotent; in particular $G''$ is $p$-nilpotent. Thus we may assume that $P \not\triangleleft G$ and $P_1 = F(G)$ the Fitting subgroup of $G$ (recall that $O_p'(G) = 1$ and that $F(G) = \langle O_p(G) \rangle$ for all $p$ divides $|G|$). Consider the subgroup $\Phi(P_1)$ and assume that $\Phi(P_1) \neq 1$. Hence if $|\Phi(P_1)| < |D|$, then $(G/\Phi(P_1))'$ is $p$-nilpotent by induction on $|G|$; in particular $G'$ is $p$-nilpotent. So assume that $|\Phi(P_1)| \geq |D|$. If $|\Phi(P_1)| = |D|$, then $P/\Phi(P_1)$ is not cyclic. Let $L/\Phi(P_1)$ be a proper subgroup of $P/\Phi(P_1)$ such that $|L/\Phi(P_1)| = p$ ($L$ is not cyclic; otherwise $\Phi(P_1)$ is cyclic and this implies that there exists $L_1 \leq \Phi(P_1)$ such that $L_1 < G$; in particular $G/C_G(L_1)$ is isomorphic to a subgroup of $Aut(L_1)$ and so $G' \leq C_G(L_1)$ and we conclude then that $G'$ is $p$-nilpotent). As $|L/\Phi(P_1)| = p$, then there exists a maximal subgroup $L_1$ of $L$ such that $|L_1| = |\Phi(P_1)| = |D|$ and $L_1 \neq \Phi(P_1)$. But $L_1\Phi(P_1)$ is $S$-permutable in $G$, then $L_1\Phi(P_1)/\Phi(P_1) = L/\Phi(P_1)$ is $S$-permutable in $G/\Phi(P_1)$. By Theorem 2.1, $(G/\Phi(P_1))' = G'\Phi(P_1)/\Phi(P_1)$ is $p$-nilpotent and so $G'$ is $p$-nilpotent. Thus we may assume that $\Phi(P_1) = 1$ and $P_1$ is elementary abelian. Since all subgroups $H$ of $P_1$ with $|H| = |D|$ are normal in $M$, we have by ([1], Lemma 2.9) that all subgroups of $P_1$ of order $p$ are normal in $M$. So $P_1 \cap Z(P) \neq 1$. Let $L \leq P_1 \cap Z(P)$ such that $|L| = p$. Then $L < G$ and since $G/C_G(L)$ is isomorphic to a subgroup of $Aut(L)$, we have that $G' \leq C_G(L)$, in particular $G'L/L$ is $p$-nilpotent and so $G'$ is $p$-nilpotent. This completes the proof of the theorem.

References


