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Curvature Inequalities for Submanifolds of S-space form

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Abstract. In this paper we establish new results of squared mean curvature and Ricci curvature for the sub manifolds of S-space from that is the generalization of complex and contact structures. Obtained results are discussed for invariant, anti invariant and CR sub manifolds of S-space from.

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1. Introduction

One of the main and useful idea in submanifolds conjectures is to derive relationship among squared mean curvature and Ricci curvature of submanifolds, was explained by Chen [6], [7]. After this many authors derived Chen inequalities for complex and contact space forms [1], [11], [12] and on hyper surfaces of a Lorentzian manifold [9].
After the generalization of complex and contact space forms into S-space form [4], it is natural to study the inequalities of Ricci curvature for submanifolds of S-space forms. Geometry of S-space forms were studied by many authors i.e. [10], [13].
In this paper we find relations between squared mean curvature and Ricci curvature for the sub manifolds of S-space form and discuss this relation for invariant, anti invariant and CR sub manifolds of S-space form. After introduction, second section contains basics of S-space forms and submanifolds. Third section contains main results.

2. Preliminaries

This section presents some well known facts related to S-space form and sub manifolds.

Yano[14] presented that almost complex and almost contact structures can be generalized as f - structure on a smooth manifold of dimension $2m + s$. The idea for the f-structure is to consider a tensor field with condition $f^3 + f = 0$, of type (1,1) and rank $2m$.

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Consider manifold M^{2m+s} along an f-structure of rank $2m$. We take s structural vector fields $\xi_1, \xi_2, \ldots, \xi_s$ on M such as:

$$f\xi_\alpha = 0, \qquad \eta_\alpha \circ f = 0, \qquad f^2 = -I + \sum \xi_\alpha \otimes \eta_\alpha, \tag{1}$$

where η_α and ξ_α are the dual forms to each other, therefore complemented frames exist on f-structure. For f- manifold we define a Riemannian metric g as

$$g(Y, X) = g(fY, fZ) + \sum \eta_\alpha(Y)\eta_\alpha(Z)$$

for vector fields Y and Z on M [4].

An f-structure f is normal, if there exist complemented frames and

$$[f, f] + 2\sum \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f, f]$ is Nijenhuis torsion of f. Let fundamental 2-form B be defined as $B(Y, Z) = g(Y, fZ)$, Y, $Z \in T(M)$. f-structure that is normal and $d\eta_1 = \cdots = d\eta_s = B$ is known as an S-structure. A smooth manifold along with an S-structure known as an S-manifold. Blair described such types of manifolds in [4].

For Sasakian manifolds we take $s = 1$. For $s \geq 2$ we may have some attractive applications discussed in [4].

If M is an S-manifold, then we consider the formulas [4]:

$$\widetilde{\nabla}_Y \xi_\alpha = -fY, \qquad Y \in T(M), \quad \alpha = 1, \ldots, s, \tag{2}$$

$$(\widetilde{\nabla}_Y f)Z = \sum_\alpha \{g(fY, fZ)\xi_\alpha + \eta_\alpha(Z)f^2 Y\}, \quad Y, Z \in T(M), \tag{3}$$

where $\widetilde{\nabla}$ is the Riemannian connection of g. The projection tensor $-f^2$ determine the distribution L and $f^2 + I$ determine the complementary distribution $\mathfrak{M}$ which is determined and spanned by $\xi_1, \ldots, \xi_s$. It can be observe that if $Y \in L$ then $\eta_\alpha(Y) = 0$ for all α, and for $Y \in \mathfrak{M}$, we have $fY = 0$. A plane section Π on M is said to be f - section if it is established by a vector $Y \in L(p)$, $p \in M$, such that $\{Y, fY\}$ span the section. We take the Sectional curvature of Π as the f-sectional curvature. If M is an S-manifold of constant f-sectional curvature k, then its curvature tensor is as:

$$\begin{aligned}
\widetilde{R}(Y, Z)U \;=\; & \sum_{\alpha,\beta} \Big\{ \eta^\alpha(Y)\eta^\beta(U)f^2 Z - \eta^\alpha(Z)\eta^\beta(U)f^2 Y - g(fY, fU)\eta^\alpha(Z)\xi_\beta \\
& + g(fZ, fU)\eta^\alpha(Y)\xi_\beta \} + \frac{k + 3s}{4} \{-g(fZ, fU)f^2 Y + g(fY, fU)f^2 Z\} \\
& + \frac{k - s}{4} \{g(Y, fU)fZ - g(Z, fU)fY + 2g(Y, fZ)fU\},
\end{aligned} \tag{4}$$

Y, Z, U $\in T(M)$. Such a manifold $M(k)$ will be called an S-space form. Examples of S-space forms are The Euclidean space E^{2n+s} and the hyperbolic space H^{2n+s} .

Consider immersed submanifold M^m of N^{2n+s}. Then M is an invariant submnaifold if $\xi_\alpha \in TM$ for any α and $fY \in TM$ for any $Y \in TM$. It is said to be anti-invariant submanifold if $fY \in TM^\perp$ for any $Y \in TM$. For a vector field $Y \in TM^\perp$, it can be written as $fY = tY + SY$, where tY represents tangent component of fY, SY shows normal component of fY. If S does not disappear, then its an f-structure [5].
For a vector field $Z \in TM$, it can be written as $fZ = PZ + NZ$, where PZ is tangent component of fZ, NZ is normal component of fZ.

Let $dim(M) \geq s$ and we take the structure vector fields $\xi_1, \xi_2, \ldots, \xi_s$ as tangents to M. M is known as CR-submaifold of N if we have two differentiable distributions D and $D^\perp$ on M, $TM = D + D^\perp$ satisfying

- D and $D^\perp$ are mutually orthogonal to each other.
- If $fD_p = D_p$, for any $p \in M$ then the distribution D is known as invariant under f.
- If $fD_p^\perp \subseteq T_pM^\perp$, for any $p \in M$ then the distribution $D^\perp$ is known as anti invariant under f.

The Gauss equation for the submanifold M^m into a $(m+q)$-dimensional Riemannian manifold N^{m+q} is

$$\widetilde{R}'(Y,Z,U,W) = R'(Y,Z,U,W) + g(h'(Y,U), h'(Z,W)) - g(h'(Y,W), h'(Z,U)).$$

Consider k-plan section of T_pM denoted by D and let X be a unit vector in D. We can choose orthonormal basis $e_1, e_2, \ldots, e_k$ of D and we consider $e_1 = X$, then Ricci curvature Ric_D of D at X can be define as

$$Ric_D(x) = K_{12} + K_{13} + \cdots + K_{1k},$$

where K_{ij} represents sectional curvature of 2-plane section spanned by e_i, e_j. Therefore the scalar curvature τ of the k-plane section D can be written as

$$\tau(D) = \sum_{1 \leq i < j \leq k} K_{ij}.$$

□.

3. Main Results

Theorem 1. *Let M^m, be a sub manifold of S-space form $N^{2n+s}(K)$ of constant f-sectional curvature k, with structural vector fields $\xi_1, \xi_2, \ldots, \xi_s$ tangent to M^m, then for each unit vector $X \in T_pM$, orthogonal to $\xi_1, \xi_2, \ldots, \xi_s$*

$$Ric(X) \leq \frac{1}{4}\left[m^2 \parallel H \parallel^2 + (m-1)(k+3s) + \frac{3}{2} \parallel PX \parallel^2 (k-s) - \frac{1}{2}(3s-1)(k+3s-4)\right] \quad (5)$$

Proof. Let M^m be a sub manifold of S-space form $N^{2n+s}(K)$ with constant f-sectional curvature K. For a point $p \in M$, take a unit vector $X \in T_pM$. Choose an orthonormal basis
$e_1, e_2, \ldots, e_{m-s}, e_{m-s+1}, \ldots, e_{m-s+s}, e_{m-s+s+1}, \ldots, e_{2n+s}$ on T_pN. Then it is clear that $e_1, e_2, \ldots, e_{m-s}, e_{m-s+1}, \ldots, e_{m-s+s}$ are unit tangent vectors to M at p. We take $X = e_1$. From the curvature tensor of S-space form $N^{2n+s}(K)$, after summation $1 \leq i, j \leq m$ we have

$$\widetilde{R}'(e_i, e_j, e_j, e_i) = \frac{1}{4}(k+3s)m(m-1) + \frac{k+3s-4}{4}(2s-2ms) + \frac{3}{4}(k-1) \parallel P \parallel^2 .$$

Using Gauss equation

$$\frac{1}{4}(k+3s)m(m-1) + \frac{k+3s-4}{4}(2s-2ms) + \frac{3}{4}(k-1) \parallel P \parallel^2 = 2\tau - m^2 \parallel H \parallel^2 + \parallel h \parallel^2, \quad (6)$$

where H represents mean curvature, h represents second fundamental form and τ is scalar curvature. Now we take the notation for second fundamental form h as $h^r_{ij} = g(h(e_i, e_j), e_r)$, $\parallel h \parallel^2 = \sum_{i,j=1}^m g(h(e_i, e_j), h(e_i, e_j))$ and $\parallel P \parallel^2 = \sum_{i,j=1}^m g^2(Pe_i, e_j)$. Also we know that mean curvature $H(p) = \frac{1}{m}\sum_{i=1}^m h(e_i, e_i)$. Then from (6), we have

$$\begin{aligned} n^2 \parallel H \parallel^2 &= 2\tau + \frac{1}{2}\sum_{r=m+1}^{2n+s}\left[(h^r_{11} + \cdots + h^r_{mm})^2 + (h^r_{11} - h^r_{22} - \cdots - h^r_{mm})^2\right] \\ &+ 2\sum_{r=m+1}^{2n+s}\sum_{i<j}(h^r_{ij})^2 - 2\sum_{r=m+1}^{2n+s}\sum_{2\leq i<j\leq n} h^r_{ii}h^r_{jj} \\ &- \frac{1}{4}(k+3s)m(m-1) - \frac{k+3s-4}{4}(2s-2ms) - \frac{3}{4}(k-s) \parallel P \parallel^2 \quad (7) \end{aligned}$$

Since sectional curvature for sub manifold M can be represented as

$$\begin{aligned} \sum_{2\leq i,j\leq n} K_{ij} &= \frac{1}{4}(k+3s)(m-2)(m-1) - \frac{k+3s-4}{4}(2(m-2)s - (s-1)) \\ &+ \frac{1}{4}(k-s)\left(3 \parallel P \parallel^2 - 3 \parallel Pe_1 \parallel^2\right). \quad (8) \end{aligned}$$

By Gauss equation, on S-space form we can write

$$\begin{aligned} \sum_{2\leq i,j\leq n} \tilde{K}_{ij} &= \frac{1}{4}(k+3s)(m-2)(m-1) - \frac{k+3s-4}{4}(2(m-2)s - (s-1)) \\ &+ \frac{1}{4}(k-s)\left(3 \parallel P \parallel^2 - 3 \parallel Pe_1 \parallel^2\right) + \sum_{r=m+1}^{2m+s}\sum_{2\leq i<j\leq n}(h^r_{ii}h^r_{jj} - (h^r_{ij})^2) \quad (9) \end{aligned}$$

Substituting (9) in (7), we get

$$\begin{aligned}\frac{1}{2}n^2 \parallel H \parallel^2 &= 2Ric(X) + \frac{1}{2}\sum_{r=m+1}^{2n+s}(h_{11}^r - h_{22}^r - \cdots - h_{mm}^r)^2 + \sum_{r=m+1}^{2n+s}\sum_{j=1}^{m}(h_{1j}^r)^2 \\ &- 2(m-1)\frac{k+3s}{4} - 3 \parallel PX \parallel^2 \frac{k-s}{4} + (3s-1)\frac{k+3s-4}{4}, \qquad (10)\end{aligned}$$

therefore

$$\frac{1}{2}n^2 \parallel H \parallel^2 \geq 2Ric(X) - 2(m-1)\frac{k+3s}{4} - 3 \parallel PX \parallel^2 \frac{k-s}{4} + (3s-1)\frac{k+3s-4}{4}, \quad (11)$$

that gives required result.

Corollary 1. *In Theorem 1, equality holds at point $p \in M$, if*

(i) $H(p) = 0$ and X is normal to T_pM

(ii) p is totally geodesic point

Proof. (1) When $H(p) = 0$, we get $h_{11}^r = h_{22}^r = \cdots = h_{mm}^r = 0$ and when X is normal to T_pM, we have $h_{12}^r = h_{13}^r = \cdots = h_{1m}^r = 0$, $r \in \{m+1, \ldots, 2n+s\}$ and from (10) equality holds.
(2)When p is totally geodesic point, $h_{ij}^r = 0$ for all i, j, $r \in \{m+1, \ldots, 2n\}$ and from (10) equality holds.

Corollary 2. *Let M be an m-dimensional invariant sub manifold tangent to structure vector fields $\xi_1, \xi_2, \ldots, \ldots, \xi_s$ in S-space form $N^{2n+s}(k)$, then for each unit vector $X \in T_pM$ orthogonal to $\xi_1, \xi_2, \ldots, \ldots, \xi_s$, we have*

$$Ric(X) \leq \frac{1}{4}\left[(m-1)(k+3s) + \frac{3}{2}(k-s) - \frac{1}{2}(3s-1)(k+3s-4)\right] \quad (12)$$

Proof. We know that every invariant sub manifold of S-space form is minimal [14] i.e. mean curvature vanishes and since X in unit tangent vector to M, $\parallel fX \parallel = \parallel PX \parallel = \parallel X \parallel = 1$, therefore from (5), we get result.

Corollary 3. *Let M be an m-dimensional anti-invariant sub manifold tangent to structure vector fields $\xi_1, \xi_2, \ldots, \ldots, \xi_s$ in S-space form $N^{2n+s}(k)$, then for each unit vector $X \in T_pM$ orthogonal to $\xi_1, \xi_2, \ldots, \ldots, \xi_s$, we have*

$$Ric(X) \leq \frac{1}{4}\left[m^2 \parallel H \parallel^2 + (m-1)(k+3s) - \frac{1}{2}(3s-1)(k+3s-4)\right]. \quad (13)$$

Proof. Here for anti invariant sub manifolds, $\parallel PX \parallel = 0$ and we have required result.

Corollary 4. *Let M be an m-dimensional CR-submanifold of S-space form $N^{2n+s}(K)$ then*
(i) For each unit vector $X \in D_p$,

$$Ric(X) \leq \frac{1}{4}\left[m^2 \parallel H \parallel^2 +(m-1)(k+3s) + \frac{3}{2}(k-s) - \frac{1}{2}(3s-1)(k+3s-4)\right]. \quad (14)$$

(i) For each unit vector $X \in D_p^{\perp}$,

$$Ric(X) \leq \frac{1}{4}\left[m^2 \parallel H \parallel^2 +(m-1)(k+3s) - \frac{1}{2}(3s-1)(k+3s-4)\right]. \quad (15)$$

□

Let M^m, be a submanifold of $N^{2n+s}(K)$ s-space form of constant f-sectional curvature k, using the results of Theorem 1, we can find a relation between the k-Ricci curvature of M^m and the squared mean curvature $\parallel H \parallel^2$. Before this result first we define shape operator.
Let $p \in M^m$ and $\{e_1, e_2, \ldots, e_m\}$ an orthonormal basis of T_pM^m. We choose the ortho normal basis $\{e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_{2n+s}\}$ on T_pN^{2n+s}. Then shape operator takes the form (see [12])

$$A_{e_{m+1}} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & 0 & a_m \end{pmatrix}$$

$A_{e_r} = (h_{ij}^r)$, $i, j = 1, \ldots, m$; $r = m+2, \ldots, 2n+s$, $traceA_{e_r} = 0$.
We can take

$$m^2 \parallel H \parallel^2 = \left(\sum_{i=1}^{m} a_i\right)^2 = \sum_{i=1}^{m} a_i^2 + 2\sum_{i<j} a_i a_j \leq m \sum_{i=1}^{m} a_i^2.$$

Since we know that $0 \leq \sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_{i=1}^{m} a_i^2 - 2\sum_{i<j} a_i a_j$, therefore

$$\sum_{i=1}^{m} a_i^2 \geq m \parallel H \parallel^2 . \quad (16)$$

Proposition 3.1. *Let M^m, be a submanifold of s-space form $N^{2n+s}(K)$ of constant f-sectional curvature k, tangent to $\xi_1, \xi_2, \ldots, \xi_s$ then we have*

$$m^2 \parallel H \parallel^2 \geq \frac{2\tau}{m(m-1)} - \frac{1}{4}(k+3s) - \frac{k+3s-4}{4}\frac{(2s-2ms)}{m(m-1)} - \frac{3}{4}(k-1)\frac{\parallel PX \parallel^2}{m(m-1)}$$

Proof. We choose orthonormal basis $e_1, e_2, \ldots, e_{m-s}, e_{m-s+1}, \ldots, e_{m-s+s}, e_{m-s+s+1}, \ldots, e_{2n+s}$ on T_pN with e_{m+1} parallel to mean curvature vector $H(p)$. From (6)

$$\begin{aligned} m^2 \parallel H \parallel^2 &= 2\tau + \parallel h \parallel^2 - \frac{1}{4}(k+3s)m(m-1) - \frac{k+3s-4}{4}(2s-2ms) - \frac{3}{4}(k-1) \parallel P \parallel^2 \\ &= 2\tau + \sum_{i=1}^{m} a_i^2 + \sum_{r=m+2}^{2n} \sum_{i,j=1}^{m} (h_{ij}^r)^2 - \frac{1}{4}(k+3s)m(m-1) \\ &- \frac{k+3s-4}{4}(2s-2ms) - \frac{3}{4}(k-1) \parallel PX \parallel^2 \end{aligned}$$

by (16)

$$\begin{aligned} m^2 \parallel H \parallel^2 &\geq 2\tau + m \parallel H \parallel^2 - \frac{1}{4}(k+3s)m(m-1) \\ &- \frac{k+3s-4}{4}(2s-2ms) - \frac{3}{4}(k-1) \parallel PX \parallel^2 \end{aligned}$$

which proves the required result.

Remark 1.

- *For $s = 0$, we have the results of Theorem 1, Corollary 3, Corollary 2, Proposition 3.1 for Khaler manifold.*

- *For $s = 1$, we have the results of Theorem 1, Corollary 3, Corollary 2, Proposition 3.1 for Sasakian Manifolds.*

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