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Counting $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes

Basri Çalışkan[1,*], Kemal Balıkçı[2]

[1] *Department of Mathematics, Faculty of Arts and Science, Osmaniye Korkut Ata University, Osmaniye, Turkey*
[2] *Department of Electrical and Electronic, Engineering Faculty, Osmaniye Korkut Ata University, Osmaniye, Turkey*

Abstract. In Algebraic Coding Theory, all linear codes can be described by generator matrices. Any linear code has different generator matrices. It is important to find the number of the generator matrices for the construction of these codes. In this paper, we study $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes, which are an extension of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. We count the number of arbitrary $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes. Then we investigate connections to $\mathbb{Z}_2\mathbb{Z}_4$ and $\mathbb{Z}_2\mathbb{Z}_8$-additive codes with $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes, and give some illustrative examples.

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1. Introduction

Coding Theory is important in modern communication systems and has become applicable to many areas such as data storage devices, mobile phones and the Internet. One of the main problems in communication is "How accurately can the symbols of communication be transmitted?". This problem is concerned with the accuracy of transference from sender to receiver of sets of symbols; that is, it involves transmission of finite discrete symbols [12]. For this purpose, linear codes are widely used in coding theory. A linear code of length n over F_q is a subspace C of the vector space F_q^n where F_q is a finite field of size q which is prime or a power of a prime. Although the finite fields are widely used in coding theory, the studies of the codes on different rings have attracted the interest of the researchers [9].

Let $\mathbb{Z}_m$ be the ring of integers modulo m. $\mathbb{Z}_m^n$ is called the set of Cartesian product of n copies of $\mathbb{Z}_m$. Any nonempty subset C of $\mathbb{Z}_m^n$ is a code and a submodule of a $\mathbb{Z}_m^n$ is called a linear code of length n over $\mathbb{Z}_m$. Specially, for m=2 and $m = 4$ the codes are called binary ($\mathbb{Z}_2$) and quaternary codes ($\mathbb{Z}_4$), respectively. It was shown that specific

*Corresponding author.
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Email addresses: bcaliskan@osmaniye.edu.tr (B. Çalışkan), kbalikci@osmaniye.edu.tr (K. Balıkçı)

good non-linear binary codes which simplifies encoding and decoding can be seen as binary images of linear codes over $\mathbb{Z}_4$ under the Gray map [9].

Additive codes were first defined by Delsarte [7] in terms of association schemes. He defined additive codes as subgroups of the underlying abelian group in a translation association scheme. According to this, for the binary Hamming scheme, when the underlying abelian group is of order 2^n, the only structures for the abelian group are those of the form $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ where α and β are positive integers ($n = \alpha + 2\beta$). Later, translation invariant propelinear codes were first introduced by Pujol et al.[11]. They showed that all these binary codes are group-isomorphic to subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Q}_8^\sigma$, being $\mathbb{Q}_8$ the non-abelian quaternion group with eight elements. Recently, in particular, $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes, which are a generalization of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, have been extensively studied [1–6, 8].

In order to examine the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes of a given length and type, it is necessary to construct the generator matrices of these codes and to find the number of these generator matrices. In fact, the counting problem is the calculation of the number of subspaces created by the rows of the generator matrices over the finite fields. Recently, in this regard, there has been some researches of particular types over the different rings. For example, while Dougherty et al. [8] counted the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, Aydogdu et al. [3] performed the count of the $\mathbb{Z}_2\mathbb{Z}_8$-additive codes. In this paper, we aim to focus on the number of matrices that generate different (not necessarily equivalent) $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes. First of all, we will introduce the structure of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes and then give the basic theorem of our article and a simple example. In addition, we will demonstrate the relationship of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes with $\mathbb{Z}_2\mathbb{Z}_4$ and $\mathbb{Z}_2\mathbb{Z}_8$-additive codes.

2. Preliminaries

In this section, we briefly introduce the backgrounds of our proceeding works.

2.1. $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-Additive Codes

Definition 1. *Let $\mathbb{Z}_2$, $\mathbb{Z}_4$ and $\mathbb{Z}_8$ be the rings of integers modulo 2, 4 and 8, respectively. Then, C is called a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code if it is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_8^\theta$ where α, β and θ are positive integers [2].*

According to this definition, the first α coordinates of C consist of entries from $\mathbb{Z}_2$, the next β coordinates are elements from $\mathbb{Z}_4$ and remaining θ coordinates are the elements of the ring $\mathbb{Z}_8$. From the Fundamental Theorem of Finite Abelian Groups, such an additive code C which is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_8^\theta$ is group isomorphic to the abelian structure

$$\mathbb{Z}_2^{k_0} \times \mathbb{Z}_4^{k_1} \times \mathbb{Z}_2^{k_2} \times \mathbb{Z}_8^{k_3} \times \mathbb{Z}_4^{k_4} \times \mathbb{Z}_2^{k_5}.$$

By this isomorphism, such a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code is classified as of type $(\alpha, \beta, \theta; k_0, k_1, k_2, k_3, k_4, k_5)$ [2].

2.2. Generator and Parity-check matrices of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes

Let C be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha,\beta,\theta;k_0,k_1,k_2,k_3,k_4,k_5)$. Then, it is shown that C is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code which has the following generator matrix in standard form:

$$\left(\begin{array}{cc|ccc|cccc} I_{k_0} & \overline{A}_{01} & 0 & 0 & 2T_1 & 0 & 0 & 0 & 4T_2 \\ 0 & \overline{S}_1 & I_{k_1} & B_{01} & B_{02} & 0 & 0 & 2T_3 & 2T_4 \\ 0 & 0 & 0 & 2I_{k_2} & 2B_{12} & 0 & 0 & 0 & 4T_5 \\ 0 & \overline{S}_2 & 0 & S_{01} & S_{02} & I_{k_3} & A_{01} & A_{02} & A_{03} \\ 0 & \overline{S}_3 & 0 & 0 & 2S_{12} & 0 & 2I_{k_4} & 2A_{12} & 2A_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4I_{k_5} & 4A_{23} \end{array}\right) \tag{1}$$

where $\overline{A}_{01}$, $\overline{S}_1$, $\overline{S}_2$, $\overline{S}_3$ are matrices with all entries from $\mathbb{Z}_2$ and B_{02}, B_{12}, S_{02} and S_{12} are matrices over $\mathbb{Z}_4$. Also, T_4, T_5 and A_{i3} are matrices over $\mathbb{Z}_8$ for $0 \le i \le 2$. All the entries in B_{01}, S_{01} and T_1 are in $\{0,1\} \subseteq \mathbb{Z}_4$. Likewise, A_{01} and T_2 are matrices over $\mathbb{Z}_4$ whose all entries are from $\{0,1\}$. T_3, A_{12} and A_{02} are matrices over $\mathbb{Z}_8$, but all values are the elements of the set $\{0,1,2,3\}$. Also, C has $2^{k_0}2^{2k_1}2^{k_2}2^{3k_3}2^{2k_4}2^{k_5}$ codewords [2].

In (1), k_0, k_2 and k_5 represent the number of order 2 generators that are contributed through $\mathbb{Z}_2$, $\mathbb{Z}_4$ and $\mathbb{Z}_8$ parts, respectively. Also, k_1 and k_4 represent the number of order 4 generators that are contributed through $\mathbb{Z}_4$ and $\mathbb{Z}_8$ parts, respectively. k_3 represents the number of order 8 generators that are contributed through the $\mathbb{Z}_8$ part. Note that, the order 2 elements from the $\mathbb{Z}_4$ and $\mathbb{Z}_8$ parts have the first α coordinates all zero [2].

The inner product for the elements $\mathbf{u},\mathbf{v} \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta} \times \mathbb{Z}_8^{\theta}$ as follows

$$\langle \mathbf{u}\cdot\mathbf{v}\rangle = 4\left(\sum_{i=1}^{\alpha} u_iv_i\right) + 2\left(\sum_{j=\alpha+1}^{\alpha+\beta} u_jv_j\right) + \sum_{k=\alpha+\beta+1}^{\alpha+\beta+\theta} u_kv_k.$$

Definition 2. *The set of all vectors which are orthogonal to every vector in C is called the dual code of C, and denoted by $C^{\perp}$. The dual code $C^{\perp}$ can be defined in the usual way with respect to inner product*

$$C^{\perp} = \left\{\boldsymbol{v} \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta} \times \mathbb{Z}_8^{\theta} \middle| \langle \boldsymbol{u}\cdot\boldsymbol{v}\rangle = \boldsymbol{0} \text{ for all } \boldsymbol{u} \in C\right\}.$$

A generator matrix for $C^{\perp}$ is called a parity-check matrix of C [2].

Let C be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code with the generator matrix (1), then we have the following parity-check matrix:

$$\left(\begin{array}{cc|ccc|c} -\overline{A}_{01}^t & I_{\alpha-k_0} & -2S_1^t & 0 & 0 & \\ -T_1^t & 0 & B_{02}^t - B_{12}^tB_{01}^t & B_{12}^t & I_{\beta-k_1-k_2} & \\ 0 & 0 & -2B_{01}^t & 2I_{k_2} & 0 & P \\ -T_2^t & 0 & -T_4^t + T_3^tA_{23}^t + T_5^tB_{01}^t & -T_5^t & 0 & \\ 0 & 0 & -2T_3^t & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \end{array}\right) \tag{2}$$

where P is the matrix:

$$P = \begin{pmatrix} 4\overline{S_2}^t - 2\overline{S_3}^t A_{01}^t & -2\overline{S_3}^t & 0 & 0 \\ -2S_{01}^t B_{12}^t - 2S_{02}^t + 2S_{12}^t A_{01}^t & -2S_{12}^t & 0 & 0 \\ -4S_{01}^t & 0 & 0 & 0 \\ -A_{03}^t + A_{13}^t A_{01}^t + A_{23}^t A_{02}^t - A_{23}^t A_{12}^t A_{01}^t + 2S_{01}^t T_5^t & -A_{13}^t + A_{23}^t A_{12}^t & -A_{23}^t & I_{\theta-k_3-k_4-k_5} \\ -2A_{02}^t + 2A_{12}^t A_{01}^t & -2A_{12}^t & 2I_{k_5} & 0 \\ -4A_{01}^t & 4I_{k_4} & 0 & 0 \end{pmatrix}. \quad (3)$$

So, the dual code $C^{\perp}$ is of type $(\alpha, \beta, \theta; \alpha - k_0, \beta - k_1 - k_2, k_2, \theta - k_3 - k_4 - k_5, k_5, k_4)$.

3. The Number of Generator Matrices for $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-Additive Codes

In this section, we shall count the number of arbitrary $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes of type $(\alpha, \beta, \theta; k_0, k_1, k_2, k_3, k_4, k_5)$. For this purpose, we will give two lemmas and the definition of Gaussian coefficient. We will also present our main theorem and give some examples.

Lemma 1. *The number of ways of choosing elements (or vectors) to generate a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha, \beta, \theta; k_0, k_1, k_2, k_3, k_4, k_5)$ is $\prod_{j=1}^{6} N_j$, where*

$$\begin{aligned} N_1 &= \prod_{i=0}^{k_0-1} \left(2^{\alpha} - 2^{i}\right) 2^{\beta+\theta}, \\ N_2 &= \prod_{i=0}^{k_1-1} \left(4^{\beta} - 2^{\beta+i}\right) 2^{\alpha+2\theta}, \\ N_3 &= \prod_{i=0}^{k_2-1} \left(2^{\beta+\theta} - 2^{\theta+k_1+i}\right), \\ N_4 &= \prod_{i=0}^{k_3-1} \left(8^{\theta} - 4^{\theta}2^{i}\right) 2^{\alpha+2\beta}, \\ N_5 &= \prod_{i=0}^{k_4-1} \left(4^{\theta} - 2^{\theta+k_3+i}\right) 2^{\alpha+2\beta}, \\ N_6 &= \prod_{i=0}^{k_5-1} \left(2^{\theta} - 2^{k_3+k_4+i}\right). \end{aligned}$$

Proof. Recall the generating matrix given in (1), we will examine six major parts for proof. In $\mathbb{Z}_2^{\alpha}\mathbb{Z}_4^{\beta}\mathbb{Z}_8^{\theta}$, because of the number of all vectors of order 2 is $2^{\alpha}2^{\beta}2^{\theta}$, we can choose first element of order 2 that contributes through $\mathbb{Z}_2$ part in $(2^{\alpha} - 1)\, 2^{\beta}2^{\theta}$ ways. Next, $(2^{\alpha} - 2)\, 2^{\beta}2^{\theta}$ selections can be made for the second element. Similarly, the last element can be selected by $\left(2^{\alpha} - 2^{k_0-1}\right) 2^{\beta}2^{\theta}$ ways. Thus, the k_0 elements of order 2 that contribute only to the binary part are selected in N_1 different ways.

Next, we will choose k_1 vectors of order 4 that from the $\mathbb{Z}_4$ part. However, in those vectors, there should not be vectors of order 4 that contribute to the $\mathbb{Z}_8$ part. There are

$2^\alpha 4^\beta 4^\theta$ vectors of order 4 in all space. Since the number of elements of order 4 in the $\mathbb{Z}_4$ part is $2^\alpha\left(4^\beta-2^\beta\right)4^\theta$, the first element can be selected in $2^\alpha\left(4^\beta-2^\beta\cdot 1\right)4^\theta$ ways. The number of choosing for the next element is $2^\alpha\left(4^\beta-2^\beta\cdot 2\right)4^\theta$ and $2^\alpha\left(4^\beta-2^\beta 2^{k_1-1}\right)4^\theta$ for the last element. So, there are N_2 different ways for k_1 elements of order 4 that contribute to the $\mathbb{Z}_4$ part.

Next, we need to choose k_2 vectors of order 2 that are not in the space generated by k_1 vectors of order 4. This imposes that the first α entries of such elements to be all zero. Thus, in order to pick such an element first we subtract elements of order 2 that are obtained through already chosen k_1 elements for the $\mathbb{Z}_4$ part. So, there are $2^\beta 2^\theta$ such vectors in the $\mathbb{Z}_4$ part. Then, first element can be chosen in $2^\beta 2^\theta-2^\theta 2^{k_1}$ ways, next choice comes from $2^\beta 2^\theta-2\cdot 2^\theta 2^{k_1}$ and inductively we reach N_3.

Now, there are $2^\alpha 4^\beta 8^\theta$ vectors in total but the number of all vectors of order 8 are $2^\alpha 4^\beta 8^\theta-2^\alpha 4^\beta 4^\theta$. So, we choose the first one of order 8 in $2^\alpha 4^\beta\left(8^\theta-4^\theta\right)$ different ways. Then we choose the remaining k_3-1 elements of order 8 in a similar way as before. Hence, the last element can be chosen in $2^\alpha 4^\beta\left(8^\theta-4^\theta 2^{k_3-1}\right)$ ways. So, we obtain N_4.

Next, to choose vectors of order 4 that are contributed through the $\mathbb{Z}_8$ part. There are $2^\alpha 4^\beta\left(4^\theta-2^\theta 2^{k_3}\right)$ selections for the first vector. Note that, the elements of order 4 that are formed from the k_3 elements of order 8 by taking their 2 multiples need to be considered. Next, $2^\alpha 4^\beta\left(4^\theta-2\cdot 2^\theta 2^{k_3}\right)$ selections can be made for the second element. Similarly, the last element can be selected by $2^\alpha 4^\beta\left(4^\theta-2^{k_4-1}\cdot 2^\theta 2^{k_3}\right)$ ways. Thus, the k_4 elements of order 4 that contribute only to the $\mathbb{Z}_8$ part are selected in N_5 different ways.

Finally, we need to choose k_5 vectors of order 2 that are not generated by k_3 and k_4 vectors of order 2. Note that, the first entries of such elements to be all zero. So, there are 2^θ vectors of order 2. In order to pick such an element first we subtract order 2 elements that are obtained through already chosen k_3 and k_4 elements for the $\mathbb{Z}_8$ part. So, we have $2^\theta-2^{k_3+k_4}$ choices. The second element comes from $2^\theta-2\cdot 2^{k_3+k_4}$ and inductively we reach N_6.

Lemma 2. *The number of distinct generator matrices of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha,\beta,\theta;k_0,k_1,k_2,k_3,k_4,k_5)$ is $\prod_{j=1}^{6} D_j$, where*

$$\begin{aligned}
D_1 &= 2^{k_0(k_1+k_2+k_3+k_4+k_5)}\prod_{i=0}^{k_0-1}\left(2^{k_0}-2^i\right),\\
D_2 &= 2^{k_1(k_0+k_1+k_2+2k_3+2k_4+k_5)}\prod_{i=0}^{k_1-1}\left(2^{k_1}-2^i\right),\\
D_3 &= 2^{k_2(k_1+3k_3+2k_4+k_5)}\prod_{i=0}^{k_2-1}\left(2^{k_2}-2^i\right),\\
D_4 &= 2^{k_3(k_0+2k_1+k_2+2k_3+2k_4+k_5)}\prod_{i=0}^{k_3-1}\left(2^{k_3}-2^i\right),
\end{aligned}$$

$$D_5 = 2^{k_4(k_0+2k_1+k_2+2k_3+k_4+k_5)} \prod_{i=0}^{k_4-1} \left(2^{k_4} - 2^i\right),$$
$$D_6 = 2^{k_5(k_3+k_4)} \prod_{i=0}^{k_5-1} \left(2^{k_5} - 2^i\right).$$

Proof. As we did in Lemma 1, we will apply the same procedure in a group of the type $(k_0, k_1, k_2, k_3, k_4, k_5)$. In order to choose first element of order 2 that contributes through the $\mathbb{Z}_2$ part, we subtract all elements of order 2 in the group from the ones that are not coming through the $\mathbb{Z}_2$ part, i.e. $2^{k_0+k_1+k_2+k_3+k_4+k_5} - 2^{k_1+k_2+k_3+k_4+k_5}$. Next, the second element can be chosen in $2^{k_0+k_1+k_2+k_3+k_4+k_5} - 2 \cdot 2^{k_1+k_2+k_3+k_4+k_5}$ different ways and inductively we reach at D_1.

Next, to choose an element of order 4 in the group, we have $\left(4^{k_1} - 2^{k_1}\right) 2^{k_0+k_2+2k_3+2k_4+k_5}$ choices. Then, there are $\left(4^{k_1} - 2 \cdot 2^{k_1}\right) 2^{k_0+k_2+2k_3+2k_4+k_5}$ choices for second element of order 4. So, inductively, we have D_2.

Now, to choose an element of order 2 in the group that only contributes through the $\mathbb{Z}_4$ part. There are $\left(2^{k_1+k_2} - 2^{k_1}\right) 2^{3k_3+2k_4+k_5}$ choices for the first element. Next, the second element can be chosen $\left(2^{k_1+k_2} - 2 \cdot 2^{k_1}\right) 2^{3k_3+2k_4+k_5}$ different ways and inductively we reach at D_3.

After the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ parts, we now calculate the selections for the $\mathbb{Z}_8$ part. First, in order to choose an element of order 8 in the group, we have $\left(8^{k_3} - 4^{k_3}\right) 2^{k_0+2k_1+k_2+2k_4+k_5}$ choices. Next, we can choose second element in $\left(8^{k_3} - 2 \cdot 4^{k_3}\right) 2^{k_0+2k_1+k_2+2k_4+k_5}$ different ways. Similarly, if we continue, we reach at D_4.

Next, to choose an element of order 4 within the group that only contributes through the $\mathbb{Z}_8$ part. We have $\left(4^{k_4} - 2^{k_4}\right) 2^{k_0+2k_1+k_2+2k_3+k_5}$ choices. Therefore, there are $\left(4^{k_4} - 2 \cdot 2^{k_4}\right) 2^{k_0+2k_1+k_2+2k_3+k_5}$ different choices for the second element. So, inductively, the last element can be chosen in $\left(4^{k_4} - 2^{k_4-1} 2^{k_4}\right) 2^{k_0+2k_1+k_2+2k_3+k_5}$ different ways. As a result, we reach at D_5.

Finally, to choose an element of order 2 within the group that solely contributes through the $\mathbb{Z}_8$ part. For this, there are $2^{k_3+k_4+k_5} - 2^{k_3+k_4}$ choices. The second element can be chosen in $2^{k_3+k_4+k_5} - 2 \cdot 2^{k_3+k_4}$ different ways. So, the last element can be chosen in $2^{k_3+k_4+k_5} - 2^{k_5-1} 2^{k_3+k_4}$ different ways. In this way, D_6 is obtained and the proof is completed.

Definition 3. *[10] Let n and q be two positive integers*

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1} = \frac{q^n - 1}{q - 1}$$

and the q-factorial is defined as

$$[n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q$$

and

$$\begin{aligned}\prod_{j=0}^{a-1}\left(2^b-2^j\right) &= \left(2^b-1\right)\left(2^b-2\right)\left(2^b-2^2\right)\cdots\left(2^b-2^{a-2}\right)\left(2^b-2^{(a-1)}\right)\\ &= 2^{1+2+\ldots+(a-1)}\left(2^b-1\right)\left(2^{b-1}-1\right)\cdots\left(2^{b-(a-1)}-1\right)\\ &= 2^{\binom{a}{2}}\frac{[b]_2!}{[b-a]_2!}.\end{aligned}$$

Definition 4. *[10] Let n, k and q be non-negative integers such that $k \leq n$. Then,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Here, $[0]_q! = 1$ and if $k = 0$ then the q binomial coefficient is equal to 1. Also we have an algebraic expression of Gaussian coefficients as follows;

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_2 = \begin{bmatrix} n \\ k_1 \end{bmatrix}_2 \cdot \begin{bmatrix} n-k_1 \\ k_2 \end{bmatrix}_2 \cdots \begin{bmatrix} n-\sum_{i=1}^{m-1} \\ k_m \end{bmatrix}_2.$$

Theorem 1. *The number of distinct $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes of type $(\alpha, \beta, \theta; k_0, k_1, k_2, k_3, k_4, k_5)$ is*

$$N_{2\times4\times8} = 2^{\delta}\begin{bmatrix} \alpha \\ k_0 \end{bmatrix}_2 \begin{bmatrix} \beta \\ k_1, k_2 \end{bmatrix}_2 \begin{bmatrix} \theta \\ k_3, k_4, k_5 \end{bmatrix}_2$$

where,

$$\begin{aligned}\delta &= k_0\left[(\beta-s)+(\theta-t)\right]\\ &\quad +k_1\left[(\alpha-k_0)+(\beta-s)+2(\theta-t)+k_5\right]\\ &\quad +k_2\left[(\theta-t)-2k_3-k_4\right]\\ &\quad +k_3\left[(\alpha-k_0)+2(\beta-s)+2(\theta-t)+k_2+k_5\right]\\ &\quad +k_4\left[(\alpha-k_0)+2(\beta-s)+(\theta-t)+k_2\right]\end{aligned}$$

$s = k_1 + k_2$ *and* $t = k_3 + k_4 + k_5$.

Proof. In order to prove this theorem we count ordered generators for the code of the type $(\alpha, \beta, \theta; k_0, k_1, k_2, k_3, k_4, k_5)$. Firstly, we count the ordered generators by choosing them from the all space $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta} \times \mathbb{Z}_8^{\theta}$ which gives say A. Let the number of codes of type $(k_0, k_1, k_2, k_3, k_4, k_5)$ be $N_{2\times4\times8}$. Secondly, we choose the ordered generators from these codes which gives say B. So, we have the following equation

$$N_{2\times4\times8} = \frac{A}{B}.$$

Note that, Lemma 1 gives the numerator A and Lemma 2 gives the denominator B. From Definition 3, we have,

$$\begin{aligned}
N_1 &= 2^{k_0(\beta+\theta)+\binom{k_0}{2}} \frac{[\alpha]_2!}{[\alpha-k_0]_2!},\\
N_2 &= 2^{k_1(\alpha+\beta+2\theta)+\binom{k_1}{2}} \frac{[\beta]_2!}{[\beta-k_1]_2!},\\
N_3 &= 2^{k_2(\theta+k_1)+\binom{k_2}{2}} \frac{[\beta-k_1]_2!}{[\beta-k_1-k_2]_2!},\\
N_4 &= 2^{k_3(\alpha+2\beta+2\theta)+\binom{k_3}{2}} \frac{[\theta]_2!}{[\theta-k_3]_2!},\\
N_5 &= 2^{k_4(\alpha+2\beta+\theta+k_3)+\binom{k_4}{2}} \frac{[\theta-k_3]_2!}{[\theta-k_3-k_4]_2!},\\
N_6 &= 2^{k_5(k_3+k_4)+\binom{k_5}{2}} \frac{[\theta-k_3-k_4]_2!}{[\theta-k_3-k_4-k_5]_2!}.
\end{aligned}$$

Also, from Definition 3, we have,

$$\begin{aligned}
D_1 &= 2^{k_0(k_1+k_2+k_3+k_4+k_5)+\binom{k_0}{2}} [k_0]_2!,\\
D_2 &= 2^{k_1(k_0+k_1+k_2+2k_3+2k_4+k_5)+\binom{k_1}{2}} [k_1]_2!,\\
D_3 &= 2^{k_1+k_2(3k_3+2k_4+k_5)+\binom{k_2}{2}} [k_2]_2!,\\
D_4 &= 2^{k_3(k_0+2k_1+k_2+2k_3+2k_4+k_5)+\binom{k_3}{2}} [k_3]_2!,\\
D_5 &= 2^{k_4(k_0+2k_1+k_2+2k_3+k_4+k_5)+\binom{k_4}{2}} [k_4]_2!,\\
D_6 &= 2^{k_5(k_3+k_4)+\binom{k_5}{2}} [k_5]_2!\cdot
\end{aligned}$$

Then,

$$N_{2\times4\times8} = \frac{A}{B} = \frac{\prod_{j=1}^{6} N_j}{\prod_{j=1}^{6} D_j}.$$

So, we have

$$N_{2\times4\times8} = 2^{\delta} \begin{bmatrix} \alpha \\ k_0 \end{bmatrix}_2 \begin{bmatrix} \beta \\ k_1 \end{bmatrix}_2 \begin{bmatrix} \beta-k_1 \\ k_2 \end{bmatrix}_2 \begin{bmatrix} \theta \\ k_3 \end{bmatrix}_2 \begin{bmatrix} \theta-k_3 \\ k_4 \end{bmatrix}_2 \begin{bmatrix} \theta-k_3-k_4 \\ k_5 \end{bmatrix}_2.$$

Hence the number of distinct $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes of type $(\alpha, \beta, \theta; k_0, k_1, k_2, k_3, k_4, k_5)$ is

$$N_{2\times4\times8} = 2^{\delta} \begin{bmatrix} \alpha \\ k_0 \end{bmatrix}_2 \begin{bmatrix} \beta \\ k_1, k_2 \end{bmatrix}_2 \begin{bmatrix} \theta \\ k_3, k_4, k_5 \end{bmatrix}_2$$

where, δ, s and t are as defined above.

Example 1. *Let C be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(2, 1, 1, 1, 1, 0, 1, 0, 0)$, then all possible generator matrices are 12 matrices that generate different codes. Here $\alpha = 2$, $\beta = \theta = 1$, $k_0 = k_1 = k_3 = 1$ and $k_2 = k_4 = k_5 = 0$.*

Here, N_1, N_2 and N_4 are calculated as 12, 32 and 64, respectively. Also, D_1, D_2 and D_4 are 4, 16 and 32, respectively. Because of $k_2 = k_4 = k_5 = 0$, N_3, N_5, N_6, D_3, D_5 and D_6 do not exist. So, we skip these terms, hence

$$N_{2\times4\times8} = \frac{N_1 N_2 N_4}{D_1 D_2 D_4} = \frac{24576}{2048} = 12.$$

These 12 generator matrices can be obtained as follows:

- $\begin{bmatrix} 1 & x & 0 & 0 \\ 0 & y & 1 & 0 \\ 0 & z & 0 & 1 \end{bmatrix}$ here, we have 8 possible matrices,
- $\begin{bmatrix} 0 & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{bmatrix}$ here, we have 4 possible matrices,

where x, y and z are either 0 or 1.

From Theorem 1, we have the following corollary.

Corollary 1. *The dual of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha, \beta, \theta; k_0, k_1, k_2, k_3, k_4, k_5)$ is of type $(\alpha, \beta, \theta; \alpha - k_0, \beta - k_1 - k_2, k_2, \theta - k_3 - k_4 - k_5, k_5, k_4)$ and the number of generator matrices of this dual code is*

$$\overline{N}_{2\times4\times8} = 2^{\overline{\delta}} \begin{bmatrix} \alpha \\ \alpha - k_0 \end{bmatrix}_2 \begin{bmatrix} \beta \\ \beta - s, k_2 \end{bmatrix}_2 \begin{bmatrix} \theta \\ \theta - t, k_5, k_4 \end{bmatrix}_2$$

where

$$\begin{aligned} \overline{\delta} &= (\alpha - k_0)(k_1 + k_3) \\ &\quad + (\beta - s)(k_0 + k_1 + 2k_3 + k_5) \\ &\quad + k_2 (k_3 - 2(\theta - t) - k_5) \\ &\quad + (\theta - t)(k_0 + 2k_1 + k_2 + 2k_3 + k_4) \\ &\quad + k_5 (k_0 + 2k_1 + k_2 + k_3). \end{aligned}$$

where $s = k_1 + k_2$ and $t = k_3 + k_4 + k_5$.

Example 2. *Let us show that for the code in the Example 1, the number of generator matrices of the dual code is 12. Since* $\alpha - k_0 = 2 - 1$, $\beta - k_1 - k_2 = 1 - 1 - 0 = 0$, $k_2 = 0$, $\theta - k_3 - k_4 - k_5 = 1 - 1 - 0 - 0 = 0$ *and* $k_5 = k_4 = 0$*; the type of the dual code is* $(2, 1, 1, 1, 0, 0, 0, 0, 0)$ *and* $\overline{\delta} = (2 - 1)(1 + 1) + 0 + 0 + 0 + 0 = 2$. *So,*

$$\overline{N}_{2\times4\times8}(2,1,1,1,0,0,0,0,0) = 2^2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}_2 \begin{bmatrix} 1 \\ 0,0 \end{bmatrix}_2 \begin{bmatrix} 1 \\ 0,0,0 \end{bmatrix}_2 = 12.$$

4. Connections to $\mathbb{Z}_2\mathbb{Z}_4$ and $\mathbb{Z}_2\mathbb{Z}_8$-Additive Codes

Here, we relate the number of the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes with the number of additive codes over $\mathbb{Z}_2\mathbb{Z}_4$ and $\mathbb{Z}_2\mathbb{Z}_8$, respectively. Hence, we also obtain formulas for counting the number of matrices that generate all these codes.

Corollary 2. *Let* C *be a* $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-additive code of type* $(\alpha, \beta, \theta; k_0, k_1, k_2, k_3, k_4, k_5)$*. If* $\theta = k_3 = k_4 = k_5 = 0$*, then we get a* $\mathbb{Z}_2\mathbb{Z}_4$*-additive code of type* $(\alpha, \beta; k_0, k_1, k_2)$*. The number of these distinct codes is as follows*

$$\begin{aligned} N_{2\times4\times8}(\alpha,\beta,0;k_0,k_1,k_2,0,0,0) &= 2^{k_0(\beta-k_1-k_2)+k_1(\alpha+\beta-k_0-k_1-k_2)} \begin{bmatrix} \alpha \\ k_0 \end{bmatrix}_2 \begin{bmatrix} \beta \\ k_1,k_2 \end{bmatrix}_2 \\ &= N_{2\times4}(\alpha,\beta;k_0,k_1,k_2) \end{aligned}$$

Proof. It is known [5] that the generator matrix for a $\mathbb{Z}_2\mathbb{Z}_4$-additive code C of type $(\alpha, \beta; \gamma, \delta, \kappa)$ has the following standard form

$$\begin{pmatrix} I_\kappa & T_b & 2T_2 & 0 & 0 \\ 0 & 0 & 2T_1 & 2I_{\gamma-\kappa} & 0 \\ 0 & S_b & S_q & R & I_\delta \end{pmatrix} \tag{4}$$

where T_b, S_b are matrices over $\mathbb{Z}_2$; T_1, T_2, R are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \in \mathbb{Z}_4$ and S_q is a matrix over $\mathbb{Z}_4$. Also, from [13], we know that the generator matrix of C of type $(\alpha, \beta; k_0, k_1, k_2)$ is in the form of

$$\begin{pmatrix} I_{k_0} & \overline{A}_{01} & 0 & 0 & 2T_{02} \\ 0 & S_1 & I_{k_1} & A_{01} & A_{02} \\ 0 & 0 & 0 & 2I_{k_2} & 2A_{12} \end{pmatrix} \tag{5}$$

which is permutation equivalent to a matrix of the form (4). So, from the generator matrices (4) and (5) we have the equations such that $\kappa = k_0$, $\delta = k_1$ and $\gamma - \kappa = k_2$.

In Theorem 1, it is clear that if $\theta = k_3 = k_4 = k_5 = 0$, then we have a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha, \beta, 0; k_0, k_1, k_2, 0, 0, 0)$. Also, the number of these codes is

$$\begin{aligned} N_{2\times4\times8}(\alpha,\beta,0;k_0,k_1,k_2,0,0,0) &= 2^{k_0(\beta-k_1-k_2)+k_1(\alpha+\beta-k_0-k_1-k_2)} \begin{bmatrix} \alpha \\ k_0 \end{bmatrix}_2 \begin{bmatrix} \beta \\ k_1,k_2 \end{bmatrix}_2 \\ &= N_{2\times4}(\alpha,\beta;k_0,k_1,k_2) \end{aligned} \tag{6}$$

In (6), if we replace $\kappa = k_0$, $\delta = k_1$ and $\gamma - \kappa = k_2$, then it is obtained the number of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes C of type $(\alpha, \beta; \gamma, \delta, \kappa)$ as in [8], (Theorem 3.3).

Corollary 3. *Let C be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha, \beta, \theta; k_0, k_1, k_2, k_3, k_4, k_5)$. If $\beta = k_1 = k_2 = 0$, then we get a $\mathbb{Z}_2\mathbb{Z}_8$-additive code of type $(\alpha, \theta; k_0, k_3, k_4, k_5)$. The number of these distinct codes is as follows*

$$\begin{aligned} N_{2\times4\times8}(\alpha, 0, \theta; k_0, 0, 0, k_3, k_4, k_5) &= 2^{k_0(\theta-k_3-k_4-k_5)+k_3[(\alpha-k_0)+2(\theta-k_3-k_4-k_5)+k_5]} \begin{bmatrix} \alpha \\ k_0 \end{bmatrix}_2 \begin{bmatrix} \theta \\ k_3, k_4, k_5 \end{bmatrix}_2 \\ &= N_{2\times8}(\alpha, \theta; k_0, k_3, k_4, k_5). \end{aligned}$$

Proof. In Theorem 1, it is clear that If $\beta = k_1 = k_2 = 0$, then we have a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha, 0, \theta; k_0, 0, 0, k_3, k_4, k_5)$. Also, the number of these codes is

$$\begin{aligned} N_{2\times4\times8}(\alpha, 0, \theta; k_0, 0, 0, k_3, k_4, k_5) &= 2^{k_0(\theta-k_3-k_4-k_5)+k_3[(\alpha-k_0)+2(\theta-k_3-k_4-k_5)+k_5]} \begin{bmatrix} \alpha \\ k_0 \end{bmatrix}_2 \begin{bmatrix} \theta \\ k_3, k_4, k_5 \end{bmatrix}_2 \\ &= N_{2\times8}(\alpha, \theta; k_0, k_3, k_4, k_5). \end{aligned} \quad (7)$$

In (7), if we replace both $k_3 = k_1$, $k_4 = k_2$, $k_5 = k_3$ and $k_1 + k_2 + k_3 = l$, then it is obtained the number of $\mathbb{Z}_2\mathbb{Z}_8$-additive codes of type $(\alpha, \theta; k_0, k_3, k_4, k_5)$ as in [3], (Theorem 2.1).

Example 3.

$$\begin{aligned} N_{2\times4}(3, 4; 2, 1, 2) &= N_{2\times4\times8}(3, 4, 0; 2, 1, 2, 0, 0, 0) = 11760 \\ N_{2\times8}(3, 4; 2, 0, 1, 2) &= N_{2\times4\times8}(3, 0, 4; 2, 0, 0, 0, 1, 2) = 11760 \end{aligned}$$

5. Conclusions

In this work we established a formula that gives the number of distinct $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes and their dual codes. Also, we showed the relationships among the numbers of $\mathbb{Z}_2\mathbb{Z}_4$, $\mathbb{Z}_2\mathbb{Z}_8$ and $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes. Also, we gave some examples that show the relationships.

Moreover, it is clear that in Theorem 1, if the parameters of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes of type $(\alpha, \beta, \theta; k_0, k_1, k_2, k_3, k_4, k_5)$ are appropriated, the numbers of $\mathbb{Z}_2$, $\mathbb{Z}_4$ and $\mathbb{Z}_8$ linear codes can be obtained.

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