On Topologies Induced by Graphs Under Some Unary and Binary Operations

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Abstract. Let \( G = (V(G), E(G)) \) be any simple undirected graph. The open hop neighborhood of \( v \in V(G) \) is the set \( N^2_G(v) = \{ u \in V(G) : d_G(u, v) = 2 \} \). Then \( G \) induces a topology \( \tau_G \) on \( V(G) \) with base consisting of sets of the form \( F^2_G[A] = V(G) \setminus N^2_G[A] \), where \( N^2_G[A] = A \cup \{ v \in V(G) : N^2_G(v) \cap A \neq \emptyset \} \) and \( A \) ranges over all subsets of \( V(G) \). In this paper, we describe the topologies induced by the complement of a graph, the join, the corona, the composition and the Cartesian product of graphs.

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1. Introduction

Let \( G = (V(G), V(H)) \) be any simple undirected graph. The distance \( d(u, v) \) between two vertices \( u \) and \( v \) in \( G \) is the length of a shortest path joining \( u \) and \( v \). Let \( v \in V(G) \). The neighborhood of \( v \) is the set \( N(v) \) consisting of all \( u \in V(G) \) which are adjacent with \( v \) and the closed neighborhood is \( N[v] = N(v) \cup \{ v \} \). For any \( A \subseteq V(G) \), \( N(A) = \{ x : xa \in E(G) \text{ for some } a \in A \} \) is called the neighborhood of \( A \) and \( N[A] = N(A) \cup A \) is called the closed neighborhood of \( A \). Moreover, for each \( v \in V(G) \), the open hop neighborhood of \( v \) is the set \( N^2_G(v) = \{ u \in V(G) : d_G(u, v) = 2 \} \) and the closed hop neighborhood of \( v \) is the set \( N^2_G[v] = \{ v \} \cup N^2_G(v) \). Also, for any \( A \subseteq V(G) \), \( N^2_G[A] = \{ v \in V(G) : N^2_G(v) \cap A \neq \emptyset \} \) is called the open hop neighborhood of \( A \) and the set \( N^2_G[A] = A \cup N^2_G(A) \) is the called closed hop neighborhood of \( A \). Denote by \( F^2_G[A] \) the complement of \( N^2_G[A] \), i.e., \( F^2_G[A] = V(G) \setminus N^2_G[A] \).

In 1983, Diesto and Gervacio in [5] proved that given a simple graph \( G = (V(G), E(G)) \), \( G \) induces a topology on \( V(G) \), denoted by \( \tau_G \), with base consisting of sets of the form

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$F_G(A) = V(G) \setminus N_G(A)$, where $N_G(A) = A \cup \{x : xa \in E \text{ for some } a \in A\}$ and $A$ ranges over all subsets of $V(G)$. Their construction was further investigated in [2], [3] and [6]. In particular, Canoy and Lemence in [2] described the topologies induced by the complement of a graph, the join of graphs, composition and Cartesian product of graphs.

In [1], Canoy and Gimeno presented another way of constructing a topology $\tau_G$ from a connected graph $G$ by considering the family $\Omega(G) = \{F_G^2[A] : A \subseteq V(G)\}$ where $F_G^2[A] = \{x \in V(G) : x \notin A \text{ and } d_G(x, a) \neq 2 \text{ for all } a \in A\}$. They showed that this family is a base for some topology $\tau_G$ on $V(G)$. This construction is also studied by Nianga et al, in [4] for any graph $G$. It is also shown that the family $B_G = \{F_G^2[A] : A \subseteq V(G)\}$ and $S_G = \{F_G^2[v] : v \in V(G)\}$ are, respectively, base and subbase for the topology $\tau_G$ on $V(G)$.

Concepts on Graph Theory and Topology are taken from [7] and [8], respectively.

2. Results

**Definition 1.** The complement of graph $G$, denoted by $\overline{G}$ is the graph with $V(G) = V(\overline{G})$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$, where $u, v \in V(G) = V(\overline{G})$.

**Theorem 1.** Let $G$ be any graph and $\overline{G}$ its complement. Then for each $v \in V(G)$,

$$F_G^2[v] = \begin{cases} F_G[v] \cup \left( \bigcap_{u \in F_G[v]} N_G(u) \right), & \text{if } F_G[v] \neq \emptyset \\ N_G(v), & \text{if } F_G[v] = \emptyset. \end{cases} \tag{1}$$

**Proof.** Let $G$ be any graph and $\overline{G}$ its complement. Let $v \in V(G)$ and set $A = \bigcap_{u \in F_G[v]} N_G(u)$. Suppose $F_G[v] = \emptyset$. Then $N_G(v) = V(G) \setminus \{v\}$. Hence, $v$ is an isolated vertex in $\overline{G}$. Thus, $F_G^2[v] = N_G(v)$. Suppose $F_G[v] \neq \emptyset$. Let $u \in F_G[v]$. Then $u \neq v$ and $u \notin N_G(v)$. Hence, $u \neq v$ and $u \in N_{\overline{G}}(v)$. Thus, $u \in F_G^2[v]$. Next, let $w \in A$. Then $w \in N_G(u)$ for all $u \in F_G[v]$. Since $u \notin N_G(v)$, it follows that $w \neq v$. Also, $w \notin N_{\overline{G}}(u)$ for all $u \in N_{\overline{G}}(v)$. It implies that $d_{\overline{G}}(w, v) \neq 2$. Hence, $w \in F_G^2[v]$. Consequently, $F_G[v] \cup \left( \bigcap_{u \in F_G[v]} N_G(u) \right) \subseteq F_G^2[v]$. Next, let $x \in F_G^2[v]$. Then $x \neq v$ and $x \notin N_G(v)$. If $x \in F_G[v]$, then we are done. Suppose $x \notin F_G[v]$. Then $x \notin N_G(v)$. Suppose further that there exists $u \in F_G[v]$ such that $x \notin N_G(u)$. Thus, $u \in N_G(v)$ and $x \in N_{\overline{G}}(u)$. Also, since $x \in N_G(v), x \neq N_{\overline{G}}(v)$. Thus, $d_{\overline{G}}(x, v) = 2$, that is, $x \in N_{\overline{G}}^2(v)$, a contradiction. Therefore, $x \in N_G(u)$ for all $u \in F_G[v]$. This shows that $x \in A$. Accordingly, $F_G^2[v] \subseteq F_G[v] \cup \left( \bigcap_{u \in F_G[v]} N_G(u) \right)$. This establishes the desired equality. \hfill $\Box$

**Theorem 2.** Let $G$ be any graph and $\overline{G}$ its complement. If $v$ is an isolated vertex of $G$ (or of $\overline{G}$), then $\{v\} \in \tau_G \cap \tau_{\overline{G}}$.

**Proof.** Suppose $v$ is an isolated vertex of $G$ (or of $\overline{G}$). Then $\{v\} = F_G^2[V(G) \setminus \{v\}] = F_{\overline{G}}^2[V(\overline{G}) \setminus \{v\}]$ and so, $\{v\} \in B_G$ and $\{v\} \in B_{\overline{G}}$. Thus, $\{v\} \in \tau_G$ and $\{v\} \in \tau_{\overline{G}}$. Therefore, $\{v\} \in \tau_G \cap \tau_{\overline{G}}$. \hfill $\Box$
Remark 1. The converse of theorem 17 is not true.

Consider $G = P_3 = [a, b, c, d, e]$. Then $\{e\} = F^2_G[a, b]$ and $\{e\} = F^2_G[a, c]$. However, $e$ is not an isolated vertex of $G$ nor of $\overline{G}$.

Definition 2. The join $G_1 + G_2$ of graphs $G_1$ and $G_2$ is the graph $G$ with $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}.$$  

Theorem 3. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs and let $\emptyset \neq A \subseteq V(G)$ and $\emptyset \neq B \subseteq V(H)$. Then

(i) $F^2_{G + H}[A] = V(H) \cup \cup_{a \in A}N_G(a)$;

(ii) $F^2_{G + H}[B] = V(G) \cup \cup_{b \in B}N_H(b)$ and

(iii) $F^2_G[\emptyset] = V(G) \cup V(H)$.

Proof. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. Let $\emptyset \neq A \subseteq V(G)$ and $\emptyset \neq B \subseteq V(H)$.

(i) Note that

$$N^2_{G + H}[A] = A \cup \{v \in V(G + H) : d_{G + H}(v, a) = 2 \text{ for some } a \in A\}.$$  

Since $V(H) \subseteq N_{G + H}(A)$,

$$N^2_{G + H}[A] = A \cup \{v \in V(G) : d_{G + H}(v, a) = 2 \text{ for some } a \in A\} \subseteq A \cup \{v \in V(G) : d_G(v, a) \neq 1 \text{ for some } a \in A\}.$$  

Hence,

$$F^2_{G + H}[A] = V(H) \cup \cup_{a \in A}N_G(a).$$  

(ii) Similarly,

$$F^2_{G + H}[B] = V(G) \cup \cup_{b \in B}N_H(b).$$  

(iii) Clearly, $F^2_{G + H}[\emptyset] = V(G) \cup V(H)$. 

Remark 2. Let $G$ be any graph and let $A_1, A_2 \subseteq V(G)$. Then

$$N^2_G[A_1 \cup A_2] = N^2_G[A_1] \cup N^2_G[A_2].$$  

Theorem 4. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. Then for any $A \subseteq V(G + H)$ such that $A \cap V(G) = A_G \neq \emptyset$ and $A \cap V(H) = A_H \neq \emptyset$,

$$F^2_{G + H}[A] = F^2_{G + H}[A_G] \cap F^2_{G + H}[A_H].$$
Case 2. Suppose \( A \subseteq V(G + H) \). Suppose \( A \cap V(G) = A_G \neq \emptyset \) and \( A \cap V(H) = A_H \neq \emptyset \). Then \( x \in F^2_{G+H}[A] \) if and only if \( x \notin N^2_{G+H}(A) \). By Remark 2, \( x \in F^2_{G+H}[A] \) if and only if \( x \in F^2_{G+H}[A] \cap F^2_{G+H}[A] \).

The next theorem follows from Theorem 3 (i) and (ii).

Corollary 1. Let \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \) be graphs. Then for any \( v \in V(G) \cup V(H) \),

\[
F^2_{G+H}[v] = \begin{cases} V(H) \cup N_G(v), & \text{if } v \in V(G) \\ V(G) \cup N_G(v), & \text{if } v \in V(H). \end{cases}
\]  

Definition 3. The corona \( G \circ H \) of graphs \( G \) and \( H \) is the graph obtained by taking one copy of \( G \) and \( |V(G)| \) copies \( H \) and then forming the sum \( (v) + H^v = v + H^v \) for each \( v \in V(G) \), where \( H^v \) is a copy of \( H \) corresponding to the vertex \( v \).

Theorem 5. Let \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \) be graphs. Then for any \( a \in V(G \circ H) \),

\[
F^2_{G \circ H}[a] = \begin{cases} \mathcal{F}^2_G[a] \cup \bigcup_{v \in V(G) \setminus N_G(a)} V(H^v), & \text{if } a \in V(G) \\ N_{H^v}(a) \cup [V(G) \setminus N_G(w)] \cup \bigcup_{v \in V(G) \setminus \{w\}} V(H^v), & \text{if } a \in V(H^w). \end{cases}
\]  

Proof. Let \( x \in F^2_{G \circ H}[a] \). Then \( x \neq a \) and \( x \notin N^2_{G \circ H}(a) \). Consider the following cases:

Case 1. Suppose \( a \in V(G) \). If \( x \in V(G) \), then \( x \notin N^2_G(a) \) since \( x \notin N^2_{G \circ H}(a) \). Hence, \( x \notin F^2_G[a] \). Suppose \( x \notin V(G) \). Let \( u \in V(G) \) such that \( x \in V(H^u) \). If \( u = a \), then \( x \in V(H^u) \) and \( u \in V(G) \setminus N_G(a) \). Suppose \( u \neq a \). Since \( x \notin N^2_G(a) \) and \( d_{G \circ H}(a, y) = 2 \) for all \( y \in V(H^2) \) with \( z \in N_G(a) \), it follows that \( u \in V(G) \setminus N_G(a) \). Thus,

\[
F^2_{G \circ H}[a] \subseteq \mathcal{F}^2_G[a] \cup \bigcup_{v \in V(G) \setminus N_G(a)} V(H^v) = X.
\]

Now, let \( w \in X \). If \( w \in \mathcal{F}^2_G[a] \), then \( w \notin N^2_G[a] \). Hence, \( w \notin N^2_{G \circ H}[a] \). This implies that \( w \notin F^2_G[a] \). Suppose \( w \in \bigcup_{v \in V(G) \setminus N_G(a)} V(H^v) \). Then there exists \( v \in V(G) \setminus N_G(a) \) such that \( w \in V(H^v) \). It follows that \( w \neq a \) and \( d_{G \circ H}(w, a) \neq 2 \). Thus, \( w \in F^2_{G \circ H}[a] \). Therefore,

\[
\mathcal{F}^2_G[a] \cup \bigcup_{v \in V(G) \setminus N_G(a)} V(H^v) \subseteq F^2_{G \circ H}[a].
\]

Case 2. Suppose \( a \in V(H^w) \) for some \( w \in V(G) \). If \( x = w \), then \( x \in V(G) \setminus N_G(w) \). Suppose \( x \neq w \). If \( x \in V(G) \), then \( d_G(x, w) \neq 1 \) because \( d_{G \circ H}(x, a) \neq 2 \). Hence, \( x \in V(H^w) \) for some \( q \in V(G) \). If \( q = w \), then \( x \in V(H^w) \). Since \( x \neq a \) and \( a \in V(H^w) \), \( a \in N_{H^w}(a) \) (otherwise, \( d_{G \circ H}(a, x) = 2 \)). Suppose \( q \neq w \). Then \( x \in V(H^q) \) and \( q \in V(G) \). Thus,

\[
z \in N_{H^w}(a) \cup [V(G) \setminus N_G(w)] \cup \bigcup_{v \in V(G) \setminus \{w\}} V(H^v) = Y.
\]
Suppose now that \( p \in Y \). If \( p \in N_{H^w(a)} \), then \( d_{G \circ H}(p,a) = d_{H^w}(p,a) = 1 \). Hence, \( p \in F_{G \circ H}^2[a] \). If \( p \in V(G) \setminus N_G(w) \), then \( d_{G \circ H}(p,w) = d_G(p,w) \neq 1 \). Hence, \( p \neq a \) and \( d_{G \circ H}(a,p) \neq 2 \). This implies that \( p \in F_{G \circ H}^2[a] \). Finally, if \( p \in \cup_{v \in V(G) \setminus \{w\}} V(H^v) \), then there exists \( r \in V(G) \backslash \{w\} \) such that \( p \in V(H^r) \). Since

\[
d_{G \circ H}(a,p) = d_{G \circ H}(a,w) + d_{G \circ H}(r,w) + d_{G \circ H}(r,p) = 2 + d_{G \circ H}(r,w) \geq 3,
\]

it follows that \( p \in F_{G \circ H}^2[a] \). Therefore,

\[
N_{H^w}(a) \cup V(G) \setminus N_G(w) \cup \left[ \cup_{v \in V(G) \setminus \{w\}} V(H^v) \right] \subseteq F_{G \circ H}^2[a].
\]

Accordingly, the desired equality follows.

**Definition 4.** The lexicographic product (composition) of graphs \( G \) and \( H \), denoted by \( G[H] \), is the graph with \( V(G[H]) = V(G) \times V(H) \) and \( (u,v)(u',v') \in E(G[H]) \) if and only if either \( uu' \in E(G) \) or \( u = u' \) and \( vv' \in E(H) \).

**Theorem 6.** Let \( G = (V(G),E(G)) \) and \( H = (V(H),E(H)) \) be any two graphs and let \((v,a) \in V(G[H])\). Then

\[
F_{G[H]}^2[(v,a)] = (F_G^2[v] \times V(H)) \cup (\{v\} \times F_H^2[a]).
\]

**Proof.** Note that \((x,q) \in F_{G[H]}^2[(v,a)]\) if and only if \((x,q) \neq (v,a)\) and \(d_{G[H]}((x,q),(v,a)) \neq 2\). Consider the following cases:

- **Case 1.** Suppose \( x = v \). Then \( q \neq a \). Since

\[
d_{G[H]}((v,q),(v,a)) = d_H(a,q) \neq 2, q \in F_H^2[a],
\]

\( q \in F_H^2[a] \). Hence, \((x,q) \in \{v\} \times F_H^2[a] \).

- **Case 2.** Suppose \( x \neq v \). Then

\[
d_G(x,v) = d_{G[H]}((x,q),(v,a)) \neq 2.
\]

Hence, \( x \in F_G^2[v] \) and \( (x,q) \in F_G^2[v] \times V(H) \). Therefore,

\[
F_{G[H]}^2[(v,a)] \subseteq (F_G^2[v] \times V(H)) \cup (\{v\} \times F_H^2[a]).
\]

Next, let \((w,p) \in F_G^2[v] \times V(H) \). Then \( w \in F_G^2[v] \), that is, \( w \neq v \) and \( d_G(w,v) \neq 2 \). It follows that \((w,p) \neq (v,a)\) and

\[
d_{G[H]}((w,p),(v,a)) = d_G(w,v) \neq 2.
\]

This shows that \((w,p) \in F_{G[H]}^2[(v,a)] \). Hence, \( F_G^2[v] \times V(H) \subseteq F_{G[H]}^2[(v,a)] \). Finally, let \((z,t) \in \{v\} \times F_H^2[a] \). Then \( z = v \) and \( t \in F_H^2[a] \). Hence, \( t \neq a \) and \( d_H(a,t) \neq 2 \). Consequently, \((z,t) \neq (v,a)\) and

\[
d_{G[H]}((z,t),(v,a)) = d_H(a,t) \neq 2,
\]

showing that \((z,t) \in F_{G[H]}^2[a] \). Thus, \( \{v\} \times F_H^2[a] \subseteq F_{G[H]}^2[(v,a)] \). This establishes the desired equality.
Definition 5. The Cartesian Product of two graphs $G_1$ and $G_2$ denoted by $G_1 \Box G_2$ is a graph with $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and two vertices $a = (u_1, u_2)$ and $b = (v_1, v_2)$ are adjacent in $G_1 \Box G_2$ if and only if either $u_1 = v_1$ and $u_2 = v_2$ or $u_1 v_1 \in E(G_1)$.

Theorem 7. Let $K = G \Box H = (V(K), E(K))$, where $G = (V(G), E(G))$ and $H = (V(H), E(H))$. Then for each $(v, a) \in V(K)$,

$$F_K^2[(v, a)] = [F_G^2[v] \times \{a\}] \cup \{(v) \times F_H^2[a]\} \cup [F_G[v] \times V(H) \setminus \{a\}] \cup [N_G(v) \times F_G[a]].$$

Proof. Let $(v, a) \in V(K) = V(G \Box H)$ and $(x, q) \in F_K^2[(v, a)]$. Then $(v, a) \neq (x, q)$ and $d_K((v, a), (x, q)) \neq 2$. Now, consider the following cases:

Case 1. Assume that $x = v$. Then $q \neq a$ and $d_H(q, a) = d_K((x, q), (x, a)) \neq 2$ and so, $q \in F_H^2[a]$. Hence, $(x, q) \in \{(v) \times F_H^2[a]\}$.

Case 2. Assume that $x \neq v$.

Subcase 1. Let $q = a$. Then $d_G(x, v) = d_K((x, q), (v, q)) \neq 2$ and thus, $x \in F_G^2[v]$. It follows that $(x, q) \in F_G^2[v] \times \{a\}$.

Subcase 2. Let $q \neq a$. Suppose that $x \in N_G(v)$. If $q \in N_H(a)$, then

$$d_K((x, q), (v, a)) = d_G(x, v) + d_H(q, a) = 2,$$

a contradiction. Thus, $q \in V(H) \setminus N_H[a]$. Hence, $(x, q) \in N_G(v) \times F_G[a]$. Suppose $x \in N_G(v)$. Then $x \in F_G[v]$. Hence, $(x, q) \in F_G[v] \times V(H) \setminus \{a\}$. Therefore,

$$F_K^2[(v, a)] \subseteq [F_G^2[v] \times \{a\}] \cup \{(v) \times F_H^2[a]\} \cup [F_G[v] \times V(H) \setminus \{a\}] \cup [N_G(v) \times F_G[a]].$$

Next, let $(v, p) \in \{v\} \times F_H^2[a]$. Then $p \neq a$ and $d_H(a, p) \neq 2$. Hence, $(v, p) \neq (v, a)$ and $d_K((v, p), (v, a)) = d_H(a, p) \neq 2$, that is, $(v, p) \in F_K^2[(v, a)]$. If $(x, a) \in F_G^2[v] \times \{a\}$, then $x \neq v$ and $d_G(x, v) \neq 2$. Hence, $(x, a) \neq (v, a)$ and $d_K((v, a), (x, a)) = d_G(x, v) \neq 2$, that is, $(x, a) \in F_K^2[(v, a)]$. Now, $(y, b) \in N_G(v) \times F_H[a]$ implies $d_G(y, v) = 1$ and $d_H(b, a) \geq 2$. It follows that $(y, b) \neq (v, a)$ and

$$d_K((y, b), (v, a)) = d_G(y, v) + d_H(b, a) \geq 3.$$

Hence, $(y, b) \in F_K^2[(v, a)]$. Finally, $(z, t) \in [F_G[v] \times V(H) \setminus \{a\}]$ implies $d_G(z, v) \geq 2$ and $d_H(t, a) \geq 1$. This means that $(z, t) \neq (v, a)$ and

$$d_K((z, t), (v, a)) = d_G(z, v) + d_H(t, a) \geq 3.$$

Thus, $(z, t) \in F_K^2[(v, a)]$. Therefore,

$$[F_G^2[v] \times \{a\}] \cup \{(v) \times F_H^2[a]\} \cup [F_G[v] \times V(H) \setminus \{a\}] \cup [N_G(v) \times F_G[a]] \subseteq F_K^2[(v, a)].$$

This establishes the desired equality.
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