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Correction to the article: "An introduction to the theory of hyperlattices"

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Abstract. The definition of hyperlattices introduced in Mathematica Balkanica, 1977 by Konstantinidou and Mittas in their paper "An introduction to the theory of hyperlattices" should be corrected. As a result, the definition of distributive and modular hyperlattices introduced by Konstantinidou should be also corrected. In the present paper we correct these definitions and give some examples to show that these corrected forms work.

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1. Introduction

According to the bibliography, the concept of hyperlattices has been introduced by Konstantinidou and Mittas in Math. Balkanica [4]. The aim of this note is to correct the definition of hyperlattice introduced in [4], and the definitions of distributive and modular hyperlattices based on it, introduced in [5, 6], and give some examples to show that these corrected forms work.

For the sake of completeness, we first mention the definition of lattices just to see, step by step, how we could generalize it to a hyperlattice.

A lattice is a nonempty set L with two binary operations "$\wedge$" and "$\vee$" on L such that
(1) $a \wedge a = a$ and $a \vee a = a$
(2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$
(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$
(4) $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.

According to [4–7], the definition of a hyperlattice is given as follows:

Definition 1.1 [4] A hyperlattice is a nonempty set L with an hyperoperation "$\vee$" (that is a mapping that assigns to each couple a, b of elements of L a nonempty subset of L) and an operation "$\wedge$" on L such that
(1) $a \in a \vee a$ $\qquad$ $a \wedge a = a$

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(2) $a \vee b = b \vee a$ $\qquad a \wedge b = b \wedge a$
(3) $(a \vee b) \vee c = a \vee (b \vee c)$ $\qquad (a \wedge b) \wedge c = a \wedge (b \wedge c)$
(4) $a \in \Big[a \vee (a \wedge b)\Big] \cap \Big[a \wedge (a \vee b)\Big]$
(5) $a \in a \vee b \Rightarrow a \wedge b = b$.

In [5], immediately after this definition, the following is written: "The set $H = \{0, 1\}$ with *a* hyperoperation $0 \vee 0 = 0$, $0 \vee 1 = 1 \vee 0 = 1$, $1 \vee 1 = H$ and *an* operation $0 \wedge 0 = 1 \wedge 0 = 0 \wedge 1 = 0$, $1 \wedge 1 = 1$ is a hyperlattice. Respectively, for $0 \vee 0 = H$, $0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1$ and $0 \wedge 0 = 1 \wedge 0 = 0 \wedge 1 = 0$, $1 \wedge 1 = 1$."

There are two examples here. As they are similar, we deal with the first one. We write a instead of 0 and b instead of 1. We also correct it somehow by writing $a \vee a = \{a\}$, $a \vee b = b \vee a = \{b\}$ as cannot be $a \vee a = a$, $a \vee b = b$, $b \vee a = b$. There is no mention in the paper of identifying the $\{a\}$ by a. Thus this example in [5] is the following example.

Example 1.2 The set $H = \{a, b\}$ with the operation and the hyperoperation defined by Table 1 is a hyperlattice.

Table 1: The operation and the hyperoperation of the Example 1.2.

$\wedge$	a	b
a	a	a
b	a	b

(a)

$\vee$	a	b
a	$\{a\}$	$\{b\}$
b	$\{b\}$	$\{a, b\}$

(b)

To show that this is a hyperlattice, we must show that $a \in a \wedge (a \vee b)$ and $(a \vee b) \vee c = a \vee (b \vee c)$ for all $a, b, c \in H$. According to Table 1, we have $a \wedge (a \vee b) = a \wedge \{b\}$, but the $a \wedge \{b\}$ has no sense. The associativity of the hyperoperation "$\vee$" has no sense as well. So the Table 1 cannot define a hyperlattice.

According [4; Remark 1(a)], the Definition 1.1 implies that each lattice is a hyperlattice. This, stated without proof in [4], cannot be proved by the Definition 1.1 as this definition has no sense. As we see in Proposition 2.5 of the present paper, this is true.

The modular and distributive hyperlattices have been defined in [5] and [6], respectively, as follows: "A hyperlattice L is said to be modular if $a \leq b \Rightarrow a \vee (c \wedge b) = (a \vee c) \wedge b$ for any $c \in H$ (as in the case of lattices).

A hyperlattice L is said to be distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$."

This is the Remark 1 in [5]: "Obviously, in every modular hyperlattice we have $a \vee (b \wedge a) = (a \vee b) \wedge a$." This being obvious for lattices, cannot be proved by the definition of modular hyperlattices given in [5]. As we will see later in section 4, in a modular hyperlattice we have $\{v \wedge a \mid v \in a \vee b\} = a \vee (b \wedge a)$; and if for the set $\{v \wedge a \mid v \in a \vee b\}$ we use the notation $(a \vee b) \wedge a$, then we can say that in a modular hyperlattice the property $a \vee (b \wedge a) = (a \vee b) \wedge a$ holds.

After this remark, there is the Example 2(a) in [5]. It is no clear what "the obvious operation" in this example means. In the Example 2(b), the $b \wedge b$ and $c \wedge c$ are missing.

Later Konstantinidou and Serafimidis changed the property (5) of Definition 1.1 and wrote "$a \in a \vee b \Leftrightarrow a \wedge b = b$" instead of "$a \in a \vee b \Rightarrow a \wedge b = b$" [7] (the "$\Leftarrow$-part" being a consequence of Definition 1.1, should be omitted from the definition). They also defined the concepts of $\wedge$-*distributive* and $\vee$-*distributive* hyperlattices as the hyperlattices in which the properties $(a \vee b) \wedge c \subseteq (a \wedge c) \vee (b \wedge c)$ and $(a \wedge b) \vee c \subseteq (a \vee c) \wedge (b \vee c)$, respectively hold. Some authors consider only the properties (1)–(4) as the definition of the hyperlattice and study hyperlattices having the property (5) as an additional property. Trying to transfer the definition of lattices to hyperlattices it is much better not to include condition (5) in the definition of hyperlattices. In the present paper we will do so.

In the definition of hyperlattices the part related to "$\wedge$" is the same as in the theory of lattices, so it has the same properties as in lattices. Concerning the "$\vee$": We cannot write $a \in a \wedge (a \vee b)$ as a is an element, $a \vee b$ is a set and "$\wedge$" is an operation between elements. Also we cannot write $a \vee (b \vee c) = (a \vee b) \vee c$ as $b \vee c, a \vee b$ are sets, c, a elements and "$\vee$" and operation between elements (the so called "hyperoperation"). That is expressions of the form $a \in a \wedge (a \vee b)$, $a \vee (b \vee c)$ and $(a \vee b) \vee c$ have no sense. For the same reason the concepts of distributive and modular hyperlattices [5, 6] and the concepts of $\wedge$-distributive and $\vee$-distributive hyperlattices considered in [7] have no sense as well. What we have already said is about the definition of hyperlattice introduced by Konstantinidou and Mittas in [4] and used in [5–7] as well. Later some authors working on the subject, just after the Definition 1.1, they added: "Let $A, B \subseteq L$. Then define $A \vee B = \bigcup\{a \vee b \mid a \in A, b \in B\}$ and $A \wedge B = \{a \wedge b \mid a \in A, b \in B\}$" (see, for example [1, 2]). But this, written in a wrong place (and not only), make the definition still unreadable.

2. Hyperlattices

To pass from lattices to hyperlattices, we only have to transfer the properties $(a \vee b) \vee c = a \vee (b \vee c)$, $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$. The property $(a \vee b) \vee c = a \vee (b \vee c)$ can be naturally transferred as follows: $x \in u \vee c$ for some $x \in a \vee b$ if and only if $x \in a \vee v$ for some $v \in b \vee c$. The $a \wedge (a \vee b) = a$ can be transferred as follows: there exists $u \in a \vee b$ such that $a \wedge u = a$; or for every $u \in a \vee b$, $a \wedge u = a$. Finally, the property $a \vee (a \wedge b) = a$ could be transferred as $a \in a \vee (a \wedge b)$; if $x \in a \vee (a \wedge b)$, then $x = a$ or both. As we see, the concept of a lattice can be extended not only in one way. To keep the existing definition in the bibliography, the concept of a lattice can be naturally transferred to a hyperlattice by the definition below. We denote by $\mathcal{P}^*(L)$ the set of (all) nonempty subsets of L.

Definition 2.1 Let L be a nonempty set,
$\wedge : L \times L \to L \mid (a, b) \to a \wedge b$ an operation on L and
$\vee : L \times L \to \mathcal{P}^*(L) \mid (a, b) \to a \vee b$ a hyperoperation on L.

We say that $(L, \wedge, \vee)$ is a *hyperlattice* if, for every $a, b, c \in L$, the following assertions are satisfied:

(1) $a \wedge a = a$ and $a \in a \vee a$

(2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$

(3) $(a \wedge b) \wedge c = (a \wedge b) \wedge c$; and $(a \vee b) \vee c = a \vee (b \vee c)$ in the sense that $x \in u \vee c$ for some $u \in a \vee b$ if and only if $x \in a \vee v$ for some $v \in b \vee c$.

(4) $a \wedge (a \vee b) = a$ in the sense that there exists $u \in a \vee b$ such that $a \wedge u = a$; and $a \in a \vee (a \wedge b)$.

Remark 2.2 (A) The property $(a \vee b) \vee c = a \vee (b \vee c)$ is clearly equivalent to

$$\bigcup_{u \in a \vee b} u \vee c = \bigcup_{v \in b \vee c} a \vee v.$$

(B) There exists $u \in a \vee b$ such that $a \wedge u = a$ if and only if $a \in \{a \wedge u \mid u \in a \vee b\}$.

If in Definition 2.1 we add the property $a \in a \vee b \Rightarrow a \wedge b = b$, then this definition is equivalent to Definition 1.1 but only if, for any nonempty subsets A and B of L, we define the $A \vee B$ and $A \wedge B$ (there is no such a definition in [4–7]), preferable before the definition or in a correct way if it is after that (I mean, not as in [1–2]); and emphasize the fact that the element a should be identified by the singleton $\{a\}$ if and when is convenient and no confusion is possible.

Remark 2.3 (see, for example [1]) In a hyperlattice, $a \wedge b = b$ implies $a \in a \vee b$. [Indeed, by Definition 2.1(4), we have $a \in a \vee (a \wedge b) = a \vee b$]. As we see later, the converse of this statement does not hold in general.

Example 2.4 The set $L = \{a, b, c\}$ with the operation "$\wedge$" and the hyperoperation "$\vee$" given by Table 2 is a hyperlattice.

Table 2: The hyperlattice of the Example 2.4.

$\wedge$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

(a)

$\vee$	a	b	c
a	$\{a, b, c\}$	$\{b, c\}$	$\{c\}$
b	$\{b, c\}$	$\{b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$

(b)

Proposition 2.5 *Every lattice $(L, \wedge, \vee)$ is a hyperlattice.*

Proof We consider the operation $\wedge : L \times L \to L \mid (a, b) \to a \wedge b$ and the hyperoperation "$\dot\vee$" on L defined by

$$\dot\vee : L \times L \to \mathcal{P}^*(L) \mid (a, b) \to a \,\dot\vee\, b := \{a \vee b\}.$$

Then $(L, \wedge, \dot\vee)$ is a hyperlattice. Indeed: The operation $\dot\vee$ is well defined, the properties (1) and (2) of Definition 2.1 are satisfied, the operation "$\dot\vee$" is associative, indeed

$$\bigcup_{u \in a \,\dot\vee\, b} u \,\dot\vee\, c = \bigcup_{u = a \vee b} u \,\dot\vee\, c = (a \vee b) \,\dot\vee\, c = \{(a \vee b) \vee c\},$$

$$\bigcup_{v \in b \,\dot\vee\, c} a \,\dot\vee\, v = \bigcup_{v = b \vee c} a \,\dot\vee\, v = a \,\dot\vee (b \vee c) = \{a \vee (b \vee c)\} \text{ and}$$

since $(a \vee b) \vee c = a \vee (b \vee c)$, we have $\bigcup\limits_{u \in a \,\dot\vee\, b} u \,\dot\vee\, c = \bigcup\limits_{v \in b \,\dot\vee\, c} a \,\dot\vee\, v$.

If $a, b \in L$ then, for the element $u := a \vee b \in a \,\dot\vee\, b$, we have $a \wedge u = a \wedge (a \vee b) = a$; and $a \in \{a\} = \{a \vee (a \wedge b)\} = a \,\dot\vee (a \wedge b)$.

Second proof We consider the operation $\wedge : L \times L \to L \mid (a, b) \to a \wedge b$
and the hyperoperation "$\dot\vee$" on L defined by

$$\dot\vee : L \times L \to \mathcal{P}^*(L) \mid (a, b) \to a \,\dot\vee\, b := \{a, b, a \vee b\}.$$

Then $(L, \wedge, \dot\vee)$ is a hyperlattice. Indeed: The hyperoperation "$\dot\vee$" is well defined and the following assertions are satisfied:

(1) $a \in a \,\dot\vee\, a$, since $a \,\dot\vee\, a := \{a, a, a \vee a\} = \{a\}$.
(2) It is clear.
(3) We have

$$\begin{aligned} \bigcup_{u \in a \,\dot\vee\, b} u \,\dot\vee\, c &= \bigcup_{u \in \{a, b, a \vee b\}} u \,\dot\vee\, c = (a \,\dot\vee\, c) \vee (b \,\dot\vee\, c) \vee \Big((a \vee b) \,\dot\vee\, c\Big) \\ &= \{a,\ c,\ a \vee c,\ b,\ c,\ b \vee c,\ a \vee b,\ c,\ (a \vee b) \vee c\}. \end{aligned}$$

$$\begin{aligned} \bigcup_{v \in b \,\dot\vee\, c} a \,\dot\vee\, v &= \bigcup_{v \in \{b, c, b \vee c\}} a \,\dot\vee\, v = (a \,\dot\vee\, b) \vee (a \,\dot\vee\, c) \vee \Big(a \,\dot\vee (b \vee c)\Big) \\ &= \{a,\ b,\ a \vee b,\ a,\ c,\ a \vee c,\ a,\ b \vee c,\ a \vee (b \vee c)\}. \end{aligned}$$

Since L is a lattice, we have $(a \vee b) \vee c = a \vee (b \vee c)$ and so $\bigcup\limits_{u \in a \,\dot\vee\, b} u \,\dot\vee\, c = \bigcup\limits_{v \in b \,\dot\vee\, c} a \,\dot\vee\, v$.

(4) There exists $u \in a \,\dot\vee\, b$ such that $a \wedge u = a$. Indeed, for the element $u := a \vee b \in a \,\dot\vee\, b$, we have $a \wedge u = a \wedge (a \vee b) = a$; and $a \in a \,\dot\vee (a \wedge b)$ since $a \,\dot\vee (a \wedge b) = \{a, a \wedge b, a \vee (a \wedge b)\} = \{a, a \wedge b\}$. □

We apply Proposition 2.5 to the following example.

Example 2.6 We consider the lattice L defined by Figure 1.

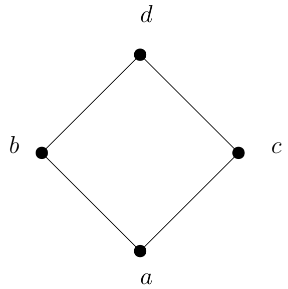

Figure 1: The lattice of the Example 2.6.

According to the first proof of Proposition 2.5, the set L with the operation and the hyperoperation of Table 3 is a hyperlattice.

Table 3: The hyperlattice of the Example 2.6
that corresponds to the first proof of Proposition 2.5.

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

(a)

$\vee$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$
c	$\{c\}$	$\{d\}$	$\{c\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

(b)

According to the second proof of Proposition 2.5, the same lattice with the operation and the hyperoperation of Table 4 is a hyperlattice. In addition, Table 4 provides us with an example of a hyperlattice for which the converse statement in Remark 2.3 does not hold. Indeed, $c \in c \vee b$ but $c \wedge b \neq b$.

Table 4: The hyperlattice of the Example 2.6
that corresponds to the second proof of Proposition 2.5.

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

(a)

$\vee$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c,d\}$	$\{b,d\}$
c	$\{a,c\}$	$\{b,c,d\}$	$\{c\}$	$\{c,d\}$
d	$\{a,d\}$	$\{b,d\}$	$\{c,d\}$	$\{d\}$

(b)

Proposition 2.7 In a hyperlattice the following are equivalent:

(1) For every $u \in a \vee b$ we have $a \wedge u = a$.

(2) $\{a \wedge x \mid x \in a \vee b\} = \{a\}$.

Proof (1) $\Longrightarrow$ (2). If $t \in \{a \wedge x \mid x \in a \vee b\}$, then $t = a \wedge x$ for some $x \in a \vee b$. Since $x \in a \vee b$, by (1), we have $a \wedge x = a$, thus we get $t = a$. On the other hand, since $a \in a \vee a$ and $a = a \wedge a$, we have $a \in \{a \wedge x \mid x \in a \vee b\}$.
(2) $\Longrightarrow$ (1). If $u \in a \vee b$, then $a \wedge u \in \{a \wedge x \mid x \in a \vee b\} = \{a\}$ and so $a \wedge u = a$. $\square$

3. Distributive hyperlattices

A lattice L is called distributive if, for any $a, b, c \in L$, we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. This concept can be naturally transferred to hyperlattices as follows:

Definition 3.1(1) A hyperlattice $(L, \wedge, \vee)$ is called *distributive* if the following assertions are satisfied:

(1) if $u \in b \vee c$, then $a \wedge u \in (a \wedge b) \vee (a \wedge c)$ and

(2) if $u \in (a \wedge b) \vee (a \wedge c)$, then there exists $v \in b \vee c$ such that $u = a \wedge v$.

In other words, $u \in (a \wedge b) \vee (a \wedge c)$ if and only if there exists $v \in b \vee c$ such that $u = a \wedge v$. That is, if $\{a \wedge v \mid v \in b \vee c\} = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$.

The equivalent definition $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ of distributive lattices can be transferred to hyperlattices in Definition 3.1(2). According to the problem we face we use either the first or the second definition.

Definition 3.1(2) A hyperlattice $(L, \wedge, \vee)$ is called *distributive* if the following assertions are satisfied:

(1) if $u \in a \vee (b \wedge c)$, then there exist $v \in a \vee b$ and $w \in a \vee c$ such that $u = v \wedge w$ and

(2) if $u \in a \vee b$ and $v \in a \vee c$, then $u \wedge v \in a \vee (b \wedge c)$.

In other words, $u \in a \vee (b \wedge c)$ if and only if there exist $v \in a \vee b$ and $w \in a \vee c$ such that $u = v \wedge w$. That is, if $\{v \wedge w \mid v \in a \vee b,\, w \in a \vee c\} = a \vee (b \wedge c)$ for all $a, b, c \in L$.

Example 3.2 (see also [1; Lemma 3.20]) The hyperlattice of Table 5 is not distributive in the sense of Definition 3.1(2). This is because for the element $c \in b \vee (a \wedge b)$, there are no $v \in b \vee a$ and $w \in b \vee b$ such that $c = v \wedge w$. It is not distributive in the sense of Definition 3.1(1) as well, since $c \in (b \wedge a) \vee (b \wedge c)$ but there is no $v \in a \vee c$ such that $c = b \wedge v$.

Table 5: The hyperlattice of the Example 3.2.

$\wedge$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

(a)

$\vee$	a	b	c
a	$\{a\}$	$\{b, c\}$	$\{c\}$
b	$\{b, c\}$	$\{b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$

(b)

Example 3.3 (see also [3; Example 1.8]) The hyperlattice defined by Table 6 is a distributive hyperlattice in the sense of Definition 3.1(1) since for all $a, b, c \in L$, we have $\{a \wedge v \mid v \in b \vee c\} = (a \wedge b) \vee (a \wedge c)$; but it is not distributive in the sense of Definition 3.1(2) as $c \in d \vee b$, $d \in d \vee c$ but $c \wedge d \notin d \vee (b \wedge c)$.

Table 6: The hyperlattice of the Example 3.3.

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

(a)

$\vee$	a	b	c	d
a	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{b\}$	$\{a, b\}$	$\{d\}$	$\{c, d\}$
c	$\{c\}$	$\{d\}$	$\{a, c\}$	$\{b, d\}$
d	$\{d\}$	$\{c, d\}$	$\{b, d\}$	$\{a, b, c, d\}$

(b)

Proposition 3.4 *If* $(L, \wedge, \vee)$ *is a distributive lattice, then the hyperlattice* $(L, \wedge, \dot{\vee})$*, where* $\dot{\vee} : (a, b) \to a \dot{\vee} b \in \{a \vee b\}$ *considered in the first part of Proposition 2.5 is a distributive hyperlattice in the sense of Definition 3.1(1) and in the sense of Definition 3.1(2).*

Proof The hyperlattice $(L, \wedge, \dot{\vee})$ is distributive in the sense of Definition 3.1(1). In fact: if $u \in b \dot{\vee} c$, then $u = b \vee c$ and $a \wedge u = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \in (a \wedge b) \dot{\vee}(a \wedge c)$.

If $u \in (a \wedge b) \dot{\vee}(a \wedge c)$, then $u = (a \wedge b) \vee (a \wedge c)$ and, for the element $v := b \vee c \in b \dot{\vee} c$, we have $u = a \wedge v$. It is distributive in the sense of Definition 3.1(2) as well. Indeed: if $u \in a \dot{\vee}(b \wedge c)$, then $u = a \vee (b \wedge c)$ and, for the elements $v := a \vee b \in a \dot{\vee} b$ and $w := a \vee c \in a \dot{\vee} c$, we have $u = a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = v \wedge w$. If $u \in a \dot{\vee} b$ and $v \in a \dot{\vee} c$, then $u = a \vee b$, $v = a \vee c$ and then $u \wedge v = (a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$. □

Proposition 3.5 *If* $(L, \wedge, \vee)$ *is a distributive lattice, then the hyperlattice* $(L, \wedge, \dot{\vee})$*, where* $\dot{\vee} : (a, b) \to a \dot{\vee} b \in \{a, b, a \vee b\}$ *considered in the second part of Proposition 2.5 is a distributive hyperlattice in the sense of Definition 3.1(1). It is not a distributive hyperlattice in the sense of Definition 3.1(2) in general, but it satisfies the property (1) of Definition 3.1(2).*

Proof Let $u \in b \dot{\vee} c$. Then $a \wedge u \in (a \wedge b) \dot{\vee}(a \wedge c)$. Indeed: if $u = b$, then $a \wedge u = a \wedge b \in (a \wedge b) \dot{\vee}(a \wedge c)$; if $u = c$, then $a \wedge u = a \wedge c \in (a \wedge b) \dot{\vee}(a \wedge c)$; if $u = b \vee c$, then $a \wedge u = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \in (a \wedge b) \dot{\vee}(a \wedge c)$.

If $u \in (a \wedge b) \dot{\vee}(a \wedge c)$, then there exists $v \in b \dot{\vee} c$ such that $u = a \wedge v$. Indeed: if $u = a \wedge b$ then, for the element $v := b \in b \dot{\vee} c$, we have $u = a \wedge v$; if $u = a \wedge c$ then, for the element $v := c \in b \dot{\vee} c$, we have $u = a \wedge v$; if $u = (a \wedge b) \vee (a \wedge c)$, then for the element $v := b \vee c \in b \dot{\vee} c$, we have $u = (a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c) = a \wedge v$.

The hyperlattice L satisfies the property (1) of Definition 3.1(2); that is, if $u \in a \dot{\vee}(b \wedge c)$, then there exist $v \in a \dot{\vee} b$ and $w \in a \dot{\vee} c$ such that $u = v \wedge w$. Indeed: if $u = a$ then, for the elements $v := a \in a \dot{\vee} b$ and $w := a \in a \dot{\vee} c$, we have $u = v \wedge w$; if $u = b \wedge c$ then, for the elements $v := b \in a \dot{\vee} b$ and $w := c \in a \dot{\vee} c$, we have $u = v \wedge w$; if $u = a \vee (b \wedge c)$ then, for the elements $v := a \vee b \in a \dot{\vee} b$ and $w := a \vee c \in a \dot{\vee} c$, we have $u = a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = v \wedge w$.

We prove the rest of the proposition by the following example.

Let us consider the distributive lattice of Figure 2.

The hyperlattice L with the operation $\wedge$ and the hyperoperation $\dot{\vee}$ defined in the second proof of Proposition 2.5 is given by Table 7 and, according to what we already said, it is a distributive hyperlattice in the sense of Definition 3.1(1) and satisfies condition (1) of Definition 3.1(2). But it is not distributive in the sense of Definition 3.1(2) as $c \in c \dot{\vee} a$, $b \in c \dot{\vee} b$ but $c \wedge b \notin c \dot{\vee}(a \wedge b)$.

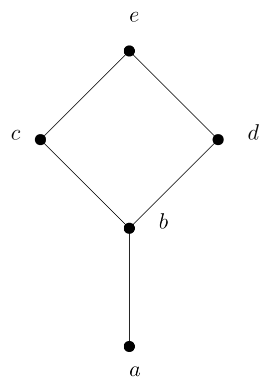

Figure 2: The distributive hyperlattice of Proposition 3.5.

Table 7: The hyperlattice in Proposition 3.5.

$\wedge$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	c	b	c
d	a	b	b	d	d
e	a	b	c	d	e

(a)

$\dot{\vee}$	a	b	c	d	e
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{a,e\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,d\}$	$\{b,e\}$
c	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d,e\}$	$\{c,e\}$
d	$\{a,d\}$	$\{b,d\}$	$\{c,d,e\}$	$\{d\}$	$\{d,e\}$
e	$\{a,e\}$	$\{b,e\}$	$\{c,e\}$	$\{d,e\}$	$\{e\}$

(b)

□

Example 3.6 Let us consider the no distributive lattice of Figure 3.
The hyperlattice $(L, \wedge, \dot{\vee})$ that corresponds to Figure 3 using the second proof of Proposition 2.5 is given by Table 8. This is not distributive hyperlattice in the sense of Definition 3.1(1) as $e \in b \dot{\vee} c$ and $d \wedge e \notin (d \wedge b) \dot{\vee} (d \wedge c)$, that is condition (1) of Definition 3.1(1) does not hold; and not distributive hyperlattice in the sense of Definition 3.1(2) as $e \in d \dot{\vee} b$, $e \in d \dot{\vee} c$ and $e \wedge e \notin d \dot{\vee} (b \wedge c) = \{d, a\}$, that is condition (2) of Definition 2.8(2) does not hold.

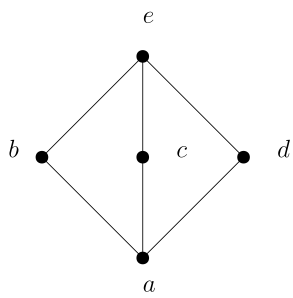

Figure 3: The no distributive lattice of the Example 3.6.

Table 8: The hyperlattice of the Example 3.6.

$\wedge$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	b
c	a	a	c	a	c
d	a	a	a	d	d
e	a	b	c	d	e

(a)

$\dot\vee$	a	b	c	d	e
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{a,e\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c,e\}$	$\{b,d,e\}$	$\{b,e\}$
c	$\{a,c\}$	$\{b,c,e\}$	$\{c\}$	$\{c,d,e\}$	$\{c,e\}$
d	$\{a,d\}$	$\{b,d,e\}$	$\{c,d,e\}$	$\{d\}$	$\{d,e\}$
e	$\{a,e\}$	$\{b,e\}$	$\{c,e\}$	$\{d,e\}$	$\{e\}$

(b)

Now we will give another proof of Proposition 2.5 in the next proposition.

Proposition 3.7 *Let $(L,\wedge,\vee)$ be a lattice and "$\dot\vee$" the hyperoperation on L defined by:*

$$\dot\vee : L\times L \to \mathcal{P}^*(L) \mid (a,b)\to a\,\dot\vee\, b := \{t\in L \mid t\le a\vee b\}.$$

Then $(L,\wedge,\dot\vee)$ is a hyperlattice.

Proof Let $x\in u\,\dot\vee\, c$ for some $u\in a\,\dot\vee\, b$. Then $x\le u\vee c$ and $u\le a\vee b$, so $x\le (a\vee b)\vee c = a\vee(b\vee c)$. For the element $v:=b\vee c\in b\,\dot\vee c$, we have $x\in a\,\dot\vee\, v$, so condition (3) of Definition 2.1 is satisfied. Let now $a,b\in L$. For the element $u:=a\vee b\in a\,\dot\vee\, b$, we have $a\wedge u = a\wedge(a\vee b)=a$; and since $a\le a\vee(a\wedge b)$, we have $a\in a\,\dot\vee(a\wedge b)$ and condition (4) of Definition 2.1 also holds. $\square$

Proposition 3.8 *If $(L,\wedge,\vee)$ is a distributive lattice, then the hyperlattice $(L,\wedge,\dot\vee)$, where $a\,\dot\vee\, b := \{t\in L \mid t\le a\vee b\}$ considered in Proposition 3.7 satisfies condition (1) of Definition 3.1(1) and condition (2) of Definition 3.1(2).*

Proof Let $u \in b \dot{\vee} c$. Then $a \wedge u \in (a \wedge b) \dot{\vee} (a \wedge c)$. Indeed: Since $u \le b \vee c$, we have $a \wedge u \le a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, so $a \wedge u \in (a \wedge b) \dot{\vee} (a \wedge c)$ and condition (1) of Definition 3.1(1) is satisfied. Let now $u \in a \dot{\vee} b$ and $v \in a \dot{\vee} c$. Then $u \wedge v \in a \dot{\vee} (b \wedge c)$. Indeed, since $u \le a \vee b$ and $v \le a \vee c$, we have $u \wedge v \le (a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$, then $u \wedge v \in a \dot{\vee} (b \wedge c)$ and condition (2) of Definition 3.1(2) also holds. $\square$

The question is: Given a distributive lattice $(L, \wedge, \vee)$ and the hyperlattice $(L, \wedge, \dot{\vee})$, where $a \dot{\vee} b := \{t \in L \mid t \le a \vee b\}$, under what conditions the hyperlattice $(L, \wedge, \dot{\vee})$ is a distributive hyperlattice in the sense of Definitions 3.1(1) and 3.1(2)? As answer is given in the rest of this section.

Proposition 3.9 *Let $(L, \wedge, \vee)$ be a distributive lattice and $(L, \wedge, \dot{\vee})$ the hyperlattice with the hyperoperation $a \dot{\vee} b := \{t \in L \mid t \le a \vee b\}$ satisfying the property*

$$u \in (a \wedge b) \dot{\vee} (a \wedge c) \text{ and } v \in b \dot{\vee} c \text{ imply } a \wedge v \le u \tag{3.1}$$

Then $(L, \wedge, \dot{\vee})$ satisfies condition (2) of Definition 3.1(1).

Proof Let $u \in (a \wedge b) \dot{\vee} (a \wedge c)$. Then $u \le (a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$. We put $v := b \vee c$ and we have $u \le a \wedge v$. On the other hand, since $u \in (a \wedge b) \dot{\vee} (a \wedge c)$ and $v \in b \dot{\vee} c$, by (3.1), we have $a \wedge v \le u$. Hence we get $u = a \wedge v$ and the hyperlattice $(L, \wedge, \dot{\vee})$ satisfies condition (2) of Definition 3.1(1). $\square$

By Propositions 3.8 and 3.9 we have the following corollary.

Corollary 3.10 *Let $(L, \wedge, \vee)$ be a distributive lattice and $(L, \wedge, \dot{\vee})$ the hyperlattice defined by $a \dot{\vee} b := \{t \in L \mid t \le a \vee b\}$ and having the property (3.1). Then $(L, \wedge, \dot{\vee})$ is a distributive hyperlattice in the sense of Definition 3.1(1).*

Proposition 3.11 *Let $(L, \wedge, \vee)$ be a distributive lattice and $(L, \wedge, \dot{\vee})$ the hyperlattice with the hyperoperation $a \dot{\vee} b := \{t \in L \mid t \le a \vee b\}$ satisfying the property*

$$u \in a \dot{\vee} (b \wedge c), v \in a \dot{\vee} b \text{ and } w \in a \dot{\vee} c \text{ imply } u \ge v \wedge w \tag{3.2}$$

Then $(L, \wedge, \dot{\vee})$ satisfies condition (1) of Definition 3.1(2).

Proof Let $u \in a \dot{\vee} (b \wedge c)$ Then $u \le a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. We put $v := a \vee b \in a \dot{\vee} b$ and $w := a \vee c \in a \dot{\vee} c$ and we have $u \le v \wedge w$. On the other hand, by (3.2) we have $u \ge v \wedge w$, then $u = v \wedge w$ and so condition (1) of Definition 3.1(2) is satisfied. $\square$

By Propositions 3.8 and 3.11 we have the following corollary.

Corollary 3.12 *Let $(L, \wedge, \vee)$ be a distributive lattice and $(L, \wedge, \dot{\vee})$ the hyperlattice defined by $a \dot{\vee} b := \{t \in L \mid t \le a \vee b\}$ and having the property (3.2). Then $(L, \wedge, \dot{\vee})$ is a distributive hyperlattice in the sense of Definition 3.1(2).*

Problem 3.13 Write a program to show that the hyperlattice considered in Proposition 3.8 does not satisfy condition (2) of Definition 3.1(1) and condition (1) of Definition 3.1(2) in general.

4. Modular hyperlattices

A lattice L is called modular if, for any $a, b, c \in L$, $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$. This concept can be naturally transferred to a hyperlattice by the following definition.

Definition 4.1 A hyperlattice L is called *modular* if the following assertions are satisfied:

(1) if $a \wedge c = a$ and $u \in a \vee (b \wedge c)$, then there exists $v \in a \vee b$ such that $u = v \wedge c$ and

(2) if $a \wedge c = a$ and $u \in a \vee b$, then $u \wedge c \in a \vee (b \wedge c)$.

In other words, if $a \wedge c = a$, then we have $u \in a \vee (b \wedge c)$ if and only if there exists $v \in a \vee b$ such that $u = v \wedge c$. That is, if $a \wedge c = a$ implies $\{v \wedge c \mid v \in a \vee b\} = a \vee (b \wedge c)$.

Since $a \wedge a = a$, in a modular hyperlattice we have $\{v \wedge a \mid v \in a \vee b\} = a \vee (b \wedge a)$.

Remark 4.2 While to a distributive lattice correspond two definitions of a distributive hyperlattice, to a modular lattice corresponds only one. Indeed, if we get the equivalent definition of a modular lattice (the dual definition) which is $a \geq c$ implies $a \wedge (b \vee c) = (a \wedge b) \vee c$, this can be transferred to hyperlattices as follows:

(1) if $a \wedge c = c$ and $u \in b \vee c$, then $a \wedge u \in (a \wedge b) \vee c$ and

(2) if $a \wedge c = c$ and $u \in (a \wedge b) \vee c$, then there exists $v \in b \vee c$ such that $u \in a \wedge v$.

In other words, if $a \wedge c = c$ implies $\{a \wedge u \mid u \in b \vee c\} = (a \wedge b) \vee c$. As we see, this is the same with the definition of modular hyperlattice given by Definition 4.1 (by interchanging a and c the two definitions coincide).

Example 4.3 We consider the no modular lattice of Figure 4.

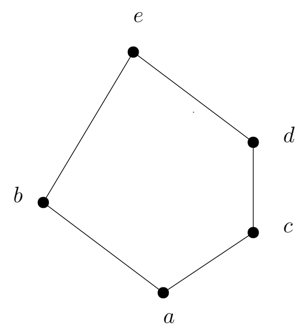

Figure 4: The no modular lattice of the example 4.3.

The hyperlattice L with the operation $\wedge$ and the hyperoperation $\dot{\vee}$ defined in the second proof of Proposition 2.5 is given by Table 9 and it is not modular as $c \wedge d = c$ and $e \in c \dot{\vee} b$ but $e \wedge d \notin c \dot{\vee} (b \wedge d)$; that is condition (2) of Definition 4.1 does not hold.

Table 9: The no modular hyperlattice of the Example 4.3.

$\wedge$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	b
c	a	a	c	c	e
d	a	a	c	d	d
e	a	b	c	d	e

(a)

$\dot{\vee}$	a	b	c	d	e
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{a,e\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c,e\}$	$\{b,d,e\}$	$\{b,e\}$
c	$\{a,c\}$	$\{b,c,e\}$	$\{c\}$	$\{c,d\}$	$\{c,e\}$
d	$\{a,d\}$	$\{b,d,e\}$	$\{c,d\}$	$\{d\}$	$\{d,e\}$
e	$\{a,e\}$	$\{b,e\}$	$\{c,e\}$	$\{d,e\}$	$\{e\}$

(b)

Proposition 4.4 *If L is a distributive hyperlattice in the sense of Definition 3.1(1), then it is modular.*

Proof Let $a \wedge c = a$ and $u \in a \vee (b \wedge c)$. Then $u \in (a \wedge c) \vee (b \wedge c) = (c \wedge a) \vee (c \wedge b)$ and, by the second property of Definition 3.1(1), there exists $v \in a \vee b$ such that $u = c \wedge v\ (= v \wedge c)$. Let now $a \wedge c = a$ and $u \in a \vee b$. Since $u \in a \vee b$, by the first property of Definition 3.1(1), we have $c \wedge u \in (c \wedge a) \vee (c \wedge b)$. Since $a \wedge c = a$, we have $u \wedge c \in a \vee (b \wedge c)$ and the proof is complete. □

According to Proposition 4.4, the distributive hyperlattice of Table 6 (Example 3.3) is modular.

Proposition 4.5 *Let $(L, \wedge, \vee)$ be a modular lattice. Then*

(1) *the hyperlattice $(L, \wedge, \dot{\vee})$, where $\dot{\vee} : (a, b) \to a \dot{\vee} b := \{a \vee b\}$ considered in the first part of Proposition 2.5 is modular;*

(2) *the hyperlattice $(L, \wedge, \dot{\vee})$, where $\dot{\vee} : (a, b) \to a \dot{\vee} b := \{a, b, a \vee b\}$ considered in the second part of Proposition 2.5 is also modular.*

Proof (1) Let $a \wedge c = a$ and $u \in a \dot{\vee} (b \wedge c)$. Since $a \le c$, $u = a \vee (b \wedge c)$ and $(L, \wedge, \vee)$ is modular, for the element $v := a \vee b \in a \dot{\vee} b$ of L, we have $u = a \vee (b \wedge c) = (a \vee b) \wedge c = v \wedge c$, so condition (1) of Definition 4.1 is satisfied. Let now $a \wedge c = a$ and $u \in a \dot{\vee} b$. Since $a \le c$, $u = a \vee b$ and $(L, \wedge, \vee)$ is modular, we have $u \wedge c = (a \vee b) \wedge c = (a \vee b) \wedge c$, then $u \wedge c \in a \dot{\vee} (b \wedge c)$ and condition (2) of Definition 4.1 also holds.

(2) Let $a \wedge c = a$ and $u \in a \dot{\vee} (b \wedge c)$. If $u = a$ then, for the element $v := a \in a \dot{\vee} b$ of L, we have $u = v \wedge c$; if $u = b \wedge c$ then, for the element $v := b \in a \dot{\vee} b$, we have $u = v \wedge c$; if $u = a \vee (b \wedge c)$ then, for the element $v := a \vee b \in a \dot{\vee} b$, we have $u = a \vee (b \wedge c) = (a \vee b) \wedge c = v \wedge c$ and condition (1) of Definition 4.1 holds. Let now $a \wedge c = a$ and $u \in a \dot{\vee} b$. If $u = a$, then $u \wedge c = a \wedge c = a \in a \dot{\vee} (b \wedge c)$; if $u = b$, then $u \wedge c = b \wedge c \in a \dot{\vee} (b \wedge c)$. Finally, let $u \in a \vee b$. Then we have $u \wedge c = (a \vee b) \wedge c$. On the other hand, since $a \leq c$ and L is modular, we have $(a \vee b) \wedge c = a \vee (b \wedge c)$. Thus we have $u \wedge c = a \vee (b \wedge c) \in a \dot{\vee} (b \wedge c)$. □

We apply Proposition 4.5 to the following example

Example 4.6 We consider the modular lattice L of Figure 5. The hyperlattice that corresponds to L via the second proof of Proposition 2.5 is given by Table 10 and, according to Proposition 4.5, this is a modular hyperlattice.

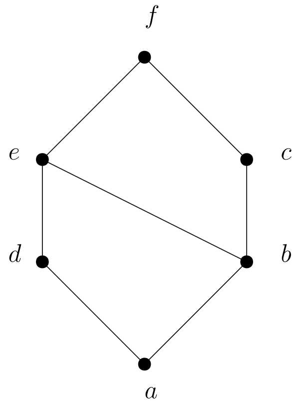

Figure 5: The modular lattice of the Example 4.6.

Table 10: The modular hyperlattice of the Example 4.6.

$\wedge$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	b	a	b	b
c	a	b	c	a	b	c
d	a	a	a	d	d	d
e	a	a	b	d	e	e
f	a	b	c	d	e	f

(a)

$\dot{\vee}$	a	b	c	d	e	f
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{a,e\}$	$\{a,f\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,d,e\}$	$\{b,e\}$	$\{b,f\}$
c	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d,f\}$	$\{c,e,f\}$	$\{c,f\}$
d	$\{a,d\}$	$\{b,d,e\}$	$\{c,d,f\}$	$\{d\}$	$\{d,e\}$	$\{d,f\}$
e	$\{a,e\}$	$\{b,e\}$	$\{c,e,f\}$	$\{d,e\}$	$\{e\}$	$\{e,f\}$
f	$\{a,f\}$	$\{b,f\}$	$\{c,f\}$	$\{d,f\}$	$\{e,f\}$	$\{f\}$

(b)

Proposition 4.7 *If $(L,\wedge,\vee)$ is a modular lattice, then the hyperlattice $(L,\wedge,\dot{\vee})$, with the hyperoperation $a\,\dot{\vee}\,b := \{t \in L \mid t \le a \vee b\}$ satisfies condition (2) of Definition 4.1.*

Proof Let $a \wedge c = a$ and $u \in a\,\dot{\vee}\,b$. Then $u \wedge c \in a\,\dot{\vee}(b \wedge c)$. Indeed: Since $u \le a \vee b$, we have $u \wedge c \le (a \vee b) \wedge c$. Since $a \le c$ and $(L,\wedge,\vee)$ is modular, we have $(a \vee b) \wedge c = a \vee (b \wedge c)$. Thus we get $u \wedge c \le a \vee (b \wedge c)$ and so $u \wedge c \in a\,\dot{\vee}(b \wedge c)$. □

We apply Proposition 4.7 to the following example.

Example 4.8 We consider the modular lattice of Figure 5 (Example 4.6). By Proposition 4.7, the hyperlattice defined by Table 11 satisfies condition (2) of Definition 4.1.

Table 11: The hyperlattice of the Example 4.8.

$\wedge$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	b	a	b	b
c	a	b	c	a	b	c
d	a	a	a	d	d	d
e	a	a	b	d	e	e
f	a	b	c	d	e	f

(a)

$\dot{\vee}$	a	b	c	d	e	f
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,d\}$	$\{a,b,d,e\}$	S
b	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,d,e\}$	$\{a,b,d,e\}$	S
c	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	S	S	S
d	$\{a,d\}$	$\{a,b,d,e\}$	S	$\{a,d\}$	$\{a,b,d,e\}$	S
e	$\{a,b,d,e\}$	$\{a,b,d,e\}$	S	$\{a,b,d,e\}$	$\{a,b,d,e\}$	S
f	S	S	S	S	S	S

(b)

The question is: Are there modular lattices for which the hypersemigroup defined in Proposition 4.7 satisfies condition (1) of Definition 4.1? As answer is given by the following proposition.

Proposition 4.9 *Let* $(L, \wedge, \vee)$ *be a modular lattice and* $(L, \wedge, \dot{\vee})$ *the hyperlattice with the hypeoperation* $a \dot{\vee} b := \{t \in L \mid t \leq a \vee b\}$ *satisfying the property*

$$v \in a \dot{\vee} b \text{ and } u \in a \dot{\vee} (b \wedge c) \text{ imply } v \wedge c \leq u \tag{4.1}$$

Then $(L, \wedge, \dot{\vee})$ *satisfies condition (1) of Definition 4.1.*

Proof Let $a \wedge c = a$ and $u \in a \dot{\vee} (b \wedge c)$. Then we have $u \leq a \vee (b \wedge c) = (a \vee b) \wedge c$ and, for the element $v := a \vee b$, we have $u \leq v \wedge c$. On the other hand, since $v \in a \dot{\vee} b$ and $u \in a \dot{\vee} (b \wedge c)$, by (4.1), we have $v \wedge c \leq u$. Hence we obtain $u = v \wedge c$ and property (1) of Definition 4.1 is satisfied. $\square$

By Propositions 4.7 and 4.9 we have the following

Corollary 4.10 *Let* $(L, \wedge, \vee)$ *be a modular lattice and "*$\dot{\vee}$*" the hyperoperation on L defined by* $a \dot{\vee} b := \{t \in L \mid t \leq a \vee b\}$ *and having the property (4.1). Then* $(L, \wedge, \dot{\vee})$ *is a modular hyperlattice.*

Remark 4.11 The hyperlattice of Table 11 does not satisfy condition (4.1); indeed, we have $b \in d \dot{\vee} b$ and $d \in d \dot{\vee} (b \wedge c)$ but $b \wedge c \not\leq d$.
However it seems to be modular; write a program to check it.

Problem 4.12 Write a program to show that the hyperlattice considered in Proposition 4.7 does not satisfy condition (1) of Definition 4.1 in general.

Problem 4.13 Write a program to check if the hyperlattices of the examples of the paper satisfy the relation $a \in a \vee b \Rightarrow a \wedge b = b$.

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