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Some Theorems on Tauber's Generalized Stirling, Lah, and Bell Numbers

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Abstract. In this paper, the authors obtain some properties for Tauber's generalized Stirling and Lah numbers, including other forms of recurrence relations, orthogonality and inverse relations, rational generating function and explicit formula in symmetric function form. Moreover, the authors derive a new explicit formula, which is analogous to the Qi formula.

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1. Introduction

The nth Bell numbers, denoted by B_n, are known by their combinatorial interpretation as the number of ways to partition an n-set. This interpretation is based on the definition of Bell numbers as the sum of the classical Stirling numbers of the second kind [2]. That is,

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \tag{1}$$

where $\left\{ {n \atop k} \right\}$ denotes the Stirling numbers of the second kind (the Karamata-Knuth notation), which can be interpreted as the number of ways to partition an n-set into k nonempty subsets.

The Bell numbers were expressed by Spivey [5] as

$$B_{m+n} = \sum_{k=0}^{n} \sum_{j=0}^{m} j^{n-k} \left\{ {m \atop j} \right\} \binom{n}{k} B_k. \tag{2}$$

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The above formula was proven combinatorially by considering another way of counting the number of ways to partition the set with $m+n$ objects.

Recently, Feng Qi [4] established a new explicit formula for Bell numbers, which is expressed in terms of Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ and Lah numbers $L(n,k)$. The formula is given by

$$B_n = \sum_{k=1}^{n} (-1)^{n-k} \left[\sum_{l=1}^{k} L(k,l) \right] \left\{ {n \atop k} \right\} \tag{3}$$

which may be referred to as the Qi formula for Bell numbers. The formula has been derived in two ways:

- using Faa di Bruno's formula and the formula for the n-th derivative of $e^{\frac{1}{x}}$ containing Lah numbers [3, 9];
- using the inverse relation for the classical Stirling numbers of the first and second kind.

These methods have been applied to obtain new explicit formula for different generalizations of Stirling numbers.

Certain generalization of Stirling numbers was introduced by S. Tauber [6–8] as the coefficients $C_{k,n}^{m}$ and $D_{k,n}^{m}$ of the following expansions of the given two sequences of polynomials $Q_1(x,n)$ and $Q_2(x,n)$ for $n = 0, 1, 2, \ldots$,

$$Q_k(x,n) = \sum_{m=0}^{n} C_{k,n}^{m} x^m \tag{4}$$

$$x^n = \sum_{m=0}^{n} D_{k,n}^{m} Q_k(x,m) \tag{5}$$

for k=1, 2. Also, these numbers are equal to zero if $n < m$, $m < 0$, $n < 0$. These generalized Stirling numbers of the first and second kind on Q- polynomials satisfy the following triangular recurrence relations in [8],

$$C_n^m = M(n) C_{n-1}^m + N(n) C_{n-1}^{m-1} \tag{6}$$

$$D_n^m = -\frac{M(m+1)}{N(m+1)} D_{n-1}^m + \frac{1}{N(m+1)} D_{n-1}^{m-1}, \tag{7}$$

where M and N are two functions of the variable n, such that $M(0) \neq 0$ and for n a positive integer or zero $N(0) \neq 0$. In the paper [6] of Tauber, a generalization of Lah numbers $L_{k,h,n}^{m}$ was introduced as the coefficient of the following relation,

$$Q_k(x,n) = \sum_{m=0}^{n} L_{k,h,n}^{m} Q_h(x,m) \tag{8}$$

for two sequences of polynomials Q_k and Q_h where $k \neq h$, and $k, h \in \{1, 2\}$. These numbers satisfy the following triangular recurrence relation,

$$L^m_{k,h,n} = \left[M_k(n) - \frac{N_k(n)M_h(m+1)}{N_h(m+1)} \right] L^m_{k,h,n-1} + \frac{N_k(n)}{N_h(m)} L^{m-1}_{k,h,n-1}, \tag{9}$$

where $M_k(n)$ and $N_k(n)$ are the subscripted version of the functions $M(n)$ and $N(n)$ that correspond to $Q_k(x, n)$. Furthermore, the generalized Lah numbers $L^m_{k,h,n}$ have been expressed in terms of generalized Stirling numbers of the first kind and the second kind as follows

$$L^m_{k,h,n} = \sum_{s=m}^{n} C^s_{k,n} D^m_{h,s}. \tag{10}$$

The generalized Bell numbers, denoted by $B_{h,n}$, may be defined as the sum of the generalized Stirling numbers of the second kind $D^m_{h,n}$,

$$B_{h,n} = \sum_{m=0}^{n} D^m_{h,n}. \tag{11}$$

The Lah numbers can be defined as the coefficients of the following relations:

$$(-x)_n = \sum_{k=0}^{n} L_{n,k}(x)_k \tag{12}$$

$$(x)_n = \sum_{k=0}^{n} L_{n,k}(-x)_k \tag{13}$$

where $(x)_n$ is the falling factorial, defined by

$$(x)_n = \begin{cases} x(x-1)\ldots(x-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}.$$

Also,

$$(-x)_n = (-1)^n (x)(x+1)\ldots(x+n-1) = (-1)^n x^{(n)}$$

where $x^{(n)}$ is the rising factorial, defined by

$$(x)^{(n)} = \begin{cases} x(x+1)\ldots(x+n-1), & n \geq 1 \\ 1, & n = 0 \end{cases}.$$

Given polynomial $Q_k(x, n) = ((-1)^{k-1}x)_n$, Tauber's generalized Lah numbers $L^m_{k,h,n}$ in (8) reduce to the ordinary Lah numbers $L_{n,m}$ in (12) and (13) for $h = 1$ and $k = 2$. That is,

$$L^m_{2,1,n} = L_{n,m} = L^m_{1,2,n}$$

Moreover, (4) and (5) give

$$\left\{C^m_{1,n}, D^m_{1,n}\right\} = \left\{(-1)^{n-m}\begin{bmatrix} n \\ m \end{bmatrix}, \begin{Bmatrix} n \\ m \end{Bmatrix}\right\}$$
$$\left\{C^m_{2,n}, D^m_{2,n}\right\} = \left\{(-1)^{n}\begin{bmatrix} n \\ m \end{bmatrix}, (-1)^n\begin{Bmatrix} n \\ m \end{Bmatrix}\right\},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ is the Karamata-Knuth notation for the unsigned Stirling numbers of the first kind.

The Whitney numbers of the first kind $w_\alpha(n,m)$ and second kind $W_\alpha(n,m)$ of Dowling lattices are defined in the paper of Benoumhani [1] as the coefficient of the following equations,

$$\alpha^n \left(\frac{x-1}{\alpha}\right)_n = \sum_{m=0}^{n} w_\alpha(n,m)x^m \tag{14}$$

$$x^n = \sum_{m=0}^{n} \alpha^m W_\alpha(n,m)\left(\frac{x-1}{\alpha}\right)_m \tag{15}$$

which can be written as

$$(x-1|\alpha)_n = \sum_{m=0}^{n} w_\alpha(n,m)x^m \tag{16}$$

$$x^n = \sum_{m=0}^{n} W_\alpha(n,m)(x-1|\alpha)_m \tag{17}$$

where α is a positive integer and $(x|\alpha)_m = \prod_{i=0}^{m-1}(x-i\alpha)$. It can easily be shown that

$$\sum_{k=j}^{n} W_\alpha(n,k)w_\alpha(k,j) = \sum_{k=j}^{n} w_\alpha(n,k)W_\alpha(k,j) = \delta_{n,j} \tag{18}$$

$$f_n = \sum_{k=0}^{n} w_\alpha(n,k)g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} W_\alpha(n,k)f_k. \tag{19}$$

When $Q_k(x,n) = ((-1)^{k-1}x-1|\alpha)_n$, equations (4) and (5) yield equations (16) and (17), respectively. This implies that

$$\left\{C^m_{1,n}, D^m_{1,n}\right\} = \{w_\alpha(n,m), W_\alpha(n,m)\} \text{ and}$$
$$\left\{C^m_{2,n}, D^m_{2,n}\right\} = \{(-1)^m w_\alpha(n,m), (-1)^n W_\alpha(n,m)\}.$$

From the definition of Whitney numbers, we may also define Whitney-Lah numbers denoted by $L^W_{n,m}(\alpha)$ as the coefficients of the following relations,

$$(-x-1|\alpha)_n = \sum_{m=0}^{n} L^W_{n,m}(\alpha)(x-1|\alpha)_m \tag{20}$$

$$(x-1|\alpha)_n = \sum_{m=0}^{n} L_{n,m}^{W}(\alpha)(-x-1|\alpha)_m. \tag{21}$$

By assigning $Q_k(x,n) = ((-1)^{k-1}x-1|\alpha)_n$, (8) gives

$$L_{2,1,n}^{m} = L_{n,m}^{W}(\alpha) = L_{1,2,n}^{m}.$$

Hence, with $k=2$ and $h=1$, (10) yields

$$L_{n,j}^{W}(\alpha) = \sum_{k=j}^{n}(-1)^k w_\alpha(n,k)W_\alpha(k,j). \tag{22}$$

The Dowling numbers, denoted by $D_n(\alpha)$, can be defined as the sum of Whitney numbers of the second kind $W_\alpha(n,m)$, i.e.

$$D_n(\alpha) = \sum_{m=0}^{n} W_\alpha(n,m). \tag{23}$$

Using the inverse relation in (19), we have

$$W_\alpha(n,k) = \sum_{j=0}^{n}(-1)^n W_\alpha(n,j)L_{j,k}^{W}(\alpha)$$

Thus, the Dowling numbers equal

$$D_n(\alpha) = \sum_{j=0}^{n}(-1)^{n-j}\left[\sum_{k=0}^{j}(-1)^j L_{j,k}^{W}(\alpha)\right]W_\alpha(n,j). \tag{24}$$

In this paper, we establish more properties of Tauber's generalized Stirling numbers as well as some properties of the generalized Bell numbers.

2. Some Recurrence Relations

Tauber's generalized Stirling numbers of the first and second kind and Tauber's generalized Lah numbers are known to have triangular recurrence relations. In this section, we establish other forms of recurrence relations for Tauber's generalized Stirling numbers of the first and second kind and Tauber's generalized Lah numbers. The following theorems contain the vertical recurrence relation.

Theorem 2.1. *Tauber's generalized Stirling numbers of the first kind C_m^n satisfy the following vertical recurrence relation,*

$$C_n^m = \sum_{i=1}^{n-m+1}\left[\prod_{j=0}^{i-1} M(n+1-j)\right] N(n-i+1)C_{n-i}^{m-1} \tag{25}$$

where

$$M(n+1) = 1, M(n+2) = 1.$$

Proof. From the triangular recurrence relation (6), that is

$$C_n^m = M(n)C_{n-1}^m + N(n)C_{n-1}^{m-1} = N(n)C_{n-1}^{m-1} + M(n)C_{n-1}^m.$$

Since

$$C_{n-1}^m = M(n-1)C_{n-2}^m + N(n-1)C_{n-2}^{m-1}$$

then

$$C_n^m = N(n)C_{n-1}^{m-1} + M(n)N(n-1)C_{n-2}^{m-1} + M(n)M(n-1)C_{n-2}^m.$$

Continuing in this manner,

$$\begin{aligned}C_n^m =& N(n)C_{n-1}^{m-1} + M(n)N(n-1)C_{n-2}^{m-1}\\ &+ \ldots + M(n)\ldots M(m+1)N(m)C_{m-1}^{m-1} + M(n)\ldots M(m+1)M(m)C_{m-1}^m.\end{aligned}$$

By definition, the last term is equal to zero. Thus,

$$C_n^m = \sum_{i=1}^{n-m+1} \left[M(n+1)\ldots M(n-i+2)N(n-i+1)\right] C_{n-i}^{m-1}$$

where

$$M(n+1) = 1, M(n+2) = 1.$$

Theorem 2.2. *Tauber's generalized Stirling numbers of the second kind* D_n^m *satisfy the following vertical recurrence relation,*

$$D_n^m = \sum_{i=1}^{n-m+1} (-1)^{i-1} \frac{(M(m+1))^{i-1}}{(N(m+1))^i} D_{n-i}^{m-1}. \tag{26}$$

Proof. From the triangular recurrence relation (7),

$$D_n^m = -\frac{M(m+1)}{N(m+1)} D_{n-1}^m + \frac{1}{N(m+1)} D_{n-1}^{m-1}$$

Since

$$D_{n-1}^m = -\frac{M(m+1)}{N(m+1)} D_{n-2}^m + \frac{1}{N(m+1)} D_{n-2}^{m-1},$$

then

$$D_n^m = \frac{1}{N(m+1)} D_{n-1}^{m-1} - \frac{M(m+1)}{(N(m+1))^2} D_{n-2}^{m-1} + \frac{(M(m+1))^2}{(N(m+1))^2} D_{n-2}^m.$$

Continuing in this manner,

$$\begin{aligned}D_n^m =& \frac{1}{N(m+1)} D_{n-1}^{m-1} - \frac{M(m+1)}{(N(m+1))^2} D_{n-2}^{m-1}\\ &+ \ldots + (-1)^{n-m} \frac{(M(m+1))^{n-m}}{(N(m+1))^{n-m+1}} D_{m-1}^{m-1}\end{aligned}$$

$$+ (-1)^{n-m+1}\frac{(M(m+1))^{n-m+1}}{(N(m+1))^{n-m+1}}D^m_{m-1}.$$

By definition, the last term is equal to zero. Thus,

$$D^m_n = \sum_{i=1}^{n-m+1}(-1)^{i-1}\frac{(M(m+1))^{i-1}}{(N(m+1))^{i}}D^{m-1}_{n-i}.$$

Theorem 2.3. *Tauber's generalized Lah numbers* $L^m_{k,h,n}$ *satisfy the following vertical recurrence relation,*

$$L^m_{k,h,n} = \sum_{i=1}^{n-m+1}\left\{\prod_{j=0}^{i-1}\left[M_k(n+1-j) - \frac{N_k(n+1-j)M_h(m+1)}{N_h(m+1)}\right]\right\}\frac{N_k(n-i+1)}{N_h(m)}L^{m-1}_{k,h,n-i}$$

where

$$\left[M_k(n+1) - \frac{N_k(n+1)M_h(m+1)}{N_h(m+1)}\right] = 1.$$

Proof. From the triangular recurrence relation (9), that is

$$\begin{aligned}L^m_{k,h,n} &= \left[M_k(n) - \frac{N_k(n)M_h(m+1)}{N_h(m+1)}\right]L^m_{k,h,n-1} + \frac{N_k(n)}{N_h(m)}L^{m-1}_{k,h,n-1}\\ &= \frac{N_k(n)}{N_h(m)}L^{m-1}_{k,h,n-1} + \left[M_k(n) - \frac{N_k(n)M_h(m+1)}{N_h(m+1)}\right]L^m_{k,h,n-1}.\end{aligned}$$

Since

$$L^m_{k,h,n-1} = \left[M_k(n-1) - \frac{N_k(n-1)M_h(m+1)}{N_h(m+1)}\right]L^m_{k,h,n-2} + \frac{N_k(n-1)}{N_h(m)}L^{m-1}_{k,h,n-2},$$

then

$$\begin{aligned}L^m_{k,h,n} =& \frac{N_k(n)}{N_h(m)}L^{m-1}_{k,h,n-1} + \frac{N_k(n-1)}{N_h(m)}\left[M_k(n) - \frac{N_k(n)M_h(m+1)}{N_h(m+1)}\right]L^{m-1}_{k,h,n-2}\\ &+ \left[M_k(n) - \frac{N_k(n)M_h(m+1)}{N_h(m+1)}\right]\left[M_k(n-1) - \frac{N_k(n-1)M_h(m+1)}{N_h(m+1)}\right]\\ &\times\left[L^m_{k,h,n-2}\right].\end{aligned}$$

Continuing in this manner,

$$\begin{aligned}L^m_{k,h,n} =& \frac{N_k(n)}{N_h(m)}L^{m-1}_{k,h,n-1} + \left[M_k(n) - \frac{N_k(n)M_h(m+1)}{N_h(m+1)}\right]\frac{N_k(n-1)}{N_h(m)}L^{m-1}_{k,h,n-2} + \ldots +\\ &\left[M_k(n) - \frac{N_k(n)M_h(m+1)}{N_h(m+1)}\right]\ldots\left[M_k(m+1) - \frac{N_k(m+1)M_h(m+1)}{N_h(m+1)}\right]\end{aligned}$$

$$\frac{N_k(m)}{N_h(m)}L^{m-1}_{k,h,m-1}+$$
$$\left[M_k(n)-\frac{N_k(n)M_h(m+1)}{N_h(m+1)}\right]\dots\left[M_k(m+1)-\frac{N_k(m+1)M_h(m+1)}{N_h(m+1)}\right]$$
$$\left[M_k(m)-\frac{N_k(m)M_h(m+1)}{N_h(m+1)}\right]L^{m}_{k,h,m-1}.$$

By definition, the last term is equal to zero. Thus,

$$\begin{aligned}L^m_{k,h,n} = &\sum_{i=1}^{n-m+1}\left[M_k(n+1)-\frac{N_k(n+1)M_h(m+1)}{N_h(m+1)}\right]\\ &\dots\left[M_k(n-i+2)-\frac{N_k(n-i+2)M_h(m+1)}{N_h(m+1)}\right]\frac{N_k(n-i+1)}{N_h(m)}L^{m-1}_{k,h,n-i}\end{aligned}$$

where

$$\left[M_k(n+1)-\frac{N_k(n+1)M_h(m+1)}{N_h(m+1)}\right]=1.$$

3. Some Explicit Formulas

Consider the generating function $\psi_m(t)$ for Tauber's generalized Stirling numbers of the second kind given by

$$\psi_m(t)=\sum_{n\geq m}D^m_{k,n}t^n \tag{27}$$

where $\psi_0(t)=1$. Using the recurrence relation in (7), we have

$$\begin{aligned}\psi_m(t) &= \sum_{n\geq m}\left\{-\frac{M(m+1)}{N(m+1)}D^m_{k,n-1}+\frac{1}{N(m+1)}D^{m-1}_{k,n-1}\right\}t^n\\ &= -\frac{M(m+1)}{N(m+1)}t\sum_{n\geq m}D^m_{k,n-1}t^{n-1}+\frac{t}{N(m+1)}\sum_{n\geq m}D^{m-1}_{k,n-1}t^{n-1}\\ &= -\frac{M(m+1)}{N(m+1)}t\psi_m(t)+\frac{t}{N(m+1)}\psi_{m-1}(t).\end{aligned}$$

Hence,

$$\psi_m(t)=\frac{t}{N(m+1+M(m+1)t}\psi_{m-1}(t).$$

By backward substitution, we have

$$\psi_m(t)=\frac{t^m}{\prod_{i=0}^{m-1}\left[N(m+1-i)+M(m+1-i)t\right]}.$$

This result is stated formally in the following theorem.

Theorem 3.1. *The rational generating function for Tauber's generalized Stirling numbers of the second kind is given by*

$$\sum_{n\geq m} D^m_{k,n} t^n = \frac{t^m}{\prod_{i=0}^{m-1}\left[N(m+1-i)+M(m+1-i)t\right]}.$$

As a consequence of this rational generating function, we have the following explicit formula in symmetric function form.

Theorem 3.2. *Tauber's generalized Stirling numbers of the second kind are given by*

$$D^m_{k,n} = \frac{1}{\prod_{i=0}^{m-1} N(m+1-i)} \sum_{s_1+s_2+\ldots+s_m=n-m} \prod_{i=0}^{m} \left[-\frac{M(m+1-i)}{N(m+1-i)}\right]^{s_i}.$$

Proof. Using the rational generating function in Theorem 3.1, we get

$$\sum_{n\geq m} D^m_{k,n} t^n = \frac{t^m}{\prod_{i=0}^{m-1}\left[N(m+1-i)+M(m+1-i)t\right]}$$

$$\begin{aligned}
&= \frac{t^m}{\prod_{i=0}^{m-1} N(m+1-i)} \prod_{i=0}^{m-1} \frac{1}{\left[1-\left(-\frac{M(m+1-i)}{N(m+1-i)}\right)t\right]} \\
&= \frac{t^m}{\prod_{i=0}^{m-1} N(m+1-i)} \prod_{i=0}^{m-1} \sum_{n\geq 0} \left(-\frac{M(m+1-i)}{N(m+1-i)}\right)^n t^n \\
&= \frac{t^m}{\prod_{i=0}^{m-1} N(m+1-i)} \sum_{n\geq m} \sum_{s_1+s_2+\ldots+s_m=n-m} \prod_{i=0}^{m} \left[-\frac{M(m+1-i)}{N(m+1-i)}\right]^{s_i} t^{s_i} \\
&= \sum_{n\geq m} \left\{ \frac{t^m}{\prod_{i=0}^{m-1} N(m+1-i)} \sum_{s_1+s_2+\ldots+s_m=n-m} \prod_{i=0}^{m} \left[-\frac{M(m+1-i)}{N(m+1-i)}\right]^{s_i} \right\} t^n
\end{aligned}$$

Comparing coefficients of t^n completes the proof of the theorem.

One of the objectives of this study is to establish a new explicit formula for Tauber's generalized Bell numbers, which is analogous to the Qi formula. To derive the desired formula, we need to establish the following relations.

Theorem 3.3. *Tauber's generalized Stirling numbers of the first and second kind satisfy the following orthogonality relation for a given sequence of polynomials $Q_k(x,n)$,*

$$\sum_{m=s}^{n} C^m_{k,n} D^s_{k,m} = \sum_{m=s}^{n} D^m_{k,n} C^s_{k,m} = \delta^s_n = \begin{cases} 0, \ if & n \neq s \\ 1, \ if & n = s \end{cases}$$

where $k = 1, 2$.

Proof. Note that (5) can be written as

$$x^m = \sum_{s=0}^{m} D_{k,m}^{s} Q_k(x,s). \tag{28}$$

Substitute (28) into (4), then

$$\begin{aligned} Q_k(x,n) = \sum_{m=0}^{n} C_{k,n}^{m} x^m &= \sum_{m=0}^{n} C_{k,n}^{m} \sum_{s=0}^{m} D_{k,m}^{s} Q_k(x,s) \\ &= \sum_{m=0}^{n} \sum_{s=0}^{m} C_{k,n}^{m} D_{k,m}^{s} Q_k(x,s) \\ &= \sum_{s=0}^{n} \left\{ \sum_{m=s}^{n} C_{k,n}^{m} D_{k,m}^{s} \right\} Q_k(x,s). \end{aligned}$$

Thus,

$$\sum_{m=s}^{n} C_{k,n}^{m} D_{k,m}^{s} = \delta_n^s = \begin{cases} 0, \; if & n \neq s \\ 1, \; if & n = s \end{cases}.$$

Similarly, (4) can be written as

$$Q_k(x,m) = \sum_{s=0}^{m} C_{k,m}^{s} x^s. \tag{29}$$

Substitute (29) into (5), then

$$\begin{aligned} x^n = \sum_{m=0}^{n} D_{k,n}^{m} Q_k(x,m) &= \sum_{m=0}^{n} D_{k,n}^{m} \sum_{s=0}^{m} C_{k,m}^{s} x^s \\ &= \sum_{m=0}^{n} \sum_{s=0}^{m} D_{k,n}^{m} C_{k,m}^{s} x^s \\ &= \sum_{s=0}^{n} \left\{ \sum_{m=s}^{n} D_{k,n}^{m} C_{k,m}^{s} \right\} x^s. \end{aligned}$$

Thus,

$$\sum_{m=s}^{n} D_{k,n}^{m} C_{k,m}^{s} = \delta_n^s = \begin{cases} 0, \; if & n \neq s \\ 1, \; if & n = s \end{cases}.$$

Consequently,

$$\sum_{m=s}^{n} C_{k,n}^{m} D_{k,m}^{s} = \sum_{m=s}^{n} D_{k,n}^{m} C_{k,m}^{s} = \delta_n^s. \tag{30}$$

Theorem 3.4. *The Tauber's generalized Stirling numbers of the first and second kind are inverses,*

$$f_n = \sum_{m=0}^{n} C_{k,n}^{m} g_m \Longleftrightarrow g_n = \sum_{m=0}^{n} D_{k,n}^{m} f_m \tag{31}$$

for $k = 1, 2$.

Proof.

$$\begin{aligned}
\sum_{m=0}^{n} D_{k,n}^{m} f_m &= \sum_{m=0}^{n} D_{k,n}^{m} \sum_{s=0}^{m} C_{k,m}^{s} g_s \\
&= \sum_{m=0}^{n} \sum_{s=0}^{m} D_{k,n}^{m} C_{k,m}^{s} g_s \\
&= \sum_{s=0}^{n} \left\{ \sum_{m=s}^{n} D_{k,n}^{m} C_{k,m}^{s} \right\} g_s \\
&= \sum_{s=0}^{n} \delta_n^s g_s = \delta_n^n g_n = g_n
\end{aligned}$$

Conversely,

$$\begin{aligned}
\sum_{m=0}^{n} C_{k,n}^{m} g_m &= \sum_{m=0}^{n} C_{k,n}^{m} \sum_{s=0}^{m} D_{k,m}^{s} f_s \\
&= \sum_{m=0}^{n} \sum_{s=0}^{m} C_{k,n}^{m} D_{k,m}^{s} f_s \\
&= \sum_{s=0}^{n} \left\{ \sum_{m=s}^{n} C_{k,n}^{m} D_{k,m}^{s} \right\} f_s \\
&= \sum_{s=0}^{n} \delta_n^s f_s = \delta_n^n f_n = f_n
\end{aligned}$$

Theorem 3.5. *The Bell numbers $B_{h,n}$ can be computed in terms of Tauber's generalized Lah and Stirling numbers of the second kind. That is,*

$$B_{h,n} = \sum_{s=0}^{n} \left\{ \sum_{m=0}^{n} L_{k,h,s}^{m} \right\} D_{k,n}^{s}. \tag{32}$$

Proof. From the inverse relation (31) of Tauber's generalized Stirling numbers of the first and second kind,

$$f_n = \sum_{s=0}^{n} C_{k,n}^{s} g_s \Longleftrightarrow g_n = \sum_{s=0}^{n} D_{k,n}^{s} f_s. \tag{33}$$

Consider (10) and let

$$g_s = D_{h,s}^m$$

and

$$f_n = L_{k,h,n}^m,$$

by (33),

$$D_{h,n}^m = \sum_{s=0}^{n} D_{k,n}^s L_{k,h,s}^m. \tag{34}$$

Substituting (34) to (11) yields

$$B_{h,n} = \sum_{m=0}^{n} \sum_{s=0}^{n} D_{k,n}^s L_{k,h,s}^m$$

which implies

$$B_{h,n} = \sum_{s=0}^{n} \left\{ \sum_{m=0}^{n} L_{k,h,s}^m \right\} D_{k,n}^s. \tag{35}$$

Note that (32) is analogous to the Qi formula. For $h = 2$, and $k = 1$, the first five values of $B_{2,n}$ are

$$\begin{aligned} B_{2,0} &= 1 \\ B_{2,1} &= 1 \\ B_{2,2} &= 0 \\ B_{2,3} &= -1 \\ B_{2,4} &= 1. \end{aligned}$$

By switching the values of k and h ($i.e., k = 2 \; and \; h = 1$), the explicit formula for generalized Bell numbers

$$B_{h,n} = \sum_{s=0}^{n} \left\{ \sum_{m=0}^{n} L_{k,h,s}^m \right\} D_{k,n}^s$$

generates the ordinary Bell numbers for $n = 1, 2, \ldots,$

$$\begin{aligned} B_{1,0} &= 1 \\ B_{1,1} &= 1 \\ B_{1,2} &= 2 \\ B_{1,3} &= 5 \\ B_{1,4} &= 15. \end{aligned}$$

Remark 3.6. Tauber's generalized Bell numbers $B_{h,n}$ are equal to $\mathbf{e}_i\mathbf{DLe}$, where $\mathbf{D}$ and $\mathbf{L}$ are the generalized Stirling and Lah matrices, $\mathbf{e}_i$ is the i-th unit vector, and $\mathbf{e}$ is the vector with all entries equal to 1.

Remark 3.6 is equivalent to the following matrix relation

$$\left[D^i_{h,j}\right]_{n\times n} = \left[D^i_{k,j}\right]_{n\times n}\left[L^j_{k,h,i}\right]_{n\times n}. \tag{36}$$

Now, the orthogonality relation in Theorem 3.3 implies the following matrix relation

$$\left[C^{\,j}_{k,i}\right]_{n\times n}\left[D^i_{k,j}\right]_{n\times n} = I_n, \tag{37}$$

where I_n is the identity matrix of order n. That is,

$$\left[D^i_{k,j}\right]^{-1}_{n\times n} = \left[C^{\,j}_{k,i}\right]_{n\times n},$$

which implies

$$\left[C^{\,j}_{k,i}\right]_{n\times n}\left[D^i_{h,j}\right]_{n\times n} = \left[L^j_{k,h,i}\right]_{n\times n}. \tag{38}$$

The matrix equation (38) is equivalent to equation (10).

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