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On Varieties of Pseudo Hyper GR-ideals of Pseudo Hyper GR-algebras

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Abstract. This study is based on the structure of hyper GR-algebras, an algebra that is partially related on some class of hyper BCI-algebras. This allows us to create a new structure and investigate how this two algebras are related to each other. A pseudo hyper GR-algebra involves two hyper operations and a set of axioms that come in pairs or a combination of both making it interesting like some algebras established. This paper focuses on some properties of pseudo hyper GR-algebras and its ideals. Moreover, pseudo hyper GR-ideals were defined and classified to determine their relationship to each other.

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1. Introduction

Algebraic hyperstructures were introduced by a French mathematician, Marty [7], in 1934. They represent a natural extension of classical hyperstructures in which the composition of two elements of a given set is a set, instead of an element. Afterwards, this new idea was expanded rapidly and showed itself as a new view of sets.

The introduction of hyperstructure theory led to the study of several problems of noncommutative algebra. Algebraic hyperstructure theory has multiple applications to other fields such as: geometry, graphs and hypergraphs, binary relations, lattices, groups, relation algebras, artificial intelligence, probabilities, and so on.

In 1966, Y. Imai and K. Iséki [4] initiated the notion of BCK-algebra as a generalization of the concept of set-theoretic difference and propositional calculi. Furthermore, Y.B. Jun et al. [6] applied hyperstructure theory to BCK-algebras and introduced the notion of hyper BCK-algebras as a generalization of BCK-algebra.

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In order to extend BCK-algebra to a noncommutative form, G. Georgescu and A. Iorgulescu [3] introduced the notion of pseudo BCK-algebras and studied their properties. On the other hand, R. A. Borzooei, A. Rezazadeh and R. Ameri [1] introduced the concept of hyper pseudo BCK-algebra which is a generalization of pseudo BCK-algebra.

R.A. Indangan and G.C. Petalcorin [5] defined a new class of algebraic hyperstructure called hyper GR-algebra. In this algebra, they presented a helpful understanding on how this hyper algebra differs from the rest.

In this paper we define a pseudo hyper GR-algebra analogous to that of a hyper GR-algebra and its pseudo hyper GR-ideals and their relationships.

2. Preliminaries

Let H be a nonempty set endowed with a hyperoperation " $*$ ", that is, " $*$ " is a function from $H \times H$ to $P^*(H) = P(H) \setminus \{\varnothing\}$. For two nonempty subsets A and B of H, $A * B = \bigcup_{a\in A, b\in B} a * b$. We shall use $x * y$ instead of $x * \{y\}, \{x\} * y$ or $\{x\} * \{y\}$. When A is a nonempty subset of H and $x \in H$, we agree to write $A * x$ instead of $A * \{x\}$. Similarly, we write $x * A$ for $\{x\} * A$. In effect, $A * x = \bigcup_{a\in A} a * x$ and $x * A = \bigcup_{a\in A} x * a$. A set H endowed with a family Γ of hyperoperations is called a *hyperstructure*. If Γ is singleton, that is, $\Gamma = \{f\}$, then the hyperstructure is called a *hypergroupoid*.

Definition 2.1. [2] Let $x, y \in H$ and $A, B \subseteq H$. Then

(i) $x \ll y$ if and only if $0 \in x \circledast y$; and

(ii) $A \ll B$ if and only if for any $a \in A$, there exists $b \in B$ such that $a \ll b$.

We call $\ll$ a *hyperorder* on H.

Remark 2.2. [2] For all $A, B \subseteq H$, $A \ll B$ implies $0 \in A \circledast B$.

Definition 2.3. [5] Let H be a nonempty set with a hyperoperation"$\circledast$" on H. Then $(H; \circledast, 0)$ is called a *hyper GR-algebra* if it contains a constant $0 \in H$ and for all $x, y, z \in H$, the following conditions are satisfied:

$[HGR_1]$ $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$;

$[HGR_2]$ $(x \circledast y) \circledast z = (x \circledast z) \circledast y$;

$[HGR_3]$ $x \ll x$;

$[HGR_4]$ $0 \circledast (0 \circledast x) \ll x$, for all $x \neq 0$; and

$[HGR_5]$ $(x \circledast y) \circledast z \ll y \circledast z$.

Example 2.4. [5] Let $H = \{0, 1, 2\}$. Define the operation "$\circledast$" by the Cayley table shown below.

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0,1,2\}$	$\{0,1\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,1,2\}$	$\{0,2\}$

By routine calculations, $(H;\circledast,0)$ is a hyper GR-algebra.

Definition 2.5. [5] A hyper GR-algebra H is *faithful* if for all $A, B \subseteq H$, $0 \in A \circledast B$ implies $A \ll B$.

Definition 2.6. [5] Let H be a hyper GR-algebra and S be a subset of H containing 0. If S is a hyper GR-algebra with respect to the hyperoperation $\circledast$ on H, then we say that S is a *hyper subGR-algebra* of H.

Theorem 2.7. [5] (Hyper SubGR-algebra Criterion)
Let H be a hyper GR-algebra and S be a nonempty subset of H. Then S is a *hyper subGR-algebra* of H if and only if $x \circledast y \subseteq S$, for all $x, y \in S$.

Definition 2.8. [5] Let I be a subset of a hyper GR-algebra H such that $0 \in I$. Then

(i) I is a *hyper GR-ideal* of H if for all $x, y \in H$, $x \circledast y \subseteq I$ and $y \in I$ imply that $x \in I$;

(ii) if H is faithful such that $x \circledast x \ll I$ for all $x \in H$, then I is *GR-reflexive* in H;

(iii) I is *hyper left* (*resp. hyper right*) *stable* in H if $x \circledast a \ll I$ (resp. $a \circledast x \ll I$) for all $x \in H$ and for all $a \in I$;

(iv) I is *hyper stable* in H if I is both hyper left and hyper right stable in H;

(v) I is *hyper left* (*resp. hyper right*) *stable GR-ideal* of H if

(a) I is hyper left (resp. hyper right) stable in H; and

(b) I is a hyper GR-ideal of H.

(vi) I is a *hyper stable GR-ideal* of H if I is both hyper left and hyper right stable GR-ideal of H.

Theorem 2.9. [5] If $\{I_i | i \in \Lambda\}$ is a nonempty collection of hyper GR-ideals of a hyper GR-algebra H, then so is $\bigcap_{i\in\Lambda} I_i$.

Definition 2.10. [5] Let H be a hyper GR-algebra, X a nonempty proper subset of H, and I a subset of H such that $0 \in I$. Then

(i) I is a *hyper GR-ideal* of H *related to* X if for all $x, y \in X$, $x \circledast y \subseteq I$ and $y \in I$ imply that $x \in I$;

(ii) I is *hyper left* (resp. *hyper right*) *stable* in H *related to* X if $x \circledast a \ll I$ (resp. $a \circledast x \ll I$) for all $x \in X$ and for all $a \in I$;

(iii) I is *hyper stable* in H *related to* X if I is both hyper left and hyper right stable in H related to X;

(iv) I is *hyper left* (*resp. hyper right*) *stable GR-ideal* of H *related to* X if

(a) I is hyper left (resp. hyper right) stable in H related to X; and

(b) I is a hyper GR-ideal of H related to X.

(v) I is a *hyper stable GR-ideal* of H *related to* X if I is both hyper left and hyper right stable GR-ideal of H related to X.

3. Pseudo Hyper GR-ideals

In this section we will define a pseudo hyper GR-algebra and the different types of pseudo hyper GR-ideals. Also, relationship among the twelve types of these ideals are discussed.

Definition 3.1. Let H be a nonempty set with "$\circledast$" and "$\circ$" be the two hyperoperations on H. Then $(H; \circledast, \circ, 0)$ is called a *pseudo hyper GR-algebra*, if it contains a constant $0 \in H$ and for all $x, y, z \in H$, the following conditions are satisfied:

$[PHGR_1]$ $(x \circ z) \circ (y \circ z) \ll x \circ y$ and $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$;

$[PHGR_2]$ $(x \circ y) \circledast z = (x \circledast z) \circ y$;

$[PHGR_3]$ $0 \in x \circledast x$ and $0 \in x \circ x$;

$[PHGR_4]$ $0 \circ (0 \circledast x) \ll x$, for all $x \neq 0$; and

$[PHGR_5]$ $(x \circledast y) \circledast z \ll y \circ z$.

where $x \ll y$ if and only if $0 \in x \circ y$ and $0 \in x \circledast y$, and for every $A, B \subseteq H$, $A \ll B$ means that for every $a \in A$, there exists $b \in B$ such that $a \ll b$.

Throughout this chapter, we denote a pseudo hyper GR-algebra $(H, \circledast, \circ, 0)$ simply by H, unless otherwise stated.

Example 3.2. Let $H = \{0, 1, 2, 3\}$ and consider the following Cayley tables below.

$\circledast$	0	1	2	3
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,1,2\}$	$\{0,2\}$	$\{0,1,2\}$
3	$\{0,1,2\}$	$\{0,3\}$	$\{0,1,3\}$	$\{0,3\}$

$\circ$	0	1	2	3
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
3	$\{0,3\}$	$\{0,1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$

By routine calculations, we see that $(H;\circledast,\circ,0)$ is a pseudo hyper GR-algebra.

Remark 3.3. In a pseudo hyper GR-algebra H, the following are evident:

(i) $x \ll x$;

(ii) $(x \circ y) \circledast z \ll (x \circledast z) \circ y$;

(iii) $(A \circ B) \circledast C = (A \circledast C) \circ B$; and

(iv) $A \subseteq B$ implies $A \ll B$.

Example 3.4. Let $H = \mathbb{N} \cup \{0\}$ be the set of all nonnegative integers and let the hyperoperations "$\circledast$" and "$\circ$" be defined on H as follows:

$$x \circledast y = \{0, x\} \text{ and } x \circ y = \{0, x, y\}.$$

Then H is a pseudo hyper GR-algebra.

To verify this, we need to check that the five conditions are satisfied. Note that $\{0,x,z\} \circ \{0,y,z\} = \{0,x,y,z\} \ll \{0,x,y\}$. This means that $(x \circ z) \circ (y \circ z) \ll x \circ y$. On the other hand, $\{0,x\} \circledast \{0,y\} = \{0,x\} \ll \{0,x\}$ means that $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$. Thus, $[PHGR_1]$ holds. Now, $(x \circ y) \circledast z = \{0,x,y\} \circledast z = \{0,x,y\}$, also $(x \circledast z) \circ y = \{0,x\} \circ y = \{0,x,y\}$ and so $(x \circ y) \circledast z = (x \circledast z) \circ y$, that is, $[PHGR_2]$ is satisfied. $[PHGR_3]$ follows immediately from the defined operations $\circ$ and $\circledast$ on H, that is, $x \circ x = \{0,x\}$ and $x \circledast x = \{0,x\}$ for all $x \in H$. Let $x \neq 0$, then $0 \circ (0 \circledast x) = 0 \circ \{0\} = \{0\} \ll x$, and thus, $[PHGR_4]$ holds. Finally, $\{0,x\} \circledast z = \{0,x\} \ll \{0,y,z\} = y \circ z$. Hence, $(x \circledast y) \circledast z \ll y \circ z$, that is, $[PHGR_5]$ holds. Therefore, H is a pseudo hyper GR-algebra.

Remark 3.5. Note that if the two hyperoperations are equal, that is, $\circledast = \circ$, then a pseudo hyper-GR algebra H becomes a hyper GR-algebra.

Definition 3.6. Let H be a pseudo hyper GR-algebra and S be a subset of H containing 0. If S itself is a pseudo hyper GR-algebra with respect to the hyperoperations $\circledast$ and $\circ$ on H, then S is called a *pseudo hyper subGR-algebra* of H.

Theorem 3.7. (Pseudo Hyper SubGR-algebra Criterion)
Let S be a nonempty subset of a pseudo hyper GR-algebra H. Then S is a pseudo hyper subGR-algebra if and only if both $x \circledast y \subseteq S$ and $x \circ y \subseteq S$ for all $x, y \in S$.

Proof. Suppose that S is a pseudo hyper subGR-algebra of H. By Definition 3.6, S is closed under the hyperoperations $\circledast$ and $\circ$ so that $x \circledast y \subseteq S$ and $x \circ y \subseteq S$ for all $x, y \in S$.

Conversely, suppose that S has the property $x \circledast y \subseteq S$ and $x \circ y \subseteq S$ for all $x, y \in S$. Since $S \subseteq H$, all the axioms $[PHGR_1]$ to $[PHGR_5]$ of Definition 3.1 are all satisfied. It remains to show that S contains the element 0. From the above hypothesis, S is nonempty and thus, must contain an element, say c. Then by Definition 3.1 $[PHGR_3]$, $0 \in c \circledast c$ and $0 \in c \circ c$. Note that $c \circledast c \subseteq S$ and $c \circ c \subseteq S$. Thus, $0 \in S$. □

Example 3.8. For any pseudo hyper GR-algebra H, the set $S = \{0\}$ is a pseudo hyper subGR-algebra of H.

For any nonempty subset I of a pseudo hyper GR-algebra H and any element y of H, we introduce the following notations and their meanings:

$$I^{\ll}_{\circledast,y} = \{x \in H \mid x \circledast y \ll I\}.$$

$$I^{\subseteq}_{\circledast,y} = \{x \in H \mid x \circledast y \subseteq I\}.$$

$$I^{\ll}_{\circ,y} = \{x \in H \mid x \circ y \ll I\}.$$

$$I^{\subseteq}_{\circ,y} = \{x \in H \mid x \circ y \subseteq I\}.$$

Definition 3.9. Let I be a nonempty subset of a pseudo hyper GR-algebra H such that $0 \in I$. Then I is said to be a *pseudo hyper-GR ideal of H* if for any $y \in I$, $I^{\subseteq}_{\circledast,y} \subseteq I$ and $I^{\subseteq}_{\circ,y} \subseteq I$.

Example 3.10. Consider the pseudo hyper GR-algebra H in Example 3.2. Let $I = \{0, 2\}$. Observe that

$$I^{\subseteq}_{\circledast,0} = \{x \in H \mid x \circledast 0 \subseteq I\} = \{2\} \subseteq I$$

$$I^{\subseteq}_{\circledast,2} = \{x \in H \mid x \circledast 2 \subseteq I\} = \{2\} \subseteq I$$

$$I^{\subseteq}_{\circ,0} = \{x \in H \mid x \circ 0 \subseteq I\} = \{2\} \subseteq I$$

$$I^{\subseteq}_{\circ,2} = \{x \in H \mid x \circ 2 \subseteq I\} = \varnothing \subseteq I.$$

Thus, I is indeed a pseudo hyper GR-ideal.

From now on, we shall call the ideal in Definition 3.9 as pseudo hyper GR-ideal of *type* 1 for we will be considering some forms of pseudo hyper GR-ideals which will be defined analogously as in Definition 3.9.

Definition 3.11. Let I be a nonempty subset of a pseudo hyper GR-algebra H such that $0 \in I$. Then I is said to be a pseudo hyper-GR ideal of H of :

type 2, if for any $y \in I$, $I^{\subseteq}_{\circledast,y} \subseteq I$ and $I^{\ll}_{\circ,y} \subseteq I$.

type 3, if for any $y \in I$, $I^{\ll}_{\circledast,y} \subseteq I$ and $I^{\subseteq}_{\circ,y} \subseteq I$.

type 4, if for any $y \in I$, $I^{\ll}_{\circledast,y} \subseteq I$ and $I^{\ll}_{\circ,y} \subseteq I$.

type 5, if for any $y \in I$, $I^{\subseteq}_{\circledast,y} \subseteq I$ or $I^{\subseteq}_{\circ,y} \subseteq I$.

type 6, if for any $y \in I$, $I^{\subseteq}_{\circledast,y} \subseteq I$ or $I^{\ll}_{\circ,y} \subseteq I$.

type 7, if for any $y \in I$, $I^{\ll}_{\circledast,y} \subseteq I$ or $I^{\subseteq}_{\circ,y} \subseteq I$.

type 8, if for any $y \in I$, $I^{\ll}_{\circledast,y} \subseteq I$ or $I^{\ll}_{\circ,y} \subseteq I$.

type 9, if for any $y \in I$, $I^{\subseteq}_{\circledast,y} \cap I^{\subseteq}_{\circ,y} \subseteq I$.

type 10, if for any $y \in I$, $I^{\subseteq}_{\circledast,y} \cap I^{\ll}_{\circ,y} \subseteq I$.

type 11, if for any $y \in I$, $I^{\ll}_{\circledast,y} \cap I^{\subseteq}_{\circ,y} \subseteq I$.

type 12, if for any $y \in I$, $I^{\ll}_{\circledast,y} \cap I^{\ll}_{\circ,y} \subseteq I$.

Example 3.12. Let $H = \{0, 1, 2\}$ with the hyperoperations $\circledast$ and $\circ$ on H given by the Cayley table below

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{0,2\}$	$\{2\}$	$\{0,2\}$

By routine calculations, H is a pseudo hyper GR-algebra. Let $I = \{0, 1\}$. Note that

$$I^{\subseteq}_{\circledast,y} = \{0, 1\} \subseteq I \text{ and } I^{\ll}_{\circ,y} = \{0, 1\} \subseteq I.$$

Thus, I is pseudo hyper GR-ideal of type 2.
Note also that

$$I^{\ll}_{\circledast,y} = \{0, 1\} \subseteq I \text{ and } I^{\subseteq}_{\circ,y} = \{0, 1\} \subseteq I.$$

Thus, I is pseudo hyper GR-ideal of type 3.
Moreover,

$$I^{\ll}_{\circledast,y} = \{0,1\} \subseteq I \text{ and } I^{\ll}_{\circ,y} = \{0,1\} \subseteq I,$$

$$I^{\ll}_{\circledast,y} = \{0,1\} \subseteq I \text{ or } I^{\ll}_{\circ,y} = \{0,1\} \subseteq I \text{ and}$$

$$I^{\ll}_{\circledast,y} \cap I^{\ll}_{\circ,y} = \{0,1\} \subseteq I.$$

Therefore, I is pseudo hyper GR-ideal of type 4, 8 and 12 respectively.

Example 3.13. Consider the pseudo hyper GR-algebra H in Example 3.2. Let $I = \{0,3\}$. Note that for any $y \in I$, $I^{\subseteq}_{\circledast,y} = \{3\} \subseteq I$. This is enough to categorize I as a pseudo hyper GR-ideal of type 6. Also for any $y \in I$, $I^{\ll}_{\circ,y} = \{0,1,2,3\}$. Even if $I^{\ll}_{\circ,y} \nsubseteq I$, $I^{\subseteq}_{\circledast,y} \cap I^{\ll}_{\circ,y} = \{3\} \subseteq I$. Thus, I must be a pseudo hyper GR-ideal of type 10. Hence, I is an example of pseudo hyper GR-ideal of type 6 and 10 but not type 2 since $I^{\subseteq}_{\circledast,y} \subseteq I$ but $I^{\ll}_{\circ,y} \nsubseteq I$.

Example 3.14. Consider the pseudo hyper GR-algebra H in Example 3.2. Let $I = \{0,1\}$. By routine calculations, I is a pseudo hyper GR-ideal of type 5.

Example 3.15. Consider the pseudo hyper GR-algebra H in Example 3.2. Let $I = \{0,1,3\}$. By routine calculations, I is a pseudo hyper GR-ideal of type 6.

Example 3.16. Consider the pseudo hyper GR-algebra H in Example 3.2. Let $I = \{0,2\}$. By routine calculations, I is a pseudo hyper GR-ideal of type 7.

Example 3.17. Consider the pseudo hyper GR-algebra H in Example 3.4. Let $H' = \{0,1,2,3\}$. Then H' together with the hyperoperations $\circledast$ and $\circ$ given by the Cayley table below is a pseudo hyper subGR-algebra of H.

$\circledast$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$
3	$\{0,3\}$	$\{0,3\}$	$\{0,3\}$	$\{0,3\}$

$\circ$	0	1	2	3
0	$\{0\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,3\}$
2	$\{0,2\}$	$\{0,1,2\}$	$\{0,2\}$	$\{0,2,3\}$
3	$\{0,3\}$	$\{0,1,3\}$	$\{0,2,3\}$	$\{0,3\}$

Consider $I = \{0,2,3\}$. Observe that $I^{\subseteq}_{\circledast,y} = \{0,2,3\} = I^{\subseteq}_{\circ,y}$. This means that $I^{\subseteq}_{\circledast,y} \cap I^{\subseteq}_{\circ,y} = \{0,2,3\} \subseteq I$. Thus, I is a pseudo hyper GR-ideal of type 9.

Let $I = \{0,1,2\}$. O bservier that $I^{\subseteq}_{\circledast,y} = \{0,1,2\}$ and $I^{\ll}_{\circ,y} = \{0,1,2,3\}$. Thus, we have $I^{\subseteq}_{\circledast,y} \cap I^{\ll}_{\circ,y} = \{0,1,2\} \subseteq I$. Therefore, I is a pseudo hyper GR-ideal of type 10.

Let $I = \{0,1,3\}$. Observe that $I^{\ll}_{\circledast,y} = \{0,1,2,3\}$ and $I^{\subseteq}_{\circ,y} = \{0,1,3\}$. Thus, we have $I^{\ll}_{\circledast,y} \cap I^{\subseteq}_{\circ,y} = \{0,1,3\} \subseteq I$. Therefore, I is a pseudo hyper GR-ideal of type 11.

Theorem 3.18. Every pseudo hyper GR-ideal in H of type 2 is a pseudo hyper GR-ideal in H of type 1.

Proof. Let I be a pseudo hyper GR-ideal of type 2. Now, we will show that I is a pseudo hyper GR-ideal of type 1. It is enough to show that for any $y \in I$, $I^{\subseteq}_{\circ,y} \subseteq I$.

Let $y \in I$ and $x \in I^{\subseteq}_{\circ,y}$. Then, $x \circ y \subseteq I$ and by Remark 3.3 (iv), $x \circ y \ll I$. Hence $x \in I^{\ll}_{\circ,y}$. Since I is a pseudo hyper GR-ideal of type 2, $I^{\ll}_{\circ,y} \subseteq I$ and so $x \in I$. Therefore, $I^{\subseteq}_{\circ,y} \subseteq I$. □

Theorem 3.19. Every pseudo hyper GR-ideal in H of type 4 is a pseudo hyper GR-ideal in H of types 1, 2 and 8.

Proof. Let I be a pseudo hyper GR-ideal in H of type 4. We will show that I is a pseudo hyper GR-ideal of type 2. It is enough to show that for any $y \in I$, $I^{\subseteq}_{\circledast,y} \subseteq I$.

Let $y \in I$ and $x \in I^{\subseteq}_{\circledast,y}$. Then, $x \circledast y \subseteq I$ and by Remark 3.3 (iv), $x \circledast y \ll I$. Hence, $x \in I^{\ll}_{\circledast,y}$. Since I is a pseudo hyper GR-ideal of type 4, $I^{\ll}_{\circledast,y} \subseteq I$ and so $x \in I$. Thus, $I^{\subseteq}_{\circledast,y} \subseteq I$. Hence, I is a pseudo hyper GR-ideal of type 2 and by Theorem 3.18, I is a pseudo hyper GR-ideal of type 1.

Furthermore, we will show that I is a pseudo hyper GR-ideal of type 8. That is, to show that for any $y \in I$, $I^{\ll}_{\circledast,y} \subseteq I$ or $I^{\ll}_{\circ,y} \subseteq I$.

Let $y \in I$ and $x \in I^{\ll}_{\circledast,y}$. Since I is a pseudo hyper GR-ideal of type 4, $I^{\ll}_{\circledast,y} \subseteq I$ and so, $x \in I$. Therefore, $I^{\ll}_{\circledast,y} \subseteq I$. Similarly, we can show for the other case that $I^{\ll}_{\circ,y} \subseteq I$. □

Theorem 3.20. Every pseudo hyper GR-ideal in H of type 8 is a pseudo hyper GR-ideal in H of types 5, 6, 7 and 12.

Proof. Let I be a pseudo hyper GR-ideal of type 8. We will show that I is a pseudo hyper GR-ideal of type 5. We will consider two cases : when $I^{\subseteq}_{\circ,y} \subseteq I$ and when $I^{\subseteq}_{\circ,y} \nsubseteq I$. If $I^{\subseteq}_{\circ,y} \subseteq I$, then we are done. Suppose that $I^{\subseteq}_{\circ,y} \nsubseteq I$. Let $x \in I^{\subseteq}_{\circledast,y}$, where $y \in I$. Then, $x \circledast y \subseteq I$, thus by Remark 3.3 (iv), $x \circledast y \ll I$. Hence, $x \in I^{\ll}_{\circledast,y}$. Since I is a pseudo hyper GR-ideal of type 8, $I^{\ll}_{\circ,y} \subseteq I$ or $I^{\ll}_{\circledast,y} \subseteq I$. Suppose that $I^{\ll}_{\circ,y} \subseteq I$. The hypothesis $I^{\subseteq}_{\circ,y} \nsubseteq I$ implies that there exists $z \in I^{\subseteq}_{\circ,y}$ such that $z \notin I$. Moreover, $z \circ y \subseteq I$ and by Remark 3.3 (iv), $z \circ y \ll I$. Hence, $z \in I^{\ll}_{\circ,y}$ and so $z \in I$. A contradiction. Thus, $I^{\ll}_{\circ,y} \nsubseteq I$. Thus, $I^{\ll}_{\circledast,y} \subseteq I$ and so $x \in I$. Therefore, $I^{\subseteq}_{\circledast,y} \subseteq I$.

Next, we will prove that I is a pseudo hyper GR-ideal of type 6. If $I^{\ll}_{\circ,y} \subseteq I$, then we are done. Suppose $I^{\ll}_{\circ,y} \nsubseteq I$. Let $x \in I^{\subseteq}_{\circledast,y}$, where $y \in I$. Then, $x \circledast y \subseteq I$, thus by Remark 3.3 (iv), $x \circledast y \ll I$. Hence, $x \in I^{\ll}_{\circledast,y}$. Since I is a pseudo hyper GR-ideal of type 8 and $I^{\ll}_{\circ,y} \nsubseteq I$, then $I^{\ll}_{\circledast,y} \subseteq I$ and so $x \in I$. Therefore, $I^{\subseteq}_{\circledast,y} \subseteq I$.

The proof for type 7 follows similarly as in the case of type 6.

Furthermore, we will prove that I is a pseudo hyper GR-ideal of type 12. Let $y \in I$ and $x \in I^{\ll}_{\circledast,y} \cap I^{\ll}_{\circ,y}$. Then $x \in I^{\ll}_{\circledast,y}$ and $I^{\ll}_{\circ,y}$. Since I is a pseudo hyper GR-ideal of type 8, we have $I^{\ll}_{\circledast,y} \subseteq I$ or $I^{\ll}_{\circ,y} \subseteq I$ and so $x \in I$. Hence, $I^{\ll}_{\circledast,y} \cap I^{\ll}_{\circ,y} \subseteq I$. □

Theorem 3.21. Every pseudo hyper GR-ideal in H of type 6 is a pseudo hyper GR-ideal in H of types 5 and 10.

Proof. Let I be a pseudo hyper GR-ideal of type 6. Now, we will show that I is a pseudo hyper GR-ideal of type 5. If $I^{\subseteq}_{\circledast,y} \subseteq I$, then we are done. Suppose $I^{\subseteq}_{\circledast,y} \nsubseteq I$. Let $x \in I^{\subseteq}_{\circ,y}$ for any $y \in I$. Then $x \circ y \subseteq I$ and so by Remark 3.3 (iv), $x \circ y \ll I$. Hence, $x \in I^{\ll}_{\circ,y}$. Since I is a pseudo hyper GR-ideal of type 6 and $I^{\subseteq}_{\circledast,y} \nsubseteq I$, $I^{\ll}_{\circ,y} \subseteq I$ and thus, $x \in I$. Hence, $I^{\subseteq}_{\circ,y} \subseteq I$.

Next, we will show that I is a pseudo hyper GR-ideal of type 10. Let $y \in I$ and $x \in I^{\subseteq}_{\circledast,y} \cap I^{\ll}_{\circ,y}$. Then, $x \in I^{\subseteq}_{\circledast,y}$ and $x \in I^{\ll}_{\circ,y}$. Since I is a pseudo hyper GR-ideal of type 6, we have $I^{\subseteq}_{\circledast,y} \subseteq I$ or $I^{\ll}_{\circ,y} \subseteq I$ and so $x \in I$. Hence, $I^{\subseteq}_{\circledast,y} \cap I^{\ll}_{\circ,y} \subseteq I$. □

Theorem 3.22. Every pseudo hyper GR-ideal in H of type 7 is a pseudo hyper GR-ideal in H of types 5 and 11.

Proof. Let I be a pseudo hyper GR-ideal of type 7. Now, we will show that I is a pseudo hyper GR-ideal of type 5. If $I^{\subseteq}_{\circ,y} \subseteq I$, then we are done. Suppose $I^{\subseteq}_{\circ,y} \nsubseteq I$. Let $x \in I^{\subseteq}_{\circledast,y}$ for any $y \in I$. Then $x \circledast y \subseteq I$ and so by Remark 3.3 (iv), $x \circledast y \ll I$. Hence, $x \in I^{\ll}_{\circledast,y}$. Since I is a pseudo hyper GR-ideal of type 7 and $I^{\subseteq}_{\circ,y} \nsubseteq I$, $I^{\ll}_{\circledast,y} \subseteq I$ and thus, $x \in I$. Hence, $I^{\subseteq}_{\circledast,y} \subseteq I$.

Next, we will show that I is a pseudo hyper GR-ideal of type 11. Let $y \in I$ and $x \in I^{\ll}_{\circledast,y} \cap I^{\subseteq}_{\circ,y}$. Then, $x \in I^{\ll}_{\circledast,y}$ and $x \in I^{\subseteq}_{\circ,y}$. Since I is a pseudo hyper GR-ideal of type 7, we have $I^{\ll}_{\circledast,y} \subseteq I$ or $I^{\subseteq}_{\circ,y} \subseteq I$ and so $x \in I$. Hence, $I^{\ll}_{\circledast,y} \cap I^{\subseteq}_{\circ,y} \subseteq I$. □

Theorem 3.23. Every pseudo hyper GR-ideal in H of type 5 is a pseudo hyper GR-ideal in H of type 9.

Proof. Suppose that I be a pseudo hyper GR-ideal of type 5. Now, we will show that I is a pseudo hyper GR-ideal of type 9. Let $y \in I$ and $x \in I^{\subseteq}_{\circledast,y} \cap I^{\subseteq}_{\circ,y}$. Then, $x \in I^{\subseteq}_{\circledast,y}$ and $x \in I^{\subseteq}_{\circ,y}$. Since I is a pseudo hyper GR-ideal of type 5, $I^{\subseteq}_{\circledast,y} \subseteq I$ or $I^{\subseteq}_{\circ,y} \subseteq I$ and so $x \in I$. Hence, $I^{\subseteq}_{\circledast,y} \cap I^{\subseteq}_{\circ,y} \subseteq I$. □

Theorem 3.24. Every pseudo hyper GR-ideal in H of type 12 is a pseudo hyper GR-ideal in H of types 9, 10 and 11.

Proof. Suppose that I be a pseudo hyper GR-ideal of type 12. Now, we will show that I is a pseudo hyper GR-ideal of type 9. Let $y \in I$ and $x \in I^{\subseteq}_{\circledast,y} \cap I^{\subseteq}_{\circ,y}$. Then, $x \in I^{\subseteq}_{\circledast,y}$ and $x \in I^{\subseteq}_{\circ,y}$. Thus, $x \circledast y \subseteq I$ and $x \circ y \subseteq I$ and and by Remark 3.3 (iv), $x \circledast y \ll I$ and $x \circ y \ll I$. This means that $x \in I^{\ll}_{\circledast,y}$ and $x \in I^{\ll}_{\circ,y}$ or equivalently $x \in I^{\ll}_{\circledast,y} \cap I^{\ll}_{\circ,y}$. Since I is a pseudo hyper GR-ideal of type 12, $I^{\ll}_{\circledast,y} \cap I^{\ll}_{\circ,y} \subseteq I$, and so $x \in I$. Therefore, $I^{\subseteq}_{\circledast,y} \cap I^{\subseteq}_{\circ,y} \subseteq I$.

Next, we will show that I is a pseudo hyper GR-ideal of type 10. Let $y \in I$ and $x \in I^{\subseteq}_{\circledast,y} \cap I^{\ll}_{\circ,y}$. Then, $x \in I^{\subseteq}_{\circledast,y}$ and $x \in I^{\ll}_{\circ,y}$. Thus, $x \circledast y \subseteq I$ and $x \circ y \ll I$ and and by Remark 3.3 (iv), $x \circledast y \ll I$. This means that $x \in I^{\ll}_{\circledast,y}$ and $x \in I^{\ll}_{\circ,y}$ or equivalently $x \in I^{\ll}_{\circledast,y} \cap I^{\ll}_{\circ,y}$. Since I is a pseudo hyper GR-ideal of type 12, $I^{\ll}_{\circledast,y} \cap I^{\ll}_{\circ,y} \subseteq I$, and so $x \in I$. Therefore, $I^{\subseteq}_{\circledast,y} \cap I^{\ll}_{\circ,y} \subseteq I$.

The proof for type 11 follows similarly as of type 10 with some modifications. □

Theorem 3.25. Every pseudo hyper GR-ideal in H of type 10 is a pseudo hyper GR-ideal in H of type 9.

Proof. Suppose that I be a pseudo hyper GR-ideal of type 10. Now, we will show that I is a pseudo hyper GR-ideal of type 9. Let $y \in I$ and $x \in I^{\subseteq}_{\circledast,y} \cap I^{\subseteq}_{\circ,y}$. Then, $x \in I^{\subseteq}_{\circledast,y}$ and $x \in I^{\subseteq}_{\circ,y}$. Thus, $x \circ y \subseteq I$ and and by Remark 3.3 (iv), $x \circ y \ll I$ which means that $x \in I^{\ll}_{\circ,y}$. Thus, $x \in I^{\subseteq}_{\circledast,y} \cap I^{\ll}_{\circ,y}$. Since I is a pseudo hyper GR-ideal of type 10, we have $I^{\subseteq}_{\circledast,y} \cap I^{\ll}_{\circ,y} \subseteq I$ and so $x \in I$. Hence, $I^{\subseteq}_{\circledast,y} \cap I^{\subseteq}_{\circ,y} \subseteq I$. □

Theorem 3.26. Every pseudo hyper GR-ideal in H of type 11 is a pseudo hyper GR-ideal in H of type 9.

Proof. Suppose that I be a pseudo hyper GR-ideal of type 11. Now, we will show that I is a pseudo hyper GR-ideal of type 9. Let $y \in I$ and $x \in I^{\subseteq}_{\circledast,y} \cap I^{\subseteq}_{\circ,y}$. Then $x \in I^{\subseteq}_{\circledast,y}$ and $x \in I^{\subseteq}_{\circ,y}$. Thus, $x \circledast y \subseteq I$ and and by Remark 3.3 (iv), $x \circledast y \ll I$ which means that $x \in I^{\ll}_{\circledast,y}$. Thus, $x \in I^{\ll}_{\circledast,y} \cap I^{\subseteq}_{\circ,y}$. Since I is a pseudo hyper GR-ideal of type 11, we have $I^{\ll}_{\circledast,y} \cap I^{\subseteq}_{\circ,y} \subseteq I$ and so, $x \in I$. Hence, $I^{\subseteq}_{\circledast,y} \cap I^{\subseteq}_{\circ,y} \subseteq I$. □

Theorem 3.27. Let $\{I_\omega | \omega \in \Omega\}$ be a family of pseudo hyper GR-ideals of type i, $1 \leq i \leq 12$, in H. Then $\bigcap_{\omega\in\Omega} I_\omega$ is also a pseudo hyper GR-ideal of type i, $1 \leq i \leq 12$ in H.

Proof. Assume that $I = \bigcap_{\omega\in\Omega} I_\omega$. Let I_ω be a pseudo hyper GR-ideal of specific type, say type 1, for any $\omega \in \Omega$. We will prove that I is a pseudo hyper GR-ideal of type 1. Since every I_ω is a pseudo hyper GR-ideal for each ω, $0 \in I_\omega$, for all $\omega \in \Omega$ and thus, $0 \in \bigcap_{\omega\in\Omega} I_\omega = I$.

Let $y \in I$, $x \in I^{\subseteq}_{\circledast,y}$ and $z \in I^{\subseteq}_{\circ,y}$. Then $x \circledast y \subseteq I$ and $z \circ y \subseteq I$. This means that for any $u \in x \circledast y$, $u \in I$. Thus, $u \in I_\omega$ for any $\omega \in \Omega$ and so, $x \circledast y \subseteq I_\omega$. Hence, $x \in I^{\subseteq}_{\omega,\circledast,y}$ and $y \in I_\omega$, for any $\omega \in \Omega$. Since I_ω is a pseudo hyper GR-ideal of type 1, $I^{\subseteq}_{\omega,\circledast,y} \subseteq I_\omega$ so that $x \in I_\omega$ for any $\omega \in \Omega$. Hence, $x \in I$ and thus,$I^{\subseteq}_{\circledast,y} \subseteq I$. In a similar manner, we can also prove that $z \in I$ so that $I^{\subseteq}_{\circ,y} \subseteq I$.

The proof for the remaining cases ($i = 2, 3, \ldots, 12$) follows the same argument with some modifications. □

Theorem 3.28. Let D be a nonempty subset of H. Let $[D]_i$ denote the intersection of all pseudo hyper GR-ideals of type i, $1 \leq i \leq 4$ containing D. Then

$$\{x \in H | (...((x \circledast d_1) \circledast d_2) \circledast ...) \circledast d_n = \{0\}, \quad d_i \in D\} \subseteq [D]_i.$$

Proof. We will prove only the case for $i = 1$, that is for the case of pseudo hyper GR-ideals of type 1. Let $x \in H$ and suppose that the condition

$$(...((x \circledast d_1) \circledast d_2) \circledast ...) \circledast d_n = \{0\}$$

is satisfied for some $d_1, d_2, ..., d_n \in D$. Note that $0 \in [D]_1$, hence

$$(...((x \circledast d_1) \circledast d_2) \circledast ...) \circledast d_n = \{0\} \subseteq [D]_1$$

Thus, for each $d \in (...((x \circledast d_1) \circledast d_2) \circledast ...) \circledast d_{n-1}$, we have $d \circledast d_n \subseteq [D]_1$, or equivalently, $d \in ([D]_1)^{\subseteq}_{\circledast, d_n}$. Since $[D]_1$ is a pseudo hyper GR-ideal of type 1, $([D]_1)^{\subseteq}_{\circledast, d_n} \subseteq [D]_1$, and so, $d \in [D]_1$. Thus,

$$(...((x \circledast d_1) \circledast d_2) \circledast ...) \circledast d_{n-1} \subseteq [D]_1$$

Continuing this process, we obtain $\{x\} \in [D]_1$ and so, $x \in [D]_1$. Therefore, $\{x \in H | (...((x \circledast d_1) \circledast d_2) \circledast ...) \circledast d_n = \{0\},\ d_i \in D\} \subseteq [D]_1$.

□

The ideal $[D]_i$ in Theorem 3.28 is called the pseudo hyper GR-ideal generated by D.

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References

[1] A. Borzooei, A. Rezazadeh, and R. Ameri. On Hyper Pseudo BCK-algebras. *Iranian Journal of Mathematical Sciences and Informatics*, 9(1):13–29, 2014.

[2] K. Hila et. al. Algebraic Hyperstructure of Soft Sets Associated with Ternary Semihypergroups. *Italian Journal of Pure and Applied Mathematics*, 30:349–372, 2013.

[3] G. Georgescu and Iorgulescu. Pseudo BCK-algebras: an extension of BCK-algebras. In Springer, editor, *Combinatorics, Computability and Logic*, pages 97–114, London, 2001. Proceedings of DMTSC 01.

[4] Y. Imai and K. Iseki. On Axiom Systems of Propositional Calculi XIV. pages 19–22, Japan, 1966. Proceedings of Japan Acedemy.

[5] R. Indangan and G. Petalcorin. Some Results on Hyper GR-ideals of a Hyper GR-algebra. *Journal of Algebra and Applied Mathematics*, 14:101–119, 2016.

[6] Y. Jun, M. Zahedi, X. Xin, and R. Borzooei. On Hyper BCK-algebras. *Journal of Algebra and Applied Mathematics*, 8:127–136, 2000.

[7] F. Marty. Sur une generalization de la notion de groupe. In *Proceedings of the 8th Congress Math. Scandinaves*, pages 45–49. Stockholm, 1934.