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On β-Open Sets and Ideals in Topological Spaces

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Abstract. Let X be a topological space and I be an ideal in X. A subset A of a topological space X is called a β-open set if $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$. A subset A of X is called β-open with respect to the ideal I, or β_I-open, if there exists an open set U such that (1) $U - A \in I$, and (2) $A - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) \in I$. A space X is said to be a β_I-compact space if it is β_I-compact as a subset. An ideal topological space (X, τ, I) is said to be a $c\beta_I$-compact space if it is $c\beta_I$-compact as a subset. An ideal topological space (X, τ, I) is said to be a countably β_I-compact space if X is countably β_I-compact as a subset. Two sets A and B in an ideal topological space (X, τ, I) is said to be β_I-separated if $\mathrm{cl}_{\beta_I}(A) \cap B = \varnothing = A \cap \mathrm{cl}_{\beta}(B)$. A subset A of an ideal topological space (X, τ, I) is said to be β_I-connected if it cannot be expressed as a union of two β_I-separated sets. An ideal topological space (X, τ, I) is said to be β_I-connected if X β_I-connected as a subset.
In this study, we introduced the notions β_I-open set, β_I-compact, $c\beta_I$-compact, β_I-hyperconnected, $c\beta_I$-hyperconnected, β_I-connected and β_I-separated. Moreover, we investigated the concept β-open set by determining some of its properties relative to the above-mentioned notions.

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1. Introduction

Topology is an interesting area of mathematics. It is new, being conceive in the 19th century. But according to Morris [15], the influence of topology is so vast, so that it is identifiable in various branches of mathematics.

Topological ideas are present not only in mathematics but also in other areas, for example biochemistry [16] and information systems [17]. Topology as a subject has several different branches such as point set topology, algebraic topology, differential topology, etc.

The basic component of a topology space are open sets, and overtime there have been so many generalizations of it. Among them are the following. Stone [18] introduced

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the concept of regular open sets. Levine [19] introduced the concept of semi open sets. Najasted [20] introduced the concept of α-open sets. Mashhour et al. [31] introduced the concept of pre-open sets. Abd El-Monsef et al. [1] introduced the concept of β-open sets.

Apart from introducing β-open sets, Abd El-Monsef et al. [1] also introduced β-continuous mappings and β-open mappings. They studied their properties and discussed the connections of these notions with the existing ones. Since then, the concept β-open sets has been a subject of a couple of investigations. Among them were the following. Abid [22] used the concept β-open set to obtain the properties of the concept non-semi-predense set. Tahiliani [23] introduced an operation on a family of β-open sets; and using the operation, the concept β-γ-open sets was defined and investigated. Kannan and Nagaveni [5] introduced another generalization of the concept β-open sets, called $\hat{\beta}$-generalized closed sets. Mubarki et al. [25] introduced and investigated β^*-open sets, which is also a generalization of the concept β-open sets. El-Mabhouh and Mizyed [26] introduced the concept βc-open set which is a particular class of β-open sets. They also showed that βc-open sets generates the same topology as the class of θ-open sets in Alexandroff space. Akdag and Ozkan [27] adapted the concept β-open set in soft topological spaces, and defined the concepts soft β-interior and soft β-closure, and gave their properties. Arockiarani and Arokia Lancy [28] presented $g\beta$-closed sets and $gs\beta$-closed sets (which were defined indirectly in terms of the notion of β-open sets) and introduced parallel concepts in soft topological spaces.

Let X be a topological space and I be an ideal in X. A subset A of a topological space X is called a β-open set if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. A subset A of X is called β-open with respect to the ideal I, or β_I-open, if there exists an open set U such that (1) $U - A \in I$, and (2) $A - \text{cl}(\text{int}(\text{cl}(U))) \in I$. A subset A of an ideal topological space (X, τ, I) is said to be β_I-compact if every cover of A by β_I-open set has a finite sub-cover. A space X is said to be a β_I-compact space if it is β_I-compact as a subset. A subset A of an ideal topological space (X, τ, I) is said to be a compatible β_I-compact, or simply $c\beta_I$-compact, if every cover $\{U_\lambda : \lambda \in \Lambda\}$ of A by β-open set has a finite subset Λ_0 of Λ such that $A - \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in I$. An ideal topological space (X, τ, I) is said to be a $c\beta_I$-compact space if it is $c\beta_I$-compact as a subset.

A subset A of an ideal topological space (X, τ, I) is said to be countably β_I-compact if every countable cover $\{U_n : n \in \mathbb{N}\}$ of A by β_I-open set, there exists a finite subset $\{i_1, i_2, \ldots, i_k\}$ of $\mathbb{N}$ such that $A - \bigcup\{U_{i_j} : j = 1, 2, \ldots, k\} \in I$. An ideal topological space (X, τ, I) is said to be a countably β_I-compact space if X is countably β_I-compact as a subset.

The concept $*$-hyperconnectedness was introduced by Ekici et al. [2], and the concept $I*$-hyperconnectedness was introduced by Abd El-Monsef et al. [7]. As defined in [4] an ideal topological space (X, τ, I) is said to be $*$-hyperconnected if $\text{cl}^*(A) = X$ for every non-empty open subset A of X, and as defined in [3], an ideal topological space (X, τ, I) is said to be I^*-hyperconnected if $X - \text{cl}^*(A) \in I$ for every non-empty open subset A of X. Given these insights, we introduce the following parallel concept. An ideal topological space (X, τ, I) is said to be β_I^*-hyperconnected if $X - \text{cl}^*(A) \in I$ for every non-empty β_I-open subset A of X.

For the concepts that were not discussed here please refer to [6, 14, 15].

2. β-Open Sets with Respect to an Ideal

In this section, we investigated the concept β-open in a direction parallel to the investigation of semi-open sets in [32].

Lemma 1. *Let (X, τ) be a topological space and A be a subset of X. Then $int(A) = int(cl(A)) = int(cl(int(A)))$.*

Proof. Let (X, τ) be a topological space and A be a subset of X. Then we have, $\text{int}(\text{cl}(A)) = \text{int}(\text{Fr}(A) \cup \text{int}(A)) = \text{int}(A) = \text{int}(\text{Fr}(\text{int}(A)) \cup \text{int}(A)) = \text{int}(\text{cl}(\text{int}(A)))$. □

Lemma 2 characterizes β-open sets.

Lemma 2. *Let (X, τ, I) be an ideal topological space. A subset A of X is β-open if and only if there exists an open set U such that $U \subseteq A \subseteq cl(int(cl(U)))$.*

Proof. Assume that A is β-open. Then $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. Let $U = \text{int}(A)$. Then U is open and, by Lemma 1 $U \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(A))) = \text{cl}(\text{int}(\text{cl}(\text{int}(A)))) = \text{cl}(\text{int}(\text{cl}(U)))$.

Conversely, assume that there exists an open set U such that $U \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(U)))$. Since $U \subseteq A$, $\text{cl}(U) \subseteq \text{cl}(A)$. Hence, $\text{int}(\text{cl}(U)) \subseteq \text{int}(\text{cl}(A))$. And so, $\text{cl}(\text{int}(\text{cl}(U))) \subseteq \text{cl}(\text{int}(\text{cl}(A)))$ Thus, $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. □

Lemma 3 says that every open set is a β_I-open set, every element of the ideal is a β_I-open set, and every β-open set is a β_I-open set.

Lemma 3. *Let (X, τ, I) be an ideal topological space.*

(i) If A is an open set, then A is an β_I-open set.

(ii) If $A \in I$, then A is an β_I-open set.

(iii) If A is a β-open set, then A is an β_I-open set.

Proof. (1) If A is open, then we let $U = A$. Observe that $U - A = \varnothing \in I$, and $A - \text{cl}(\text{int}(\text{cl}(U))) = A - \text{cl}(U) = A - \text{cl}(A) = \varnothing \in I$. This shows that A is an β_I-open set. (2) If $A \in I$, then we let $U = \varnothing$. Observe that $U - A = \varnothing \in I$, and $A - \text{cl}(\text{int}(\text{cl}(U))) = A - \varnothing = A \in I$. This shows that A is an β_I-open set. (3) If A is β-open, then by Lemma 2 there exists an open set U such that $U \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(U)))$. Observe that $U - A = \varnothing \in I$, and $A - \text{cl}(\text{int}(\text{cl}(U))) = \varnothing \in I$. This shows that A is an β_I-open set. □

Lemma 4 says that if I is the minimal ideal, then the β_I-open sets are precisely the β-open sets.

Lemma 4. *Let (X, τ, I) be an ideal topological space. If I is not countably additive, then the following statements are equivalent.*

(i) If $I = \{\varnothing\}$.

(ii) A is a β-open set if and only if A is a β_I-open set.

Proof. Assume that $I = \{\varnothing\}$, and A be a β-open set. Then by Lemma 3, A is a β_I-open set. Conversely, let A be a β_I-open set. Then there exists an open set U such that $U - A \in I$ and $A - \text{cl}(\text{int}(\text{cl}(U))) \in I$. Since $\varnothing$, $U - A \in \varnothing$ and $A - \text{cl}(\text{int}(\text{cl}(U))) \in \varnothing$, that is, $U \subseteq A$ and $A \subseteq \text{cl}(\text{int}(\text{cl}(U)))$. Thus, $U \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(U)))$. By Lemma 2, A is a β-open set.

Next, assume that A is a β-open set if and only if A is a β_I-open set, and suppose that $I \neq \{\varnothing\}$. Let $B \in I$ with $B \neq \varnothing$. Then by Lemma 3, B is a β_I-open set. By assumption B is a β-open set. By Lemma 2, there exists an open set U_1 such that $U_1 \subseteq B \subseteq \text{cl}(\text{int}(\text{cl}(U_1)))$. Since $B \in I$ and $U_1 \subseteq B$, $U_1 \in I$. Hence, $U_1 \cup B \in I$. By Lemma 2, $U_1 \cup B$ is a β_I-open set. By assumption $U_1 \cup B$ is a β-open set. Thus, there exists an open set U_2 such that $U_2 \subseteq (U_1 \cup B) \subseteq \text{cl}(\text{int}(\text{cl}(U_2)))$. Since $U_1 \cup B \in I$ and $U_2 \subseteq U_1 \cup B$, $U_2 \in I$. Hence, $U_1 \cup U_2 \cup B \in I$. By Lemma 2, $U_1 \cup U_2 \cup B$ is a β_I-open set. By assumption $U_1 \cup U_2 \cup B$ is a β-open set. Thus, there exists an open set U_3 such that $U_3 \subseteq (U_1 \cup U_2 \cup B) \subseteq \text{cl}(\text{int}(\text{cl}(U_3)))$. Since $U_1 \cup U_2 \cup B \in I$ and $U_3 \subseteq U_1 \cup U_2 \cup B$, $U_3 \in I$. Hence, $U_1 \cup U_2 \cup U_3 \cup B \in I$. Continuing in this manner we obtain a sequence $\langle U_1, U_2, U_3, \ldots \rangle$ of set in I such that $U_1 \cup U_2 \cup U_3 \cup \cdots \in I$. This is a contradiction since I is not countably additive. Therefore, $I = \{\varnothing\}$. □

Theorem 1 says that if I is the minimal ideal, then the notions β-compact, β_I-compact and $c\beta_I$-compact coincides.

Theorem 1. *For an ideal topological space (X, τ, I), the following statements are equivalent.*

(i) (X, τ) is a β-compact space.

(ii) $(X, \tau, \{\varnothing\})$ is a β_I-compact space.

(iii) $(X, \tau, \{\varnothing\})$ is a $c\beta_I$-compact space.

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds, and let $\{U_\lambda : \lambda \in \Lambda\}$ be a cover of X by β-open sets. By Lemma 3 U_λ is a β_I-open set for all $\lambda \in \Lambda$. Since X is β-compact, there exists a finite subset Λ_0 of Λ such that $\{U_\lambda : \lambda \in \Lambda_0\}$ is still a cover of X. By Lemma 4, U_λ is a β_I-open set for all $\lambda \in \Lambda_0$. Hence, there exists a finite subcover of X by β_I-open sets. This shows that X is a β_I compact space.

(2) $\Rightarrow$ (3) Assume that (2) holds, and let $\{U_\lambda : \lambda \in \Lambda\}$ be a cover of X by β-open sets. By Lemma 3 U_λ is a β_I-open set for all $\lambda \in \Lambda$. By assumption, there exists a finite subset

Λ_0 of Λ such that $\{U_\lambda : \lambda \in \Lambda_0\}$ is still a cover of X, that is $X - \bigcup_\lambda \in \Lambda = \varnothing \in I$. This show that (3) holds.
(3) $\Rightarrow$ (1) Assume that (3) holds, and let $\{U_\lambda : \lambda \in \Lambda\}$ be a cover of X by β-open sets. By Lemma 4, if $I = \{\varnothing\}$, then a β-open sets is precisely a β_I-open set. Thus, U_λ is at the same time a β_I-open set for all $\lambda \in \Lambda$. By assumption, there exists a finite subset Λ_0 of Λ such that $X - \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in I$, that is $X - \bigcup\{U_\lambda : \lambda \in \Lambda_0\} = \varnothing$ ($X \subseteq \bigcup\{U_\lambda : \lambda \in \Lambda_0\}$). Hence, there exists a finite subset Λ_0 of Λ such that $X \subseteq \bigcup\{U_\lambda : \lambda \in \Lambda_0\}$.This show that (1) holds. □

Theorem 2 characterizes β_I-compact space.

Theorem 2. *For an ideal topological space (X, τ, I), the following statements are equivalent.*

(i) (X, τ, I) is a β_I-compact space.

(ii) For every family $\{F_\lambda : \lambda \in \Lambda\}$ of β_I-closed sets such that $\bigcap\{F_\lambda : \lambda \in \Lambda\} = \varnothing$, there exist a finite subset Λ_0 of Λ such that $\bigcap\{F_\lambda : \lambda \in \Lambda_0\} = \varnothing$.

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds, and let $\{F_\lambda : \lambda \in \Lambda\}$ be a family of β_I-closed sets such that $\bigcap\{F_\lambda : \lambda \in \Lambda\} = \varnothing$. If $\bigcap\{F_\lambda : \lambda \in \Lambda\} = \varnothing$, then $\bigcup\{F_\lambda^C : \lambda \in \Lambda\} = (\bigcap\{F_\lambda : \lambda \in \Lambda\})^C = X$. Hence, $\bigcup\{F_\lambda^C : \lambda \in \Lambda\}$ is a covering of X by β_I-open sets. By assumption there exists a finite subset Λ_0 of Λ such that $\bigcup\{F_\lambda^C : \lambda \in \Lambda_0\} = X$, that is $\bigcap\{F_\lambda : \lambda \in \Lambda_0\} = \varnothing$.
(2) $\Rightarrow$ (1) Assume that (2) holds, and let $\{U_\lambda : \lambda \in \Lambda\}$ be a cover of X by β_I-open sets. If $\{U_\lambda : \lambda \in \Lambda\}$ is a cover of X by β_I-open sets, that is $\bigcup\{U_\lambda : \lambda \in \Lambda\} = X$, then $\bigcap\{U_\lambda^C : \lambda \in \Lambda\} = (\bigcup\{U_\lambda : \lambda \in \Lambda\})^C = \varnothing$. By assumption, there exists a finite subset Λ_0 of Λ such that $\bigcap\{U_\lambda^C : \lambda \in \Lambda_0\} = \varnothing$, that is $\bigcup\{U_\lambda : \lambda \in \Lambda_0\} = X$. This show that (1) holds. □

Theorem 3 characterizes β_I-compact space.

Theorem 3. *In an ideal topological space (X, τ, I), the following statements are equivalent.*

(i) (X, τ, I) is a $c\beta_I$-compact space.

(ii) For every family $\{F_\lambda : \lambda \in \Lambda\}$ of β_I-closed sets such that $\bigcap\{F_\lambda : \lambda \in \Lambda\} = \varnothing$, there exist a finite subset Λ_0 of Λ such that $\bigcap\{F_\lambda : \lambda \in \Lambda_0\} \in I$.

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds, and let $\{F_\lambda : \lambda \in \Lambda\}$ be a family of β_I-closed sets such that $\bigcap\{F_\lambda : \lambda \in \Lambda\} = \varnothing$. If $\bigcap\{F_\lambda : \lambda \in \Lambda\} = \varnothing$, then $\bigcup\{F_\lambda^C : \lambda \in \Lambda\} = (\bigcap\{F_\lambda : \lambda \in \Lambda\})^C = X$. Hence, $\bigcup\{F_\lambda^C : \lambda \in \Lambda\}$ is a covering of X by β_I-open sets. By assumption there exists a finite subset Λ_0 of Λ such that $X - \bigcup\{F_\lambda^C : \lambda \in \Lambda_0\} \in I$, that is $\bigcap\{F_\lambda : \lambda \in \Lambda_0\} \in I$.
(2) $\Rightarrow$ (1) Assume that (2) holds, and let $\{U_\lambda : \lambda \in \Lambda\}$ be a cover of X by β_I-open sets. If $\{U_\lambda : \lambda \in \Lambda\}$ is a cover of X by β_I-open sets, that is $\bigcup\{U_\lambda : \lambda \in \Lambda\} = X$, then

$\bigcap\{U_\lambda^C : \lambda \in \Lambda\} = (\bigcup\{U_\lambda : \lambda \in \Lambda\})^C = \varnothing$. By assumption, there exists a finite subset Λ_0 of Λ such that $\bigcap\{U_\lambda^C : \lambda \in \Lambda_0\} \in I$, that is $X - \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in I$. This show that (1) holds. □

Remark 1. *[30] Let (X, τ, I) and (Y, σ, J) be ideal topological spaces.*

(i) If $f : (X, \tau, I) \to (Y, \sigma)$ is a function, then $f(I) = \{f(A) : A \in I\}$ is an ideal in Y.

(ii) If $f : (X, \tau) \to (Y, \sigma, J)$ is an injective function, then $f^{-1}(J) = \{f^{-1}(B) : B \in J\}$ is an ideal in X.

We note that a function $f : (X, \tau, I) \to (Y, \sigma, J)$ is called

(i) β_I-open function if $f(A)$ is β_J-open in Y for each β_I-open set A in X.

(ii) β_I-irresolute function if $f^{-1}(B)$ is β_I-open in X for each β_J-open set B in Y.

(iii) β_I-continuous function if $f^{-1}(B)$ is β_I-open in X for each open set B in Y.

The following Theorems are worth-noting.

Theorem 4. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is a β_I-irresolute surjective function and (X, τ, I) is a $c\beta_I$-compact space, then (Y, σ, J) is also a $c\beta_I$-compact space.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a cover of Y by β_I-open sets. Since f is a β_I-irresolute surjective function, $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is a cover of X by β_I-open sets. Since X is $c\beta_I$-compact, there exists a finite subset Λ_0 of Λ such that $X - \bigcup\{f^{-1}(U_\lambda) : \lambda \in \Lambda_0\} \in I$. By Remark 1, $Y - \cup\{f(U_\lambda) : \lambda \in \Lambda_0\} = f(X - \bigcup\{f^{-1}(U_\lambda) : \lambda \in \Lambda_0\}) \in J$. □

Theorem 5. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is a $\beta_{f^{-1}(J)}$-open bijective function and (Y, σ, J) is a $c\beta_J$-compact space, then $(X, \tau, f^{-1}(J))$ is also a $c\beta_{f^{-1}(J)}$-compact space.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a cover of X by $\beta_{f^{-1}(J)}$-open sets. Since f is an open bijective function, $\{f(U_\lambda) : \lambda \in \Lambda\}$ is a cover of Y by β_J-open sets. Since Y is a $c\beta_J$-compact space, there exists a finite subset Λ_0 of Λ such that $Y - \bigcup\{f(U_\lambda) : \lambda \in \Lambda_0\} \in J$, that is $X - \cup\{U_\lambda : \lambda \in \Lambda_0\} = f^{-1}(Y - \bigcup\{f(U_\lambda) : \lambda \in \Lambda_0\}) \in J$. This shows that $(X, \tau, f^{-1}(J))$ is a $c\beta_{f^{-1}(J)}$-compact space. □

Theorem 6. *Every $c\beta_I$-compact space is also a countably β_I-compact space.*

Proof. Let (X, τ, I) be $c\beta_I$-compact space. Let $\{U_n : n \in \mathbb{N}\}$ be a countable cover of X by β_I-open sets. Since X is a $c\beta_I$-compact space, there exists a finite subset $\{i_j : j = 1, 2, \ldots, k\}$ of $\mathbb{N}$ such that $X - \bigcup\{U_{i_j} : j = 1, 2, \ldots, k\} \in I$. This shows that (X, τ, I) is a countably β_I-compact space. □

3. Hyperconnectedness with Respect to Ideals

The concept $*$-hyperconnectedness was introduced by Ekici et al. [2], and the concept $I*$-hyperconnectedness was introduced by Abd El-Monsef et al. [7]. These insights propelled us to create a parallel concept called β_I*-hyperconnectedness, and the investigation in this section is parallel to the investigation of α-open sets in [24].

Theorem 7. *Every β_I*-hyperconnected space is also an $I*$-hyperconnected space.*

Proof. Let (X, τ, I) be a β_I^*-hyperconnected space and A be an open set. Since X is a β_I^*-hyperconnected space and every open set is a β_I-open set, $X - cl^*(A) \in I$. Thus, (X, τ, I) is also an I_*-hyperconnected space. □

Lemma 5. *The intersection of any family of ideals on X is an ideal on X.*

Theorem 8 say that if I is the minimal ideal, then the notions $*$-hyperconnectedness and I^*-hyperconnectedness are the same. Moreover, if the topological space is clopen, then the notions $*$-hyperconnectedness, I^*-hyperconnectedness and β_I^*-hyperconnectedness are the same.

Theorem 8. *Let $(X, \tau, \{\varnothing\})$ be an ideal topological space.*

(i) If $I = \{\varnothing\}$, then the concepts $*$-hyperconnectedness and I^*-hyperconnectedness are equivalent. [3]

(ii) If $I = \{\varnothing\}$ and every open set is closed, then the concepts $*$-hyperconnectedness, I^*-hyperconnectedness and β_I^*-hyperconnectedness are equivalent.

Proof. (1) If (X, τ, I) is a β_I^*-hyperconnected space and A is a non-empty open set, then $cl^*(A) = X$. Hence, $X - cl^*(A) = \varnothing \in I$. Thus, (X, τ, I) is an I^*-hyperconnected space.

Conversely, if (X, τ, I) is an I^*-hyperconnected space and A is a non-empty open set, then $X - cl^*(A) \in I$. Since $I = \{\varnothing\}$, $X - cl^*(A) = \varnothing$, that is $cl^*(A) = X$. Thus, (X, τ, I) is an $*$-hyperconnected space.

(2) Assume that $I = \{\varnothing\}$ and every open set is closed.

Claim 1. A is an open set if and only if A is a β-open set.

If A is an open set, then $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) = A$. Hence, A is a β-open set. Conversely, if A is a β-open set, then by Lemma 2 there exists an open set U such that $U \subseteq A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) = U$, that is $A = U$. Hence, A is an open set. This shows the claim.

Claim 2. The concepts I^*-hyperconnectedness and β_I^*-hyperconnectedness are equivalent. if (X, τ, I) is an I^*-hyperconnected space and A is a non-empty open set, then $X - cl^*(A) \in I$. By Claim 1 and Lemma 5, any open set is precisely a β_I-open set. Hence, $X - cl^*(A) \in I$ for every non-empty β_I-open set. Thus, (X, τ, I) is a β_I^*-hyperconnected space. Conversely, if (X, τ, I) is a β_I^*-hyperconnected space and A is a non-empty β_I-open set, then $X - cl^*(A) \in I$. By Claim 1 and Lemma 5, any open set is precisely a β_I-open set. Hence, $X - cl^*(A) \in I$ for every non-empty open set. Thus, (X, τ, I) is an I^*-hyperconnected space.

Accordingly, by statement (1) and Claim 2, statement (2) follows. □

Theorem 9. *If an ideal topological space $(X, \tau, \{\varnothing\})$ is a β_I*-hyperconnected space, then $X - cl^*(A) \in I$ for every non-empty β-open subset A of X.*

Proof. Let (X, τ, I) be a β_I^*-hyperconnected space and A is a non-empty β-open set. Since by Lemma 4 every β-open set is a β_I-open set, A is a non-empty β_I-open set also. If (X, τ, I) is a β_I^*-hyperconnected space, then $X - cl^*(A) \in I$. □

Theorem 10 characterizes a β_I^*-hyperconnected space.

Theorem 10. *For an ideal topological space (X, τ, I), the following statements are equivalent.*

(i) X is a β_I^*-hyperconnected space.

(ii) $int^*(A) \in I$ for every proper β_I-closed subset A of X.

Proof. (1) ⇒ (2) Assume that (1) holds, and let B be a β_I-closed set. If B is a β_I-closed set, then $X - B$ is a β_I-open set. Moreover, if B is a proper subset, then $B^C \neq \varnothing$. By assumption, $\text{int}^*(B) = X - \text{cl}^*(X - B) \in I$.
(2) ⇒ (1) Assume that (2) holds, and let A be a non-empty β_I-open set. If A is a non-empty β_I-open set, then $X - A$ is a proper β_I-open subset of X. By assumption, $X - \text{cl}^*(A) = X - \text{cl}^*(X - (X - A)) = \text{int}^*(X - A) \in I$. This shows that X is a β_I^*-hyperconnected space. □

4. Separation Notions with Respect to Ideals

Let (X, τ, I) be an ideal topological space and A be a subset of X. The β_I-closure of A is the smallest β_I-closed set containing A, denoted by $\text{cl}_{\beta_I}(A)$.

Recall that two sets A and B in an ideal topological space (X, τ, I) is said to be β_I-separated if $\text{cl}_{\beta_I}(A) \cap B = \varnothing = A \cap \text{cl}_\beta(B)$, and a subset A of an ideal topological space (X, τ, I) is said to be β_I-connected if it cannot be expressed as a union of two β_I-separated sets. An ideal topological space (X, τ, I) is said to be β_I-connected if X β_I-connected as a subset. A subset A of an ideal topological space (X, τ, I) is said to be β_I-connected if it cannot be expressed as a union of two β_I-separated sets. An ideal topological space (X, τ, I) is said to be β_I-connected if X β_I-connected as a subset.

Lemma 6. *Let (X, τ, I) be an ideal topological space. If A and B are non-empty disjoint subsets of X such that A is β-open and B is β_I-open, then A and B are β_I-separated sets.*

Proof. Suppose that A and B are β_I-separated sets, that is $\text{cl}_{\beta_I}(A) \cap B \neq \varnothing$ or $A \cap \text{cl}_\beta(B) \neq \varnothing$. Since A and B are non-empty disjoint subsets of X, $A \subseteq B^C$ and $B \subseteq A^C$. If A is β-open and B is β_I-open, then A^C is β-closed and B^C is β_I-closed. Hence, $B^C \cap B \supseteq \text{cl}_{\beta_I}(A) \cap B \neq \varnothing$ or $A \cap A^C \supseteq A \cap \text{cl}_\beta(B) \neq \varnothing$. This is a contradiction. □

The next statement, Lemma 7, stressed that every β_I-connected space is connected.

Lemma 7. *If an ideal topological space (X, τ, I) is β_I-connected, then (X, τ) is connected.*

Proof. Suppose that X is not connected. Let A and B be non-empty disjoint open sets such that $X = A \cup B$. Since every open set is both β-open and β_I-open, A and B are both β-open and β_I-open. Since $A = B^C$ and $B = A^C$, A and B are also both β-closed and β_I-closed. Thus $A = \mathrm{cl}_{\beta_I}(A)$ and $B = \mathrm{cl}_\beta(B)$. Hence, $\mathrm{cl}_{\beta_I}(A) \cap B = A \cap B = \varnothing$ and $A \cap \mathrm{cl}_\beta(B) = A \cap B = \varnothing$. Therefore, (X, τ, I) is not β_I-connected. $\square$

Theorem 11. *Let (X, τ, I) be an ideal topological space and Y be an open set. If A is a β_I subset of X, then $A \cap Y$ is β_{I_Y}-open subset of Y.*

Proof. If A is a β_I subset of X, then there exists an open set U' such that $U' - A \in I$ and $A - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) \in I$. Let $U = U' \cap Y$. Then

$$\begin{aligned} U - (A \cap Y) &= U \cap (A \cap Y)^C \\ &= (U' \cap Y) \cap (A^C \cup Y^C) \\ &= (U' \cap Y \cap A^C) \cup (U' \cap Y \cap Y^C) \\ &= U' \cap Y \cap A^C \\ &= (U' - A) \cap Y \in I_Y. \end{aligned}$$

Moreover,

$$\begin{aligned} (A \cap Y) - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) &= (A \cap Y) - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U' \cap Y))) \\ &= (A \cap Y) - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U'))) \cap Y \\ &= [A - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U')))] \cap Y \in I_Y. \end{aligned}$$

Therefore, $A \cap Y$ is β_{I_Y}-connected. $\square$

Remark 2. *Let (X, τ, I) be an ideal topological space. If $Y \subseteq X$, then $I_Y = \{A \cap Y : A \in I\}$ is a subset of I.*

The succeeding Theorems are worth-noting.

Theorem 12. *Let (X, τ, I) be an ideal topological space, Y be an open set, and $A \subseteq Y$. A is a β_{I_Y}-open subset of Y if and only if it is a β_I-open subset of X.*

Proof. Assume that A is a β_{I_Y}-open set. Then there exists an open set U such that $U - A \in I_Y$ and $A - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) \in I_Y$. Let $U' = U \cap Y$. Since Y is open, U' is open. Thus, by Remark 2 $U' - A \in I$ and $A - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U'))) \in I$. This shows that A is β_I-open.

Conversely, assume that A is a β_I-open set. Then there exists an open set U such that $U - A \in I$ and $A - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) \in I$. Note that $Y \cap (U - A) \in I_Y$ and $A - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) \in I_Y$, that is $(Y \cap U) - A \in I_Y$ and $A - \mathrm{cl}(\mathrm{int}(\mathrm{cl}(Y \cap U))) \in I_Y$. This shows that A is β_{I_Y}-open. $\square$

Theorem 13. *Let (X, τ, I) be an ideal topological space, Y be an open set, and $A \subseteq Y$. Then $cl_{\beta_{I_Y}}(A) = cl_{\beta_I}(A) \cap Y$.*

Proof. Let $w \notin \mathrm{cl}_{\beta_I}(A) \cap Y$. Then $w \in X - \mathrm{cl}_{\beta_I}(A)$. By Theorem 12, $(X - \mathrm{cl}_{\beta_I}(A)) \cap Y$ is a β_{I_Y}-open subset of Y. Note that w must be in $(X - \mathrm{cl}_{\beta_I}(A))$. Hence, $Y - [(X - \mathrm{cl}_{\beta_I}(A)) \cap Y]$ is a β_{I_Y}-closed set in Y, which does not contain w. Thus, $x \notin \mathrm{cl}_{\beta_I}(A)$. Therefore, $\mathrm{cl}_{\beta_{I_Y}}(A) \subseteq \mathrm{cl}_{\beta_I}(A) \cap Y$.

Next, let $z \notin \mathrm{cl}_{\beta_{I_Y}}(A)$. Then $z \in X - \mathrm{cl}_{\beta_{I_Y}}(A)$. By Theorem 12, $(Y - \mathrm{cl}_{\beta_{I_Y}}(A))$ is a β_I-open subset of X. Note that $(Y - \mathrm{cl}_{\beta_{I_Y}}(A))$ must contain w. Thus, $X - (Y - \mathrm{cl}_{\beta_{I_Y}}(A))$ is a β_I-closed set in X, which does not contain z. Thus, $\mathrm{cl}_{\beta_I}(A) = \bigcap\{F : F$ is a β_I-closed set and $A \subseteq F\}$ does not contain z, that is $z \notin \mathrm{cl}_{\beta_I}(A)$. Therefore, $\mathrm{cl}_{\beta_{I_Y}}(A) \supseteq \mathrm{cl}_{\beta_I}(A) \cap Y$. □

Theorem 14. *Let (X, τ, I) be an ideal topological space, Y be an open set, and A and B be subsets of Y. Then the following statements are equivalent.*

(i) A and B are β_{I_Y}-separated in Y.

(ii) A and B are β_I-separated in X.

Proof. (1) ⇒ (2) Assume that (1) holds. If A and B are β_{I_Y}-separated in Y, then by the assumption and by Theorem 13 $\mathrm{cl}_{\beta_I}(A) \cap B = \mathrm{cl}_{\beta_I}(A) \cap (B \cap Y) = (\mathrm{cl}_{\beta_I}(A) \cap Y) \cap B = \mathrm{cl}_{\beta_{I_Y}}(A) \cap B = \varnothing$ and $A \cap \mathrm{cl}_{\beta}(B) = (A \cap Y) \cap \mathrm{cl}_{\beta}(B) = A \cap (\mathrm{cl}_{\beta}(B) \cap Y) = A \cap \mathrm{cl}_{\beta_Y}(B) = \varnothing$. This shows that A and B are β_I-separated.
(2) ⇒ (1) Assume that (2) holds. If A and B are β_I-separated in X, then by the assumption and by Theorem 13 $\varnothing = \mathrm{cl}_{\beta_I}(A) \cap B = \mathrm{cl}_{\beta_I}(A) \cap (B \cap Y) = (\mathrm{cl}_{\beta_I}(A) \cap Y) \cap B = \mathrm{cl}_{\beta_{I_Y}}(A) \cap B$ and $\varnothing = A \cap \mathrm{cl}_{\beta}(B) = (A \cap Y) \cap \mathrm{cl}_{\beta}(B) = A \cap (\mathrm{cl}_{\beta}(B) \cap Y) = A \cap \mathrm{cl}_{\beta_Y}(B)$. This shows that A and B are β_{I_Y}-separated. □

Theorem 15. *An ideal topological space (X, τ, I) is a β_I-connected if and only if it cannot be written as a disjoint union of a non-empty β-open set and a β_I-open set.*

Proof. Suppose that (X, τ, I) is a β_I-connected space and X can be written as a disjoint union of a non-empty β-open set and a β_I-open set. Let A be a non-empty β-open set and B be a β_I-open set with $X = A \cup B$ and $A \cap B \neq \varnothing$. If $X = A \cup B$ and $A \cap B \neq \varnothing$, then $A^C = B$ and $B^C = A$. Since $A^C = B$ and $B^C = A$, A is a β_I-closed set and B be a β-closed set. Thus, $\mathrm{cl}_{\beta_I}(A) \cap B = A \cap B = \varnothing$ and $A \cap \mathrm{cl}_{\beta}(B) = A \cap B = \varnothing$. Thus, A and B are β_I-separated sets. This is a contradiction since X is a β_I-connected space.

Conversely, assume that X cannot be written as a disjoint union of a non-empty β-open set and a β_I-open set. If (X, τ, I) is not β_I-connected, then X can be written as a union of two β_I-separated sets, say A and B, with $X = A \cup B$. Thus, $\mathrm{cl}_{\beta_I}(A) \cap B = \varnothing$ and $A \cap \mathrm{cl}_{\beta}(B) = \varnothing$, that is $\mathrm{cl}_{\beta_I}(A) = B^C$ and $\mathrm{cl}_{\beta}(B) = A^C$. This implies that A is a non-empty β-open set and B is a β_I-open set. This is a contradiction since X cannot be written as a disjoint union of a non-empty β-open set and a β_I-open set. □

Theorem 16. *Let (X, τ, I) be an ideal topological space and A be an open set. If A is β_I-connected, and H and G are β_I-separated with $A \subseteq H \cup G$, then either $A \subseteq H$ or $A \subseteq G$.*

Proof. Suppose that $A \cap H \neq \varnothing$ and $A \cap G \neq \varnothing$. Since $A \subseteq H \cup G$, $A = (A \cap H) \cup (A \cap G)$. Since H and G are β_I-separated, $\mathrm{cl}_{\beta_I}(A \cap H) \cap (A \cap G) = \mathrm{cl}_{\beta_I}(H) \cap (G) = \varnothing$ and $(A \cap H) \cap \mathrm{cl}_\beta(A \cap G) = H \cap \mathrm{cl}_\beta(G) = \varnothing$. Thus, $[\mathrm{cl}_{\beta_I}(A \cap H) \cap A] \cap (A \cap G) = \varnothing$ and $(A \cap H) \cap [\mathrm{cl}_\beta(A \cap G) \cap A] = \varnothing$. By Theorem 13, $\mathrm{cl}_{\beta_{I_A}}(A \cap H) \cap (A \cap G) = \varnothing$ and $(A \cap H) \cap \mathrm{cl}_{\beta_A}(A \cap G) = \varnothing$. This implies that A is not β_I-connected. This is a contradiction. Therefore, either $A \cap H = \varnothing$ or $A \cap G = \varnothing$, that is $H \subseteq A$ or $G \subseteq A$. □

Theorem 17. *Let (X, τ, I) be an ideal topological space and, A and B be β_I-separated subsets of X. If C and D are two non-empty subsets of X such that $C \subseteq D$ and $D \subseteq B$, then C and D are also β_I-separated.*

Proof. If A and B are β_I-separated, then $\mathrm{cl}_{\beta_I}(A) \cap B = \varnothing$ and $A \cap \mathrm{cl}_\beta(B) = \varnothing$. Hence, $\mathrm{cl}_{\beta_I}(C) \cap D \subseteq \mathrm{cl}_{\beta_I}(A) \cap B = \varnothing$ and $C \cap \mathrm{cl}_\beta(D) = A \cap \mathrm{cl}_\beta(B) = \varnothing$, that is $\mathrm{cl}_{\beta_I}(C) \cap D = \varnothing = C \cap \mathrm{cl}_\beta(D)$. This shows that C and D are β_I-separated. □

Theorem 18. *If A is a β_I-connected subset of a β_I-connected ideal topological space (X, τ, I) such that $X - A$ is the union of two β_I-separated sets B and C, then $A \cup B$ and $A \cup C$ are β_I-connected.*

Theorem 19. *The continuous image a β_I-connected space is connected.*

Theorem 20. *Let (X, τ, I) be an ideal topological space. If the union of two β_I-separated sets is a β-closed set, then one of the sets is β-closed and the other is β_I-closed.*

Proof. Let A and B be β_I-separated such that $A \cup B$ is β-closed. If A and B is β_I-separated, then $\mathrm{cl}_{\beta_I}(A) \cap B = \varnothing = A \cap \mathrm{cl}_\beta(B) = \varnothing$. Moreover, if $A \cup B$ is β-closed, then $\mathrm{cl}_\beta(A \cup B) = A \cup B$. Thus, $A \subseteq A \cup B$ implies $\mathrm{cl}_{\beta_I}(A) \subseteq \mathrm{cl}_{\beta_I}(A \cup B) \subseteq \mathrm{cl}_\beta(A \cup B) = A \cup B$. Hence, $\mathrm{cl}_{\beta_I}(A) \subseteq \mathrm{cl}_{\beta_I}(A) \cap (A \cup B) = \mathrm{cl}_{\beta_I}(A) \cap A \cup \mathrm{cl}_{\beta_I}(A) \cap B = \mathrm{cl}_{\beta_I}(A) \cap A = A$, that is A is β_I-closed. Similarly, $B \subseteq A \cup B$ implies $\mathrm{cl}_\beta(B) \subseteq \mathrm{cl}_\beta(A \cup B) = A \cup B$. Hence, $\mathrm{cl}_\beta(B) \subseteq \mathrm{cl}_\beta(B) \cap (A \cup B) = \mathrm{cl}_\beta(B) \cap A \cup \mathrm{cl}_\beta(B) \cap B = \mathrm{cl}_\beta(B) \cap B = B$, that is B is β-closed. □

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References

[1] M.E. Abd El-Monsef, S.N. El-Deep, R.A. Mahmood, β-open sets and β-continuous mappings, Bul. Fac. Sci. Assiut Univ. 12 (1983) 77-90.

[2] E. Ekici, T. Noiri, $*$-Hyperconnected ideal topological spaces, Analele Stiintifice Aie Universitatii Matematica (2012) 121-129.

[3] M.E. Abd El-Monsef, A.A. Nasef, A.E. Radwan, F.A. Ibrahem, R.B. Esmaeel, Some properties of semi-open sets with respect to an ideal, (submitted).

[4] E. Ekici and T. Noiri, $*$-Hyperconnected ideal topological spaces, Analele Stiintifice Aie Universitatii Matematica (2012), 121-129.

[5] K. Kannan, N. Nagaveni, On $\widehat{\beta}$-Generalized Closed Sets in Topological Spaces, International Journal of Math. Analysis, 6(57) (2012), 2819-2828.

[6] K. Kuratowski, Topology, Vol I. New York: Academic Press, (1933).

[7] S. N. Maheshwari and S. S. Thakur, On α-compact spaces, Bulletin of The Institute of Mathematics, Academia Sinice, Vol. 13, No. 4, Dece. (1985), 341- 347.

[8] A.M. Mubarki, M.M. Al-Rshudi, M.A. Al-Juhani, β^*-Open sets and β^*-continuity in topological spaces, Journal of Taibah University of Science, 8 (2014), 142-148.

[9] S. Mel, M. Fhal, New Near Open Sets In Topological Space, Journal of Physical Mathematics, 7(4) (2016), 1-8.

[10] M.P. Chaudhary, V. Kumar, S. Chowhary, On topological sets and spaces, Global Journal of Science Frontier Research, Volume XI.

[11] S.G. Crossley, S.K. Hildebrand, Semi-Topological properties, Fund. Math., 74, 233-254.

[12] C. Dorsett, Semi-regular spaces, Soochow J. Math., 8, 45-53.

[13] C. Dorsett, Semi-normal spaces, Kyungpook Math. J., 25, 173-180.

[14] J. Dugundji, Topology, USA: Allyn and Bacon, Inc.

[15] S. Morris, Topology without tears, Version of February 20, 2012.

[16] P. Bhattacharya, B.K. Lahiri, Semi generalized closed sets in topology, Indian J Math, 29, 373-382.

[17] A. Skowron, On topology information systems, Indian Bull Polish Acta Sci Math, 36178, 87-90.

[18] M.H. Stone, Applications of the theory of Boolean rings to general topology, Indian J Math 29, 373-382.

[19] N. Levine, Semi open sets and semi continuous mappings in topological spaces,. Amr Math Monthly, 70, 36-41.

[20] O. Najastad, On some classes of nearly open sets, Pacific J Math, 15, 961-970.

[21] D. Andrijević, Semi-preopen sets, Matematički Vesnik 38.93 (1986): 24-32.

[22] M.Y. Abid, Non Semipre-denseness in Topological Spaces, Journal of Kerbala University, Vol. 5 No. 2 Scientific, June 2007.

[23] S. Tahiliani, Operation approach to β-open sets and applications, Math Commun 16 (2011), 577-591.

[24] A.A. Nasef, A.E. Radwan, R.B. Esmaeel, Some properties of α-open sets with respect to an ideal, Int Journal of Pure and Applied Mathematics, Vol. 102, No. 57 2015, 613-630.

[25] A.M. Mubarki, A. Mizyed, On the topology generated by βc-Open Sets, International Journal of Math. Sci. and Engg. Appls. (IJMSEA), Vol. 9 No. 2 Scientific, June 2007.

[26] A. El Mabhouh, M.M. Al-Rshudi, M.A. Al-Juhani, β^*-Open Sets and β^*-continuity in Topological Spaces, Journal of Kerbala University, Vol. 5 No. 1, March 2015, 223-232.

[27] M. Akdag, A. Ozkan, On soft β-open sets and soft β-continuous functions, The Scientific World Journal, June 2014.

[28] M. Arockiarani, A. Arokia Lancy, Generalized soft $g\beta$-closed sets and soft $gs\beta$-closed sets in soft topological spaces, International Journal of Mathematical Archive, 4 (2), 2013, 17-23

[29] A. Devika, R. Vani, On $\pi g^*\beta$-Closed in Topological Spaces, Journal of Applied and Computational Mathematics, 7 (3), 413, 2018.

[30] R.L. Newcomb,Topologies which are compact modulo an ideal, Ph.D. dissertation, Univ. of Cal. At Santa Barbara, (1967).

[31] A.S. Mashhour, M.E. Abd El-Monsef, S.N. El-Deeb, On pre-continuous and 3 weak pre-continuous mappings, Proc Math and phys Soc Egypt, 53, 47-53.

[32] F.I. Michael, On the semi-open sets with respect to an ideal, European Journal of Pure and Applied Mathematics, Vol. 6, No. 1, 2013, 53-58.