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Resolution of nonlinear convection - diffusion - reaction equations of Cauchy's kind by the Laplace SBA method

Youssouf Paré[1,*], Youssouf Minoungou[1], Abdoul Wassiha Nébié[1]

[1] *Département de Mathématiques, UFR SEA, Université Joseph Ki-Zerbo, Ouagadougou, Burkina Faso*

Abstract. In this paper, we proposed the solution of few non linear partial differential equations modelling diffusion, convection and reaction problems Cauchy kind. The Laplace SBA method based on combination of Laplace's transform, Adomian Decomposition Method(ADM), Picard principle and successive approximations is used for solving these equations.

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1. Introduction

Many problems are governed by partial differential equations, or by systems of partial differential equations. It is generally difficult to find exact solutions of these problems. In this article, we recall the definition, some results and determinate some models convection - diffusion - reaction non-linear equations with Cauchy condition by the Laplace - SBA method. The Laplace -SBA [6, 10] method is a numerical method based on the combination of the Laplace transform, the successive approximation method, the adomian decomposition method [1, 2, 7] and the Picard fixed point principle, the Laplace - SBA method avoids the difficulties associated with calculating Adomian polynomials unlike the Laplace - Adomian algorithm[8, 9], this method allows a rapid convergence towards the solution, if it exists.

2. Preliminary and definitions

In this section, we recall some definitions and theorems

*Corresponding author.
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Email addresses: youssoufpare@gmail.com (Y. Paré), m.youl@yahoo.fr (Y. Minoungou), nebwass@yahoo.fr (A. W. Nébié)

Definition 1. *Let $u : \mathbb{R}^+ \mapsto \mathbb{C}$ be a continuous function. The function $\mathcal{L}(u)$ defined by*

$$\mathcal{L}(u) = \int_0^\infty u(t)e^{-st}dt.$$

This theorem is a result that proves the existence and uniqueness of the solution of the functional equation $Au = f$

Theorem 1. (*Minty-Browder*)*[5, 6]*
Let E be a reflexive Banach space. Let $A : E \to E'$ an application (non-linear) continues such that :

$$\langle Au - Av; u - v\rangle > 0 \qquad \forall u, v \in E, \qquad u \neq v$$

$$\lim_{\|v\| \to \infty} \frac{\langle Av; v\rangle}{\|v\|} = \infty$$

So for everything $f \in E'$, there is $u \in E$ a unique solution to the equation $Au = f$

3. The numerical LAPLACE - SBA method

In this section, we describe how to use the Laplace - SBA algorithm to solve the equations.

Let us consider the following functional equation :

$$Au = f \tag{1}$$

Where $A : H \to H$, is an operator not necessarly linear and H is a Hilbert space adequatly chosen given the operator A.

Let :

$$A = L + R + N \tag{2}$$

Where L is an inversible operator in the Adomian sense, R the linear remainder and N a nonlinear operator. The equation (1) therefore becomes :

$$Lu + Ru + Nu = f \tag{3}$$

Applying the Laplace transform $\mathcal{L}_t$, on both sides of equations (3) , we have :

$$\mathcal{L}[Lu] + \mathcal{L}[Ru] + \mathcal{L}[Nu] = \mathcal{L}(f) \tag{4}$$

We suppose : $L(.) = \dfrac{\partial^m}{\partial t^m}(.)$ By using the differentiation property of Laplace transform, we then obtain

$$\mathcal{L}_t\left[\frac{\partial^m u}{\partial t^m}\right] = s^m \mathcal{L}_t(u) - \sum_{k=0}^{m-1} s^k u^{m-k-1}(0, x) \tag{5}$$

The equation (4) become

$$s^m \mathcal{L}_t(u) = -\mathcal{L}_t(Ru) - \mathcal{L}_t(Nu) + \sum_{j=0}^{m-1} s^j u^{m-j-1}(0,x) + \mathcal{L}_t(f) \tag{6}$$

Using the successive approximations, we have

$$\Rightarrow s^m \mathcal{L}_t\left(u^k\right) = -\mathcal{L}_t\left(Ru^k\right) - \mathcal{L}_t\left(Nu^{k-1}\right) + \sum_{j=0}^{m-1} s^j u^{m-j-1}(0,x) + \mathcal{L}_t(f) \tag{7}$$

Substituting $u^k(x,t)$ par $\sum_{n\geq 0} u_n^k(x,t)$ et $g_{k-1} = Nu^{k-1}$, we obtain

$$s^m \sum_{n\geq 0} \mathcal{L}_t\left(u_n^k\right) = -\sum_{n\geq 0} \mathcal{L}_t\left(Ru_n^k\right) - \mathcal{L}_t(g_{k-1}) + \sum_{j=0}^{m-1} s^j u^{m-j-1}(0,x) + \mathcal{L}_t(f) \tag{8}$$

We deduce the following Laplace - SBA algorithm[9]:

$$\begin{cases} \mathcal{L}_t(u_0^k) & = & -\dfrac{1}{s^m}\mathcal{L}_t(g_{k-1}) + \dfrac{1}{s^m}\displaystyle\sum_{j=0}^{m-1} s^j u^{m-j-1}(0,x) + \dfrac{1}{s^m}\mathcal{L}_t(f), \quad k \geq 1 \\ \mathcal{L}_t(u_{n+1}^k) & = & -\dfrac{1}{s^m}\mathcal{L}_t\left(Ru_n^k\right) \qquad , \; n\geq 0, \; k \geq 1 \end{cases} \tag{9}$$

Applying the inverse transform $\mathcal{L}_t^{-1}$ Laplace, we have:

$$\begin{cases} u_0^k & = & \mathcal{L}_t^{-1}\left[\dfrac{1}{s^m}\displaystyle\sum_{j=0}^{m-1} p^j u^{m-j-1}(0,x)\right] + \mathcal{L}_t^{-1}\left[\dfrac{1}{s^m}\mathcal{L}_t(f)\right] - \mathcal{L}_t^{-1}\left(\dfrac{1}{s^m}\mathcal{L}_t(g_{k-1})\right) \qquad , \; k \geq 1 \\ u_{n+1}^k & = & -\mathcal{L}_t^{-1}\left[\dfrac{1}{s^m}\mathcal{L}_t\left(Ru_n^k\right)\right] \qquad , \; n \geq 0, \; k \geq 1 \end{cases} \tag{10}$$

The Picard principle is then applied to the equation (6) : let u^0 be such that $N\left(u^0\right) = 0$,

for $k = 1$ we get :

$$\begin{cases} u_0^1 & = & \mathcal{L}_t^{-1}\left[\dfrac{1}{s^m}\displaystyle\sum_{j=0}^{m-1} p^j u^{m-j-1}(0,x)\right] + \mathcal{L}_t^{-1}\left[\dfrac{1}{s^m}\mathcal{L}_t(f)\right] - \mathcal{L}_t^{-1}\left(\dfrac{1}{s^m}\mathcal{L}_t(g_0)\right) \\ u_{n+1}^1 & = & -\mathcal{L}_t^{-1}\left[\dfrac{1}{s^m}\mathcal{L}_t\left(Ru_n^1\right)\right] \end{cases} \tag{11}$$

If the series $\left(\sum_{n=0}^{+\infty} u_n^1\right)$ converges, then $u^1 = \sum_{n=0}^{+\infty} u_n^1$. For $k = 2$, we get :

$$\begin{cases} u_0^2 &= \mathcal{L}_t^{-1}\left[\frac{1}{s^m}\sum_{j=0}^{m-1} s^j u^{m-j-1}(0,x)\right] + \mathcal{L}_t^{-1}\left[\frac{1}{s^m}\mathcal{L}_t(f)\right] - \mathcal{L}_t^{-1}\left(\frac{1}{s^m}\mathcal{L}_t(g_1)\right) \\ u_{n+1}^2 &= -\mathcal{L}_t^{-1}\left[\frac{1}{s^m}\mathcal{L}_t\left(Ru_n^1\right)\right] \end{cases} \tag{12}$$

If the series $\left(\sum_{n=0}^{+\infty} u_n^2\right)$ converges, then $u^2 = \sum_{n=0}^{+\infty} u_n^2$. This process is repeated to k. If the series $\left(\sum_{n=0}^{+\infty} u_n^k\right)$ converges, then $u^2 = \sum_{n=0}^{+\infty} u_n^k$, therefore $u = \lim_{k\to+\infty} u^k$ is the solution of the equation (1). At each stage $k \geq 1$, we check that : $N\left(u^k\right) = 0$.

4. Application of Laplace SBA

In this section, we propose the solution of some equations of convection - reaction - diffusion non-linear[4][3] by the Laplace - SBA method, we show that the choice of operator L is very decisive for the convergence speed of the algorithm.

Example 1. *: Let us consider the following nonlinear diffusion equation*

$$(E) : \begin{cases} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - u^2\frac{\partial u}{\partial x} + u^3 \\ u(0,x) &= e^x \end{cases}$$

with $u = u(t,x),\ (t,x) \in [0,+\infty[\times \mathbb{R}$.

Applying the Laplace SBA method, we have

$$\begin{cases} u_0^k(x,t) &= \mathcal{L}_t^{-1}\left[\frac{1}{s}u(0,x) + \frac{1}{s}\mathcal{L}_t\left(g_{k-1}\right)\right] \\ u_{n+1}^k(x,t) &= \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_n^k)\right)\right],\ n \geq 0 \end{cases}$$

Where $Lu = \frac{\partial u}{\partial t}$, $Ru = \frac{\partial^2 u}{\partial x^2}$

and $g_{k-1} = Nu^{k-1} = -\left(u^{k-1}\right)^2\frac{\partial u^{k-1}}{\partial x} + \left(u^{k-1}\right)^3$

Determinate $u_n^k(t,x)$, **for** $n \geq 0$

We take $u^0 = 0$

$u_0^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{1}{s}e^x\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(g_0\right)\right) \Rightarrow u_0^1(t,x) = e^x$

$u_1^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_0^1)\right)\right]$

$\Rightarrow u_1^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{1}{s^2}e^x\right) = te^x$

$u_2^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_1^1)\right)\right]$
$\Rightarrow u_2^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{1}{s^3}e^x\right) = \frac{t^2}{2!}e^x$

$u_3^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_2^1)\right)\right]$
$\Rightarrow u_3^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{1}{s^4}e^x\right) = \frac{t^3}{3!}e^x$

$u_4^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_3^1)\right)\right]$
$\Rightarrow u_4^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{1}{s^5}e^x\right) = \frac{t^4}{4!}e^x$

$u_5^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_4^1)\right)\right]$
$\Rightarrow u_5^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{1}{s^6}e^x\right) = \frac{t^5}{5!}e^x$

In recursive way, we deduce

$$u_n^1(t,x) = \frac{1}{n!}t^n e^x$$

Then

$$u^1(t,x) = \sum_{n\geq 0} u_n^1(t,x) = \sum_{n\geq 0}\frac{1}{n!}t^n e^x$$

$\Rightarrow$

$$u^1(t,x) = e^{x+t}$$

So

$$\begin{aligned} Nu^1 &= -\left(u^1\right)^2 \frac{\partial\left(u^1\right)}{\partial x} + \left(u^1\right)^3 \\ Nu^1 &= -\left(e^{x+t}\right)^3 + \left(e^{x+t}\right)^3 = 0 \end{aligned}$$

Step by step, we obtain

$$u^1(t,x) = u^2(t,x) = ... = u^k(t,x) = e^{x+t}$$

The exact solution of model is

$$u(t,x) = \lim_{k\to+\infty} u^k(t,x) = e^{x+t}$$

Conclusion : The exact solution of model is

$$u(t,x) = e^{x+t}$$

Example 2. *Let us consider the following nonlinear diffusion equation*

$$(E):\begin{cases} \dfrac{\partial u}{\partial t} & = & \varepsilon\dfrac{\partial^2 u}{\partial x^2} + \lambda\dfrac{\partial u}{\partial x} + u^3 + u^2\dfrac{\partial^2 u}{\partial x^2} \\ u(0,x) & = & \sin x\ ,\ \ (t,x)\in\Omega = [0,+\infty[\times\mathbb{R} \end{cases} \tag{13}$$

with $u = u(t,x)$, $(t,x)\in[0,+\infty[\times\mathbb{R}$, $\varepsilon > 0$ *et* $\lambda > 0$.
Applying the Laplace - SBA method, we get

$$\begin{cases} u_0^k(t,x) & = & \mathcal{L}_t^{-1}\left[\dfrac{1}{s}u(0,x) + \dfrac{1}{s}\mathcal{L}_t\left(g_{k-1}\right)\right] \\ u_{n+1}^k(t,x) & = & \mathcal{L}_t^{-1}\left[\dfrac{1}{s}\mathcal{L}_t\left(R(u_n^k)\right)\right],\ n\geq 0 \end{cases} \tag{14}$$

where $Lu = \dfrac{\partial u}{\partial t}$, $Ru = \varepsilon\dfrac{\partial^2 u}{\partial x^2} + \lambda\dfrac{\partial u}{\partial x^2}$
and $g_{k-1} = Nu^{k-1} = \left(u^{k-1}\right)^3 + \left(u^{k-1}\right)^2\dfrac{\partial^2 u^{k-1}}{\partial x^2}$.
Let us calculate $u_n^k(t,x)$, **for** $n\geq 0$ **:**

we take $u^0 = 0$
$u_0^1(t,x) = \mathcal{L}_t^{-1}\left(\dfrac{1}{s}\sin x\right) + \mathcal{L}_t^{-1}\left(\dfrac{1}{s}\mathcal{L}_t\left(g_0\right)\right)$
$\Rightarrow u_0^1(t,x) = \sin x$

$u_1^1(t,x) = \mathcal{L}_t^{-1}\left[\dfrac{1}{s}\mathcal{L}_t\left(R(u_0^1)\right)\right]$
$\Rightarrow u_1^1(t,x) = \mathcal{L}_t^{-1}\left(\dfrac{\lambda\cos x - \varepsilon\sin x}{s^2}\right)$
$\Rightarrow u_1^1(t,x) = \lambda t\cos x - \varepsilon t\sin x$

$u_2^1(t,x) = \mathcal{L}_t^{-1}\left[\dfrac{1}{s}\mathcal{L}_t\left(R(u_1^1)\right)\right]$
$\Rightarrow u_2^1(t,x) = \mathcal{L}_t^{-1}\left(\dfrac{-\lambda\left(t\varepsilon\cos x + t\lambda\sin x\right) - \varepsilon\left(t\lambda\cos x - t\varepsilon\sin x\right)}{s^3}\right)$
$\Rightarrow u_2^1(t,x) = -\dfrac{1}{2}\left(\sin x\right)t^2\lambda^2 - \left(\cos x\right)t^2\lambda\varepsilon + \dfrac{1}{2}\left(\sin x\right)t^2\varepsilon^2$
$u_3^1(t,x) = \mathcal{L}_t^{-1}\left[\dfrac{1}{s}\mathcal{L}_t\left(R(u_2^1)\right)\right]$

$$\Rightarrow u_3^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{-(\cos x)\lambda^3 + 3(\sin x)t^2\lambda^2\varepsilon + 3(\cos x)t^2\lambda\varepsilon^2 - (\sin x)\varepsilon^3}{s^4}\right)$$

$$\Rightarrow u_3^1(t,x) = -\frac{1}{3!}(\cos x)t^3\lambda^3 + \frac{1}{2!}(\sin x)t^3\lambda^2\varepsilon + \frac{1}{2!}(\cos x)t^3\lambda\varepsilon^2 - \frac{1}{3!}(\sin x)t^3\varepsilon^3$$

$$u_4^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_3^1)\right)\right]$$

$$\Rightarrow u_4^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{(\sin x)\lambda^4 + 4(\cos x)\lambda^3\varepsilon - 6(\sin x)\lambda^2\varepsilon^2 - 3(\cos x)\lambda\varepsilon^3 - (\cos x)\lambda\varepsilon^3 + (\sin x)\varepsilon^4}{s^5}\right)$$

$$\Rightarrow \left\{\begin{array}{ll} u_4^1(t,x) & = \frac{1}{24}(\sin x)t^4\lambda^4 + \frac{1}{6}(\cos x)t^4\lambda^3\varepsilon - (\sin x)t^4\lambda^2\varepsilon^2 \\ -\frac{1}{6}(\cos x)t^4\lambda\varepsilon^3 + \frac{1}{24}(\sin x)t^4\varepsilon^4 & \end{array}\right.$$

$$u_5^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_4^1)\right)\right]$$

$$\Rightarrow \left\{\begin{array}{ll} u_5^1(t,x) = & \mathcal{L}_t^{-1}\left(\dfrac{\frac{1}{24}(\cos x)\lambda^5 - \frac{5}{24}(\sin x)\lambda^4\varepsilon - \frac{7}{6}(\cos x)\lambda^3\varepsilon^2}{s^6}\right) \\ & +\mathcal{L}_t^{-1}\left(\dfrac{\frac{7}{6}(\sin x)\lambda^2\varepsilon^3 + \frac{5}{24}(\cos x)\lambda\varepsilon^4 - \frac{1}{24}(\sin x)\varepsilon^5}{s^6}\right) \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{ll} u_5^1(t,x) = & \frac{1}{24}(\cos x)t^5\lambda^5 - \frac{5}{24}(\sin x)t^5\lambda^4\varepsilon - \frac{7}{6}(\cos x)t^5\lambda^3\varepsilon^2 + \frac{7}{6}(\sin x)t^5\lambda^2\varepsilon^3 \\ & +\frac{5}{24}(\cos x)t^5\lambda\varepsilon^4 - \frac{1}{24}(\sin x)t^5\varepsilon^5 \end{array}\right.$$

Step by step, we then deduce

$$u^1(t,x) = \left\{\begin{array}{c} \sin x\left(1 - \varepsilon t + \frac{1}{2!}(\varepsilon t)^2 - \frac{1}{3!}(\varepsilon t)^3 + \frac{1}{4!}(\varepsilon t)^4 - \frac{1}{5!}(\varepsilon t)^5 + \frac{1}{6!}(\varepsilon t)^6 + ...\right) \\ +\lambda t\cos x\left(1 - \varepsilon t + \frac{1}{2!}(\varepsilon t)^2 - \frac{1}{3!}(\varepsilon t)^3 + \frac{1}{4!}(\varepsilon t)^4 - \frac{1}{5!}(\varepsilon t)^5 + \frac{1}{6!}(\varepsilon t)^6 + ...\right) \\ -\frac{1}{2}(\lambda t)^2\sin x\left(1 - \varepsilon t + \frac{1}{2!}(\varepsilon t)^2 - \frac{1}{3!}(\varepsilon t)^3 + \frac{1}{4!}(\varepsilon t)^4 - \frac{1}{5!}(\varepsilon t)^5 + \frac{1}{6!}(\varepsilon t)^6 + ...\right) \\ -\frac{1}{3!}(\lambda t)^3\cos x\left(1 - \varepsilon t + \frac{1}{2!}(\varepsilon t)^2 - \frac{1}{3!}(\varepsilon t)^3 + \frac{1}{4!}(\varepsilon t)^4 - \frac{1}{5!}(\varepsilon t)^5 + \frac{1}{6!}(\varepsilon t)^6 ...\right) \\ +\frac{1}{4!}(\lambda t)^4\sin x\left(1 - \varepsilon t + \frac{1}{2!}(\varepsilon t)^2 - \frac{1}{3!}(\varepsilon t)^3 + \frac{1}{4!}(\varepsilon t)^4 - \frac{1}{5!}(\varepsilon t)^5 + \frac{1}{6!}(\varepsilon t)^6 ...\right) \\ +\frac{1}{5!}(\lambda t)^5\cos x\left(1 - \varepsilon t + \frac{1}{2!}(\varepsilon t)^2 - \frac{1}{3!}(\varepsilon t)^3 + \frac{1}{4!}(\varepsilon t)^4 - \frac{1}{5!}(\varepsilon t)^5 + \frac{1}{6!}(\varepsilon t)^6 + ...\right) \\ +... \end{array}\right.$$

We obtain

$$u^1(t,x) = \left\{\begin{array}{c} \sin x\left(1 - \frac{1}{2!}(\lambda t)^2 + \frac{1}{4!}(\lambda t)^4 + ...\right) \times \left(1 - \varepsilon t + \frac{1}{2!}(\varepsilon t)^2 - \frac{1}{3!}(\varepsilon t)^3 + ...\right) \\ +\cos x\left(\lambda t - \frac{1}{3!}(\lambda t)^3 + \frac{1}{5!}(\lambda t)^5 + ...\right) \times \left(1 - \varepsilon t + \frac{1}{2!}(\varepsilon t)^2 - \frac{1}{3!}(\varepsilon t)^3 + ...\right) \end{array}\right.$$

$$u^1(t,x) = \sin x \cos(\lambda t) e^{-\varepsilon t} + \cos x \sin(\lambda t) e^{-\varepsilon t} = e^{-\varepsilon t} \sin(x + \lambda t)$$

The solution to step $k = 1$ is

$$u^1(t,x) = e^{-\varepsilon t} \sin(x + \lambda t)$$

So, we have

$$\begin{aligned} Nu^1 &= \left(u^1\right)^3 + \left(u^1\right)^2 \frac{\partial^2 u^1}{\partial x^2} \\ Nu^1 &= \left(e^{-\varepsilon t} \sin(x + \lambda t)\right)^3 - \left(e^{-\varepsilon t} \sin(x + \lambda t)\right)^3 = 0 \end{aligned}$$

We deduce by recursive way :

$$u^1(t,x) = u^2(t,x) = ... = u^k(t,x) = e^{-\varepsilon t} \sin(x + \lambda t)$$

and we obtain the exact solution

$$u(t,x) = \lim_{k \to +\infty} u^k(t,x) = e^{-\varepsilon t} \sin(x + \lambda t)$$

Conclusion : The exact solution of model is

$$u(t,x) = e^{-\varepsilon t} \sin(x + \lambda t)$$

Example 3. *: Let us consider the following nonlinear diffusion equation*

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t} &=& \lambda \dfrac{\partial u}{\partial x} + \gamma u + u^n + u^{n-1} \dfrac{\partial^2 u}{\partial x^2};\ n \geq 1 \\ u(0,x) &=& \cos x\ ,\ \ (t,x) \in \Omega = [0, +\infty[\times \mathbb{R} \end{array} \right. \tag{15}$$

Applying the Laplace SBA method, we deduce

$$\left\{ \begin{array}{rcl} u_0^k(t,x) &=& \mathcal{L}_t^{-1} \left[\dfrac{1}{s} u(x,0) + \dfrac{1}{s} \mathcal{L}_t\left(g_{k-1}\right) \right] \\ u_{n+1}^k(t,x) &=& \mathcal{L}_t^{-1} \left[\dfrac{1}{s} \mathcal{L}_t\left(R(u_n^k)\right) \right],\ n \geq 0 \end{array} \right. \tag{16}$$

Where $Lu = \dfrac{\partial u}{\partial t}$, $Ru = \lambda \dfrac{\partial u}{\partial x} + \gamma u$ and $g_{k-1} = Nu^{k-1} = \left(u^{k-1}\right)^n + \left(u^{k-1}\right)^{n-1} \dfrac{\partial^2 u^{k-1}}{\partial x^2}$

Let us calculate the terms $u_n^k(t,x)$ **for** $n \geq 0$.

We take $u^0 = 0$ $u_0^1(t,x) = \mathcal{L}_t^{-1}\left(\dfrac{1}{s} \cos x\right) + \mathcal{L}_t^{-1}\left(\dfrac{1}{s} \mathcal{L}_t\left(g_0\right)\right)$
$\Rightarrow u_0^1(t,x) = \cos x$

$$u_1^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_0^1)\right)\right]$$
$$\Rightarrow u_1^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{\gamma\cos x - \lambda\sin x}{s^2}\right)$$
$$\Rightarrow u_1^1(x,t) = \gamma t\cos x - \lambda t\sin x$$

$$u_2^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_1^1)\right)\right]$$
$$\Rightarrow u_2^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{-(\cos x)\lambda^2 - 2(\sin x)\lambda\gamma + (\cos x)\gamma^2}{s^3}\right)$$
$$\Rightarrow u_2^1(t,x) = -\frac{(\lambda t)^2}{2!}(\cos x) - \lambda\gamma t^2(\sin x) + \frac{(\gamma t)^2}{2!}(\cos x)$$

$$u_3^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_2^1)\right)\right]$$
$$\Rightarrow u_3^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{(\sin x)\lambda^3 - 3(\cos x)\lambda^2\gamma - 3(\sin x)\lambda\gamma^2 + (\cos x)\gamma^3}{s^3}\right)$$
$$\Rightarrow u_3^1(t,x) = \frac{1}{3!}(\sin x)t^3\lambda^3 - \frac{1}{2!}(\cos x)t^3\lambda^2\gamma - \frac{1}{2!}(\sin x)t^3\lambda\gamma^2 + \frac{1}{3!}(\cos x)t^3\gamma^3$$

$$u_4^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_3^1)\right)\right]$$
$$\Rightarrow u_4^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{(\cos x)\lambda^4 + 4(\sin x)t^3\lambda^3\gamma - 6(\cos x)\lambda^2\gamma^2 - 4(\sin x)t^3\lambda\gamma^3 + (\cos x)\gamma^4}{s^3}\right)$$
$$\Rightarrow u_4^1(t,x) = \frac{1}{4!}(\cos x)t^4\lambda^4 + \frac{1}{3!}(\sin x)t^4\lambda^3\gamma - \frac{1}{4}(\cos x)t^4\lambda^2\gamma^2 - \frac{1}{3!}(\sin x)t^4\lambda\gamma^3 + \frac{1}{4!}(\cos x)t^4\gamma^4$$
$$u_5^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_4^1)\right)\right]$$

$$\Rightarrow \left\{\begin{array}{ll} u_5^1(t,x) = & \mathcal{L}_t^{-1}\left(\frac{-(\sin x)\lambda^5 + 5(\cos x)\lambda^4\gamma + 10(\sin x)\lambda^3\gamma^2 - 10(\cos x)\lambda^2\gamma^3}{s^3}\right) \\ & +\mathcal{L}_t^{-1}\left(\frac{-5(\sin x)t^4\lambda\gamma^4 + (\cos x)\gamma^5}{s^3}\right) \end{array}\right.$$
$$\Rightarrow \left\{\begin{array}{ll} u_5^1(t,x) = & -\frac{1}{5!}(\sin x)t^5\lambda^5 + \frac{1}{4!}(\cos x)t^4\lambda^4\gamma + \frac{2}{4!}(\sin x)t^5\lambda^3\gamma^2 - \frac{2}{4!}(\cos x)t^5\lambda^2\gamma^3 \\ & -\frac{1}{4!}(\sin x)t^5\lambda\gamma^4 + \frac{1}{5!}(\cos x)t^5\gamma^5 \end{array}\right.$$

we deduce

$$\begin{cases} u_0^1(t,x) = & \cos x \\ u_1^1(t,x) = & \gamma t\cos x - \lambda t\sin x \\ u_2^1(t,x) = & -\frac{(\lambda t)^2}{2!}(\cos x) - \lambda\gamma t^2(\sin x) + \frac{(\gamma t)^2}{2!}(\cos x) \\ u_3^1(t,x) = & \frac{1}{3!}(\sin x)t^3\lambda^3 - \frac{1}{2!}(\cos x)t^3\lambda^2\gamma - \frac{1}{2!}(\sin x)t^3\lambda\gamma^2 + \frac{1}{3!}(\cos x)t^3\gamma^3 \\ u_4^1(t,x) = & \frac{1}{4!}(\cos x)t^4\lambda^4 + \frac{1}{3!}(\sin x)t^4\lambda^3\gamma - \frac{1}{4}(\cos x)t^4\lambda^2\gamma^2 - \frac{1}{3!}(\sin x)t^4\lambda\gamma^3 + \frac{1}{4!}(\cos x)t^4\gamma^4 \\ u_5^1(t,x) = & \begin{array}{l} -\frac{1}{5!}(\sin x)t^5\lambda^5 + \frac{1}{4!}(\cos x)t^4\lambda^4\gamma + \frac{2}{4!}(\sin x)t^5\lambda^3\gamma^2 - \frac{2}{4!}(\cos x)t^5\lambda^2\gamma^3 - \frac{1}{4!}(\sin x)t^5\lambda\gamma^4 \\ +\frac{1}{5!}(\cos x)t^5\gamma^5 \end{array} \end{cases}$$

By recursive way, we have

$$\begin{cases} u^1(t,x) = & \cos x\left(1+\gamma t+\frac{1}{2}t^2\gamma^2+\frac{1}{3!}t^3\gamma^3+...\right) - \frac{(\lambda t)^2}{2!}(\cos x)\left(1+\gamma t+\frac{1}{2}t^2\gamma^2+\frac{1}{3!}t^3\gamma^3+...\right) \\ & +\frac{(\lambda t)^4}{4!}(\cos x)\left(1+\gamma t+\frac{1}{2}t^2\gamma^2+\frac{1}{3!}t^3\gamma^3+...\right)+... \\ & -\lambda t\sin x\left(1+\gamma t+\frac{1}{2}t^2\gamma^2+\frac{1}{3!}t^3\gamma^3+...\right)+\frac{(\lambda t)^3}{3!}\sin x\left(1+\gamma t+\frac{1}{2}t^2\gamma^2+\frac{1}{3!}t^3\gamma^3+...\right) \\ & -\frac{(\lambda t)^5}{5!}\sin x\left(1+\gamma t+\frac{1}{2}t^2\gamma^2+\frac{1}{3!}t^3\gamma^3+...\right)+... \end{cases}$$

$$\begin{cases} u^1(t,x) \simeq & \cos x\left(1-\frac{(\lambda t)^2}{2!}+\frac{(\lambda t)^4}{4!}+...\right)\left(1+\gamma t+\frac{1}{2}t^2\gamma^2+\frac{1}{3!}t^3\gamma^3+...\right) \\ & -\sin x\left(\lambda t-\frac{(\lambda t)^3}{3!}+\frac{(\lambda t)^5}{5!}+...\right)\left(1+\gamma t+\frac{1}{2}t^2\gamma^2+\frac{1}{3!}t^3\gamma^3+...\right) \end{cases}$$

We obtain

$$u^1(t,x) = e^{\gamma t}\cos x\cos(\lambda t) - e^{\gamma t}\sin x\sin(\lambda t) = e^{\gamma t}\cos(x+\lambda t)$$

So, we have

$$\begin{aligned} Nu^1 &= \left(u^1\right)^n + \left(u^1\right)^{n-1}\frac{\partial^2 u^1}{\partial x^2} \\ Nu^1 &= \left(e^{\gamma t}\cos(x+\lambda t)\right)^n - \left(e^{\gamma t}\cos(x+\lambda t)\right)^n e^{\gamma t}\cos(x+\lambda t) = 0 \end{aligned}$$

By recursive way, we have

$$u^1(t,x) = u^2(t,x) = ... = u^k(t,x) = e^{\gamma t}\cos(x+\lambda t)$$

Conclusion : The exact solution of the model is

$$u(t,x) = \lim_{k\to+\infty} u^k(t,x) = e^{\gamma t}\cos(x+\lambda t)$$

Example 4. *Let us consider the following nonlinear diffusion equation*

$$(E):\begin{cases} \dfrac{\partial u}{\partial t} &= \lambda\dfrac{\partial u}{\partial x}+\gamma u+u^2-\left(\dfrac{\partial^2 u}{\partial x^2}\right)^2\ ;\ n\geq 1 \\ u(0,x) &= \sin x\ ,\ \ (t,x)\in\Omega=[0,+\infty[\times\mathbb{R}\end{cases} \tag{17}$$

Applying the Laplace SBA method, we obtain

$$\begin{cases} u_0^k(t,x) &= \mathcal{L}_t^{-1}\left[\dfrac{1}{s}u(x,0)+\dfrac{1}{p}\mathcal{L}_t\left(g_{k-1}\right)\right] \\ u_{n+1}^k(t,x) &= \mathcal{L}_t^{-1}\left[\dfrac{1}{s}\mathcal{L}_t\left(R(u_n^k)\right)\right],\ n\geq 0\end{cases} \tag{18}$$

Where $Lu=\dfrac{\partial u}{\partial t}$, $Ru=\lambda\dfrac{\partial u}{\partial x}+\gamma u$ and $g_{k-1}=Nu^{k-1}=\left(u^{k-1}\right)^2-\left(\dfrac{\partial^2 u^{k-1}}{\partial x^2}\right)^2$

Let us determinate $u_n^k(t,x)$ **for** $n\geq 0$

We take $u_0^1=0$ $u_0^1(t,x)=\mathcal{L}_t^{-1}\left(\dfrac{1}{s}\sin x\right)+\mathcal{L}_t^{-1}\left(\dfrac{1}{s}\mathcal{L}_t\left(g_0\right)\right)$

$\Rightarrow u_0^1(t,x)=\sin x$ $u_1^1(t,x)=\mathcal{L}_t^{-1}\left[\dfrac{1}{s}\mathcal{L}_t\left(R(u_0^1)\right)\right]$

$$\Rightarrow u_1^1(t,x)=\mathcal{L}_t^{-1}\left(\frac{\lambda\cos x+\gamma\sin x}{s^2}\right)$$

$$\Rightarrow u_1^1(t,x)=\lambda t\cos x+\gamma t\sin x$$

$$u_2^1(t,x)=\mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_1^1)\right)\right]$$

$$\Rightarrow u_2^1(t,x)=-\left(\sin x\right)\frac{(\lambda t)^2}{2!}+\left(\cos x\right)\lambda\gamma t^2+\left(\sin x\right)\frac{(\gamma t)^2}{2!}$$

$$u_3^1(t,x)=\mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_2^1)\right)\right]$$

$$\Rightarrow u_3^1(t,x)=\mathcal{L}_t^{-1}\left(\frac{-\left(\cos x\right)\lambda^3-3\left(\sin x\right)\lambda^2\gamma+3\left(\cos x\right)\lambda\gamma^2+\left(\sin x\right)\gamma^3}{s^4}\right)$$

$$\Rightarrow u_3^1(t,x)=-\frac{1}{3!}\left(\cos x\right)t^3\lambda^3-\frac{1}{2}\left(\sin x\right)t^3\lambda^2\gamma+\frac{1}{2}\left(\cos x\right)t^3\lambda\gamma^2+\frac{1}{3!}\left(\sin x\right)t^3\gamma^3$$

$$u_4^1(t,x)=\mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_3^1)\right)\right]$$

$$\Rightarrow u_4^1(t,x)=\mathcal{L}_t^{-1}\left(\frac{\left(\sin x\right)\lambda^4-4\left(\cos x\right)\lambda^3\gamma-6\left(\sin x\right)\lambda^2\gamma^2+4\left(\cos x\right)\lambda\gamma^3+\left(\sin x\right)\gamma^4}{s^5}\right)$$

$$\Rightarrow u_4^1(t,x)=\frac{1}{4!}\left(\sin x\right)t^4\lambda^4-\frac{1}{3!}\left(\cos x\right)t^4\lambda^3\gamma-\frac{1}{4}\left(\sin x\right)t^4\lambda^2\gamma^2+\frac{1}{3!}\left(\cos x\right)t^4\lambda\gamma^3+\frac{1}{4!}\left(\sin x\right)t^4\gamma^4$$

$$u_5^1(t,x)=\mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_4^1)\right)\right]$$

$\Rightarrow u_5^1(t,x) = \mathcal{L}_t^{-1}\left(\dfrac{(\cos x)\,\lambda^5 + 5\,(\sin x)\,\lambda^4\gamma - 10\,(\cos x)\,\lambda^3\gamma^2 - 10\,(\sin x)\,\lambda^2\gamma^3 + 5\,(\cos x)\,\lambda\gamma^4 + (\sin x)\,\gamma^5}{s^6}\right)$

$\Rightarrow u_5^1(t,x) = \dfrac{1}{5!}(\cos x)\,t^5\lambda^5 + \dfrac{1}{4!}(\sin x)\,t^5\lambda^4\gamma - \dfrac{2}{4!}(\cos x)\,t^5\lambda^3\gamma^2 - \dfrac{2}{4!}(\sin x)\,t^5\lambda^2\gamma^3 + \dfrac{1}{4!}(\cos x)\,t^5\lambda\gamma^4 + \dfrac{1}{5!}(\sin x)\,t^5\gamma^5$

We deduce

$$\begin{cases} u_0^1(t,x) = & \sin x \\ u_1^1(t,x) = & \lambda t\cos x + \gamma t\sin x \\ u_2^1(t,x) = & -(\sin x)\dfrac{(\lambda t)^2}{2!} + (\cos x)\,\lambda\gamma t^2 + (\sin x)\dfrac{(\gamma t)^2}{2!} \\ u_3^1(t,x) = & -\dfrac{1}{3!}(\cos x)\,t^3\lambda^3 - \dfrac{1}{2}(\sin x)\,t^3\lambda^2\gamma + \dfrac{1}{2}(\cos x)\,t^3\lambda\gamma^2 + \dfrac{1}{3!}(\sin x)\,t^3\gamma^3 \\ u_4^1(t,x) = & \dfrac{1}{4!}(\sin x)\,t^4\lambda^4 - \dfrac{1}{3!}(\cos x)\,t^4\lambda^3\gamma - \dfrac{1}{4}(\sin x)\,t^4\lambda^2\gamma^2 + \dfrac{1}{3!}(\cos x)\,t^4\lambda\gamma^3 + \dfrac{1}{4!}(\sin x)\,t^4\gamma^4 \\ u_5^1(t,x) = & \begin{array}{l} \dfrac{1}{5!}(\cos x)\,t^5\lambda^5 + \dfrac{1}{4!}(\sin x)\,t^5\lambda^4\gamma - \dfrac{2}{4!}(\cos x)\,t^5\lambda^3\gamma^2 - \dfrac{2}{4!}(\sin x)\,t^5\lambda^2\gamma^3 + \dfrac{1}{4!}(\cos x)\,t^5\lambda\gamma^4 \\ +\dfrac{1}{5!}(\sin x)\,t^5\gamma^5 \end{array} \end{cases}$$

In recursive way, we obtain

$$\begin{cases} u^1(t,x) = & \sin x\left(1+\gamma t+\dfrac{1}{2}t^2\gamma^2+...\right) - \dfrac{(\lambda t)^2}{2!}(\sin x)\left(1+\gamma t+\dfrac{1}{2}t^2\gamma^2+...\right) \\ & +\dfrac{(\lambda t)^4}{4!}(\sin x)\left(1+\gamma t+\dfrac{1}{2}t^2\gamma^2+...\right)+... \\ & +\lambda t\cos x\left(1+\gamma t+\dfrac{1}{2}t^2\gamma^2+...\right) - \dfrac{(\lambda t)^3}{3!}(\cos x)\left(1+\gamma t+\dfrac{1}{2}t^2\gamma^2+...\right) \\ & +\dfrac{(\lambda t)^5}{5!}(\cos x)\left(1+\gamma t+\dfrac{1}{2}t^2\gamma^2+...\right)+... \end{cases}$$

$\Rightarrow$

$$\begin{cases} u^1(t,x) = & \sin x\left(1-\dfrac{(\lambda t)^2}{2!}+\dfrac{(\lambda t)^4}{4!}+...\right)\left(1+\gamma t+\dfrac{(\gamma t)^2}{2}+\dfrac{(\gamma t)^3}{3!}+...\right) \\ & +\cos x\left(\lambda t-\dfrac{(\lambda t)^3}{3!}+\dfrac{(\lambda t)^5}{5!}+...\right)\left(1+\gamma t+\dfrac{1}{2}t^2\gamma^2+\dfrac{1}{3!}t^3\gamma^3+...\right) \end{cases}$$

we deduce

$$u^1(t,x) = -e^{\gamma t}\sin x\cos(\lambda t) + e^{\gamma t}\cos x\sin(\lambda t) = e^{\gamma t}\sin(x+\lambda t)$$

So, we have

$$Nu^1 = u^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 = \left(e^{\gamma t}\sin(x+\lambda t)\right)^n - \left(e^{\gamma t}\sin(x+\lambda t)\right)^{n-1}e^{\gamma t}\sin(x+\lambda t) = 0$$

Step by step, we deduce

$$u^1(t,x) = u^2(t,x) = ... = u^k(t,x) = e^{\gamma t}\sin(x+\lambda t)$$

Conclusion : The exact solution of the model is

$$u(t,x) = \lim_{k\to+\infty} u^k(t,x) = e^{\gamma t}\sin(x+\lambda t)$$

Example 5. *Let us consider the following non linear convection - diffusion reaction equation*

$$(E): \begin{cases} \dfrac{\partial u}{\partial t} & = & \varepsilon\dfrac{\partial^2 u}{\partial x^2} + \lambda\dfrac{\partial u}{\partial x} + \gamma u + u^3 + \left(\dfrac{\partial^2 u}{\partial x^2}\right)^3 \\ u(0,x) & = & \sin x\ ,\ \ (t,x)\in\Omega = [0,+\infty[\times\mathbb{R} \end{cases} \tag{19}$$

Applying the Laplace SBA method, we have

$$\begin{cases} u_0^k(t,x) & = & \mathcal{L}_t^{-1}\left[\dfrac{1}{(s-\gamma)}u(x,0) + \dfrac{1}{(s-\gamma)}\mathcal{L}_t(g_{k-1})\right] \\ u_{n+1}^k(t,x) & = & \mathcal{L}_t^{-1}\left[\dfrac{1}{(s-\gamma)}\mathcal{L}_t\left(R(u_n^k)\right)\right],\ n\geq 0 \end{cases} \tag{20}$$

Where $Lu = \dfrac{\partial u}{\partial t} - \gamma u$, $Ru = \varepsilon\dfrac{\partial^2 u}{\partial x^2} + \lambda\dfrac{\partial u}{\partial x}$ and $Nu = u^3 + \left(\dfrac{\partial^2 u}{\partial x^2}\right)^3$

In this example, we take the linear operator L in this form to accelerate convergence

Let us determinate the terms $u_n^k(t,x)$ for $n\geq 0$

We take $u^0(t,x) = 0$

$$u_0^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{1}{(s-\gamma)}\sin x\right) + \mathcal{L}_t^{-1}\left(\frac{1}{(s-\gamma)}\mathcal{L}_t(g_0)\right)$$

$$\Rightarrow u_0^1(t,x) = e^{\gamma t}\sin x$$

$$u_1^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{(s-\gamma)}\mathcal{L}_t\left(R(u_0^1)\right)\right]$$

$$\Rightarrow \mathcal{L}_t\left(R(u_0^1)\right) = \frac{\lambda\cos x - \varepsilon\sin x}{(s-\gamma)}$$

$$\Rightarrow u_1^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{\lambda\cos x - \varepsilon\sin x}{(s-\gamma)^2}\right)$$

$$\Rightarrow u_1^1(t,x) = (\lambda t\cos x - \varepsilon t\sin x)\,e^{\gamma t}$$

$$u_2^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p}\mathcal{L}_t\left(R(u_1^1)\right)\right]$$

$$\Rightarrow \mathcal{L}_t\left(R(u_1^1)\right) = \frac{(\sin x)\,\lambda^2 - 2\,(\cos x)\,\lambda\varepsilon + (\sin x)\,\varepsilon^2}{(s-\gamma)^2}$$

$$\Rightarrow u_2^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{-(\sin x)\,\lambda^2 - 2\,(\cos x)\,\lambda\varepsilon + (\sin x)\,\varepsilon^2}{(s-\gamma)^3}\right)$$

$$\Rightarrow u_2^1(t,x) = -\frac{(\lambda t)^2}{2!}(\sin x) - t^2(\cos x)\,\lambda\varepsilon + \frac{(\varepsilon t)^2}{2!}(\sin x)$$

$$u_3^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p}\mathcal{L}_t\left(R(u_2^1)\right)\right]$$

$$\Rightarrow u_3^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{3\lambda\varepsilon^2\cos x - \varepsilon^3\sin x - \lambda^3\cos x + 3\lambda^2\varepsilon\sin x}{(s-\gamma)^4}\right)$$

$$\Rightarrow u_3^1(t,x) = \left(-\frac{1}{3!}t^3\lambda^3\cos x + \frac{1}{2!}t^3\lambda\varepsilon^2\cos x + \frac{1}{2!}t^3\lambda^2\varepsilon\sin x - \frac{1}{3!}t^3\varepsilon^3\sin x\right)e^{t\gamma}$$

$$u_4^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p}\mathcal{L}_t\left(R(u_3^1)\right)\right]$$

$$\Rightarrow u_4^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{\lambda^4\sin x + \varepsilon^4\sin x - 6\lambda^2\varepsilon^2\sin x - 4\lambda\varepsilon^3\cos x + 4\lambda^3\varepsilon\cos x}{(s-\gamma)^5}\right)$$

$$\Rightarrow u_4^1(t,x) = \frac{1}{4!}t^4\lambda^4e^{t\gamma}\sin x + \frac{1}{4!}t^4\varepsilon^4e^{t\gamma}\sin x - \frac{t^4}{4}\lambda^2\varepsilon^2e^{t\gamma}\sin x - \frac{1}{3!}t^4\lambda\varepsilon^3e^{t\gamma}\cos x + \frac{1}{3!}t^4\lambda^3\varepsilon e^{t\gamma}\cos x$$

$$u_5^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p}\mathcal{L}_t\left(R(u_4^1)\right)\right]$$

$$\mathcal{L}_t\left(R(u_4^1)\right) = \frac{\lambda^5\cos x - \varepsilon^5e^{t\gamma}\sin x - 10\lambda^3\varepsilon^2\cos x + 10\lambda^2\varepsilon^3\sin x + 5\lambda\varepsilon^4\cos x - 5\lambda^4\varepsilon\sin x}{(s-\gamma)^5}$$

$$\Rightarrow u_5^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{\lambda^5\cos x - \varepsilon^5\sin x - 10\lambda^3\varepsilon^2\cos x + 10\lambda^2\varepsilon^3\sin x + 5\lambda\varepsilon^4\cos x - 5\lambda^4\varepsilon\sin x}{(s-\gamma)^6}\right)$$

$$\Rightarrow \left\{\begin{array}{ll} u_5^1(t,x) = & \frac{1}{5!}t^5\lambda^5e^{t\gamma}\cos x - \frac{1}{5!}t^5\varepsilon^5e^{t\gamma}\sin x - \frac{10}{5!}t^5\lambda^3\varepsilon^2e^{t\gamma}\cos x + \frac{10}{5!}t^5\lambda^2\varepsilon^3e^{t\gamma}\sin x \\ & +\frac{1}{4!}t^5\lambda\varepsilon^4e^{t\gamma}\cos x - \frac{1}{4!}t^5\lambda^4\varepsilon e^{t\gamma}\sin x \end{array}\right.$$

we deduce

$$\begin{cases} u_0^1(t,x) = & e^{\gamma t}\sin\mathbf{x} \\ u_1^1(t,x) = & (\lambda t\cos x - \varepsilon t\sin x)\,e^{\gamma t} \\ u_2^1(t,x) = & -\frac{(\lambda t)^2}{2!}e^{\gamma t}(\sin x) - t^2\lambda\varepsilon e^{\gamma t}(\cos x) + \frac{(\varepsilon t)^2}{2!}e^{\gamma t}(\sin x) \\ u_3^1(t,x) = & -\frac{1}{3!}t^3\lambda^3 e^{\gamma t}\cos x + \frac{1}{2!}t^3\lambda\varepsilon^2 e^{\gamma t}\cos x + \frac{1}{2!}t^3\lambda^2\varepsilon e^{\gamma t}\sin x - \frac{1}{3!}t^3\varepsilon^3 e^{\gamma t}\sin x \\ u_4^1(t,x) = & \frac{1}{4!}t^4\lambda^4 e^{t\gamma}\sin x + \frac{1}{4!}t^4\varepsilon^4 e^{t\gamma}\sin x - \frac{t^4}{4}\lambda^2\varepsilon^2 e^{t\gamma}\sin x - \frac{1}{3!}t^4\lambda\varepsilon^3 e^{t\gamma}\cos x + \frac{1}{3!}t^4\lambda^3\varepsilon e^{t\gamma}\cos x \\ u_5^1(t,x) = & \frac{1}{5!}t^5\lambda^5 e^{t\gamma}\cos x - \frac{1}{5!}t^5\varepsilon^5 e^{t\gamma}\sin x - \frac{10}{5!}t^5\lambda^3\varepsilon^2 e^{t\gamma}\cos x + \frac{10}{5!}t^5\lambda^2\varepsilon^3 e^{t\gamma}\sin x + \frac{1}{4!}t^5\lambda\varepsilon^4 e^{t\gamma}\cos x \\ & -\frac{1}{4!}t^5\lambda^4\varepsilon e^{t\gamma}\sin x \end{cases}$$

Step by step, we deduce

$$\begin{cases} u^1(t,x) = & e^{\gamma t}\sin x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)-\frac{(\lambda t)^2}{2!}e^{\gamma t}\sin x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right) \\ & +\frac{(\lambda t)^4}{4!}e^{\gamma t}\cos x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)+... \\ & \lambda t e^{\gamma t}\cos x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)-\frac{(\lambda t)^3}{3!}e^{\gamma t}\cos x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right) \\ & +\frac{(\lambda t)^5}{5!}e^{\gamma t}\cos x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)+... \end{cases}$$

We obtain

$$\begin{cases} u^1(t,x) = & e^{\gamma t}\sin x\left(1-\frac{(\lambda t)^2}{2!}+\frac{(\lambda t)^4}{4!}+...\right)\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right) \\ & e^{\gamma t}\cos x\left(\lambda t-\frac{(\lambda t)^3}{3!}+\frac{(\lambda t)^5}{5!}+...\right)\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right) \end{cases}$$

$$u^1(t,x) = e^{\gamma t}\sin x\cos(\lambda t)\,e^{-\varepsilon t} + e^{\gamma t}\cos x\sin(\lambda t)\,e^{-\varepsilon t} = e^{(\gamma-\varepsilon)t}\sin(x+\lambda t)$$

So, we have

$$\begin{aligned} Nu^1 &= u^3 + \left(\frac{\partial^2 u}{\partial x^2}\right)^3 \\ Nu^1 &= \left(e^{(\gamma-\varepsilon)t}\sin(x+\lambda t)\right)^3 + \left(-e^{(\gamma-\varepsilon)t}\sin(x+\lambda t)\right)^3 = 0 \end{aligned}$$

In recursive way, we have

$$u^1(t,x) = u^2(t,x) = ... = u^k(t,x) = e^{(\gamma-\varepsilon)t}\sin(x+\lambda t)$$

The exact solution of the model is

$$u(t,x) = \lim_{k\to+\infty} u^k(t,x) = e^{(\gamma-\varepsilon)t}\sin(x+\lambda t)$$

Example 6.

Let us consider the following non linear convection diffusion reaction model

$$(E):\begin{cases} \dfrac{\partial u}{\partial t} &= \varepsilon\dfrac{\partial^2 u}{\partial x^2}+\lambda\dfrac{\partial u}{\partial x}+\gamma u+u^n+u^{n-1}\dfrac{\partial^2 u}{\partial x^2};\ n\geq 1 \\ u(0,x) &= \cos x\ ,\ \ (t,x)\in\Omega=[0,+\infty[\times\mathbb{R} \end{cases} \tag{21}$$

Where $u=u(t,x)$ with $x\in\mathbb{R}$ and $t\geq 0$

Applying the Laplace SBA method, we obtain

$$\begin{cases} u_0^k(t,x) &= \mathcal{L}_t^{-1}\left[\dfrac{1}{(s-\gamma)}u(0,x)+\dfrac{1}{(s-\gamma)}\mathcal{L}_t\left(g_{k-1}\right)\right] \\ u_{n+1}^k(t,x) &= \mathcal{L}_t^{-1}\left[\dfrac{1}{(s-\gamma)}\mathcal{L}_t\left(R(u_n^k)\right)\right],\ n\geq 0 \end{cases} \tag{22}$$

With $Lu=\dfrac{\partial u}{\partial t}-\gamma u$, $Ru=\varepsilon\dfrac{\partial^2 u}{\partial x^2}+\lambda\dfrac{\partial u}{\partial x}$ and $Nu=u^n-u^{n-1}\dfrac{\partial^2 u}{\partial x^2}$ In this example, we put the linear operator L in the form $\dfrac{\partial u}{\partial t}-\gamma u$ to accelerate convergence.

Let us determinate the terms $u_n^k(x,t)$ **for** $n\geq 0$

We take $u^0(t,x)=0$

$u_0^1(t,x)=\mathcal{L}_t^{-1}\left(\dfrac{1}{(s-\gamma)}\cos x\right)+\mathcal{L}_t^{-1}\left(\dfrac{1}{(s-\gamma)}\mathcal{L}_t\left(g_0\right)\right)$

$\Rightarrow u_0^1(t,x)=e^{\gamma t}\cos x$

$u_1^1(t,x)=\mathcal{L}_t^{-1}\left[\dfrac{1}{(s-\gamma)}\mathcal{L}_t\left(R(u_0^1)\right)\right]$

$\Rightarrow u_1^1(t,x)=\mathcal{L}_t^{-1}\left(\dfrac{-\varepsilon\cos x-\lambda\sin x}{(s-\gamma)^2}\right)$

$\Rightarrow u_1^1(t,x)=\left(-\varepsilon t\cos x-\lambda t\sin x\right)e^{\gamma t}$

$u_2^1(t,x)=\mathcal{L}_t^{-1}\left[\dfrac{1}{p}\mathcal{L}_t\left(R(u_1^1)\right)\right]$

$\Rightarrow \mathcal{L}_t\left(R(u_1^1)\right)=\dfrac{-(\cos x)\lambda^2+2(\sin x)\lambda\varepsilon+(\cos x)\varepsilon^2}{(s-\gamma)^2}$

$\Rightarrow u_2^1(t,x)=\mathcal{L}_t^{-1}\left(\dfrac{-(\cos x)\lambda^2+2(\sin x)\lambda\varepsilon+(\cos x)\varepsilon^2}{(s-\gamma)^3}\right)$

$$\Rightarrow u_2^1(t,x) = -\frac{(\lambda t)^2}{2!}e^{\gamma t}(\cos x) + t^2 e^{\gamma t}(\sin x)\lambda\varepsilon + \frac{(\varepsilon t)^2}{2!}e^{\gamma t}(\cos x)$$

$$u_3^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p}\mathcal{L}_t\left(R(u_2^1)\right)\right]$$
$$\Rightarrow \mathcal{L}_t\left(R(u_2^1)\right) = \frac{\lambda^3\sin x - \varepsilon^3\cos x + 3\lambda^2\varepsilon\cos x - 3\lambda\varepsilon^2\sin x}{(s-\gamma)^3}$$
$$\Rightarrow u_3^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{\lambda^3\sin x - \varepsilon^3\cos x + 3\lambda^2\varepsilon\cos x - 3\lambda\varepsilon^2\sin x}{(s-\gamma)^4}\right)$$

$$\Rightarrow u_3^1(t,x) = \frac{1}{3!}t^3\lambda^3 e^{t\gamma}\sin x - \frac{1}{3!}t^3\varepsilon^3 e^{t\gamma}\cos x + \frac{1}{2!}t^3\lambda^2\varepsilon e^{t\gamma}\cos x - \frac{1}{2!}t^3\lambda\varepsilon^2 e^{t\gamma}\sin x$$
$$u_4^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{s}\mathcal{L}_t\left(R(u_3^1)\right)\right]$$
$$\Rightarrow u_4^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{\lambda^4\cos x + \varepsilon^4\cos x - 6\lambda^2\varepsilon^2\cos x + 4\lambda\varepsilon^3\sin x - 4\lambda^3\varepsilon\sin x}{(s-\gamma)^5}\right)$$
$$\Rightarrow u_4^1(t,x) = \frac{1}{4!}t^4\lambda^4 e^{t\gamma}\cos x + \frac{1}{4!}t^4\varepsilon^4 e^{t\gamma}\cos x - \frac{t^4}{4}\lambda^2\varepsilon^2 e^{t\gamma}\cos x + \frac{1}{3!}t^4\lambda\varepsilon^3 e^{t\gamma}\sin x - \frac{1}{3!}t^4\lambda^3\varepsilon e^{t\gamma}\sin x$$

$$u_5^1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p}\mathcal{L}_t\left(R(u_4^1)\right)\right]$$
$$R(u_4^1) = \varepsilon\frac{\partial^2 u_4^1}{\partial x^2} + \lambda\frac{\partial u_4^1}{\partial x}$$
$$\Rightarrow u_5^1(t,x) = \mathcal{L}_t^{-1}\left(\frac{10\lambda^2\varepsilon^3\cos x - \lambda^5\sin x - \varepsilon^5\cos x + 10\lambda^3\varepsilon^2\sin x - 5\lambda^4\varepsilon\cos x - 5\lambda\varepsilon^4\sin x}{(s-\gamma)^6}\right)$$
$$\Rightarrow \begin{cases} u_5^1(t,x) = & -\frac{1}{5!}t^5\lambda^5 e^{t\gamma}\sin x - \frac{1}{5!}t^5\varepsilon^5 e^{t\gamma}\cos x + \frac{10}{5!}t^4\lambda^3\varepsilon^2 e^{t\gamma}\sin x - \frac{1}{4!}t^4\lambda^4\varepsilon e^{t\gamma}\cos x \\ & -\frac{1}{4!}t^4\lambda\varepsilon^4 e^{t\gamma}\sin x + \frac{10}{5!}t^4\lambda^2\varepsilon^3 e^{t\gamma}\cos x \end{cases}$$

We deduce

$$\begin{cases} u_0^1(t,x) = & e^{\gamma t}\cos x \\ u_1^1(t,x) = & (-\varepsilon t\cos x - \lambda t\sin x)e^{\gamma t} \\ u_2^1(t,x) = & -\frac{(\lambda t)^2}{2!}e^{\gamma t}(\cos x) + t^2 e^{\gamma t}(\sin x)\lambda\varepsilon + \frac{(\varepsilon t)^2}{2!}e^{\gamma t}(\cos x) \\ u_3^1(t,x) = & \frac{1}{3!}t^3\lambda^3 e^{t\gamma}\sin x - \frac{1}{3!}t^3\varepsilon^3 e^{t\gamma}\cos x + \frac{1}{2!}t^3\lambda^2\varepsilon e^{t\gamma}\cos x - \frac{1}{2!}t^3\lambda\varepsilon^2 e^{t\gamma}\sin x \\ u_4^1(t,x) = & \frac{1}{4!}t^4\lambda^4 e^{t\gamma}\cos x + \frac{1}{4!}t^4\varepsilon^4 e^{t\gamma}\cos x - \frac{t^4}{4}\lambda^2\varepsilon^2 e^{t\gamma}\cos x + \frac{1}{3!}t^4\lambda\varepsilon^3 e^{t\gamma}\sin x - \frac{1}{3!}t^4\lambda^3\varepsilon e^{t\gamma}\sin x \\ u_5^1(t,x) = & -\frac{1}{5!}t^5\lambda^5 e^{t\gamma}\sin x - \frac{1}{5!}t^5\varepsilon^5 e^{t\gamma}\cos x + \frac{10}{5!}t^4\lambda^3\varepsilon^2 e^{t\gamma}\sin x - \frac{1}{4!}t^4\lambda^4\varepsilon e^{t\gamma}\cos x - \frac{1}{4!}t^4\lambda\varepsilon^4 e^{t\gamma}\sin x \\ & +\frac{10}{5!}t^4\lambda^2\varepsilon^3 e^{t\gamma}\cos x \end{cases}$$

Step by step, we obtain

$$\left\{\begin{array}{rl} u^1(t,x) = & e^{\gamma t}\cos x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)-\frac{(\lambda t)^2}{2!}e^{\gamma t}\cos x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)\\ & +\frac{(\lambda t)^4}{4!}e^{\gamma t}\cos x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)+...\\ & -\lambda te^{\gamma t}\sin x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)+\frac{(\lambda t)^3}{3!}e^{\gamma t}\sin x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)\\ & -\frac{(\lambda t)^5}{5!}e^{\gamma t}\sin x\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)+... \end{array}\right.$$

we obtain

$$\left\{\begin{array}{rl} u^1(t,x) = & e^{\gamma t}\cos x\left(1-\frac{(\lambda t)^2}{2!}+\frac{(\lambda t)^4}{4!}+...\right)\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right)\\ & -e^{\gamma t}\sin x\left(\lambda t-\frac{(\lambda t)^3}{3!}+\frac{(\lambda t)^5}{5!}+...\right)\left(1-\varepsilon t+\frac{(\varepsilon t)^2}{2!}-\frac{(\varepsilon t)^3}{3!}+....\right) \end{array}\right.$$

We deduce

$$u^1(t,x) = e^{\gamma t}\cos x\cos(\lambda t)\,e^{-\varepsilon t} - e^{\gamma t}\sin x\sin(\lambda t)\,e^{-\varepsilon t} = e^{(\gamma-\varepsilon)t}\cos(x+\lambda t)$$

So, we have

$$\begin{array}{rcl} Nu^1 & = & \left(u^1\right)^n+\left(u^1\right)^{n-1}\frac{\partial^2 u^1}{\partial x^2}\\ Nu^1 & = & \left(e^{(\gamma-\varepsilon)t}\cos(x+\lambda t)\right)^n+\left(e^{(\gamma-\varepsilon)t}\cos(x+\lambda t)\right)^{n-1}\left(-e^{(\gamma-\varepsilon)t}\cos(x+\lambda t)\right)=0 \end{array}$$

In recursive way, we deduce

$$u^1(t,x) = u^2(t,x) = ... = u^k(t,x) = e^{(\gamma-\varepsilon)t}\sin(x+\lambda t)$$

Conclusion : The exact solution of the model is

$$u(t,x) = \lim_{k\to+\infty} u^k(t,x) = e^{(\gamma-\varepsilon)t}\sin(x+\lambda t)$$

5. Conclusion

Laplace's SBA numerical method allowed us to solve some linear partial differential equations by modelling Cauchy diffusion, convection and reaction problems. It is therefore a very powerful numerical analysis tool to solve this type of problem. This method accelerates convergence to the solution and avoids Adomian polynomial calculations. Our study was limited to models of convection, diffusion and non-homogeneous reaction, a study of these models in homogeneous cases would be an important contribution to the understanding of these models.

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