EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 12, No. 3, 2019, 722-733
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Linear combination and reliability of generalized logistic random variables

Luan C. de S. M. Ozelim[1,*], Pushpa N. Rathie[2]

[1] *Department of Civil and Environmental Engineering, University of Brasilia, Brasilia- DF, 70910-900, Brazil*
[2] *Department of Statistics, University of Brasilia, Brasilia- DF, 70910-900, Brazil*

Abstract. Experimental random data, in general, present a skewed behaviour. Thus, asymmetrical generalized distributions are of interest. The generalized logistic distributions (GLDs) are good candidates to model skewed data because their probability density functions (p.d.f.) and characteristic functions are mathematically simple. In this paper, exact expressions in terms of the H-function are, for the first time, derived for the p.d.f. and for the cummulative distribution function of the linear combination of GLDs of type IV with different location, scale and shape parameters. Also, exact and approximate expressions are derived for $R = P(X < Y)$. Numerical examples illustrate the correctness of the expressions derived.

2010 Mathematics Subject Classifications: 33C60

Key Words and Phrases: Generalized Logistic Distribution, Mellin Transform, Fourier Transform, H-function, Reliability

---

1. Introduction

Modelling nature is by no means a trivial task. At first, scientists have to observe how a given phenomenon occurs and then, by means of physically justifiable premises, propose a model which is able to predict the outcomes of such phenomenon when some important inputs are known. Over the last half century, the scientific community started to model both the inputs and outcome of such models as random variables. Thus, the algebra of random variables has become increasingly of interest to not only pure but also applied scientists.

Instead of modelling failure as stresses overcoming a given strength threshold, a probabilistic approach enables one to calculate the probability of failure. The latter approach may lead to cost reduction and time consumption in the implementation of a given project. [13]

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v12i3.3444

Email addresses: luanoz@gmail.com (L. C. de S. M. Ozelim), pushpanrathie@yahoo.com (P. N. Rathie)

The application of reliability measures of the type $R = P(X < Y)$, where X is the stress to which a given structure of strength Y is subjected, have many applications in various areas, including quality control, engineering statistics, reliability, medicine, psychology, biostatistics, stochastic precedence, and probabilistic mechanical design [9, 10]. For example, if X represents the maximum pressure caused by flooding and Y represents the strength of the leg of a bridge on a stream, then R is the probability that the bridge will resist [7].

On the other hand, for medical applications, let X and Y represent the control and treatment groups, respectively. Then R measures the treatment effect [7, 18]. Alternatively, in the cases of diagnostic tests used to distinguish between diseased and non-diseased patients, the area under the receiver operating characteristics (ROC) curve, based on the sensitivity and the complement to specificity at different cut-off points of the range of possible test values, is equal to R [16, 18].

Besides, in the ecotoxicological risk assessment literature, R is also employed to quantify risk [8]. Thus, it is clear that studying reliability measures of the type $R = P(X < Y)$ is fundamental to various areas of science. In this regard, the estimation of R is of interest to provide information in decision processes. It is common for authors to assume that X and Y belong to a certain family of probability distributions with unknown parameters and then to consider the estimation problem of the reliability R [7]. One may check [7] and the references therein for estimation problems related to exponential, uniform, generalized exponential, generalized gamma, Burr, gamma and beta distributions.

Being able to build not only estimates but also the exact value of R when the statistical distribution of the variables involved are known is then important. Besides considering the random variables (RVs) previously cited, it is important to obtain the exact value of R for more general RVs. In the present paper, generalized logistic random variables are studied.

Statistical models of the logistic type have been applied to all sorts of pure and applied problems in Statistics. This comes from the fact that logistic models are flexible, being able to mimic normality as well as show skewness in some generalized models. Applications of this kind of random variable are easily seen in the literature. A complete study has been recently performed in [12]. For example, regarding applied scientists, [4] employed generalized logistic models to perform flood analysis in partial duration series.

On the other hand, regarding pure statistics papers, [19] discussed the parameter estimation for a certain type of generalized logistic distribution. Also, [3] obtained approximate maximum likelihood estimation for some generalized logistic distributions. Besides, [1] performed a comparative study of the techniques available for the estimation of the generalized logistic distribution parameters.

Consider a random variable S. Let S follow a generalized logistic distribution of type IV, from now on called GLD. Also, let one consider a location parameter $\mu \in \mathbb{R}$, a scale parameter $\sigma > 0$ and two shape parameters α and β such that $\alpha, \beta > 0$. One says $S \sim GLD(\mu, \sigma, \alpha, \beta)$ and that the probability density function (p.d.f) of S is given by:

$$f(x;\sigma,\mu,\alpha,\beta)=\frac{1}{B(\alpha,\beta)}\frac{e^{-\beta(\frac{x-\mu}{\sigma})}}{\sigma[1+e^{-(\frac{x-\mu}{\sigma})}]^{\alpha+\beta}},\quad \forall x\in R, \tag{1}$$

where $B(\alpha,\beta)$ is the Beta function, defined as:

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{2}$$

In (2), $\Gamma(x)$ stands for the Gamma function defined as:

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt. \tag{3}$$

Despite being constantly considered in pure and applied sciences, no closed-form compact representation for the sum of independent not identically distributed generalized logistic random variables has been presented yet. In the present paper, the linear combination of generalized logistic random variables with different location, scale and shape parameters is given, in a compact form, in terms of the H-function. This latter function is a generalized hypergeometric special function whose importance has been widely recognized. In special, [17] discusses the central role of this function to the study of the algebra of random variables. Besides the pure statistical applications where the linear combination itself is sought, reliability models can also be derived based on the latter by noticing that $R=P(X<Y)=P(X-Y<0)$.

As discussed, mathematical procedures are used to estimate R. Regarding logistic RVs, for example, [14] proposed methods of estimation of the shape parameters of the generalized logistic distribution and of $P(Y<X)$, when X and Y were independent random variables from two distributions having the same scale parameters but different shape parameters.

More recently, [2] derived estimators for R when both the distributions compared are generalized logistic. The authors considered the estimation of R, when X and Y are both two-parameter generalized logistic distribution with the same unknown scale but different shape parameters or with the same unknown shape but different scale parameters. They also considered the general case when the shape and scale parameters are different. In [2] the maximum likelihood estimator of R and its asymptotic distribution are obtained and it is used to construct the asymptotic confidence interval of R. Besides, [5] studied the estimation of two measures of reliability for generalized half logistic distributions.

One of the purposes of the present paper is to provide exact and approximate formulas for reliability measures $R=P(\sum_{i=1}^{N_1}X_i<\sum_{j=1}^{N_2}Y_j)$, when X_i, $i=1,...,N_1$ and Y_j, $j=1,...,N_2$ are GLD random variables with different location, scale and shape parameters. By means of standard mathematical tools, an exact expression for R is derived in terms of the H-function. Also, an approximation to R, in terms of elementary functions, is presented. The latter is based on the series expansion of the H-function.

Considering the linear combination of RVs instead of simply $R = P(X < Y)$ is of interest because in many applications, the stress and strength are described in terms of other random variables. For example, the strength of a given geomaterial may be obtained by means of the Mohr-Coulomb failure criterion. Mathematically, this criterion may be represented as [6]:

$$\tau = \sigma \tan(\phi) + c \tag{4}$$

where τ, σ and c are the shear, normal, and cohesion stresses, respectively. When both σ and c are considered as RVs, it can be seen from (4) that the shear resistance τ is nothing but the linear combination of the other random variables, as the internal friction angle ϕ may be considered constant.

Obtaining the linear combination is, thus, more interesting than simply expressing $P(X < Y)$, as the results are more general. For example, in [15], linear combinations, products and ratios of t Random Variables were studied, therefore all the theoretical and mathematical basis for studying reliability measures for that specific type of random variable is presented. To familiarize the reader, the definition of the H-function and its Mellin Transform are given in the next section.

2. The H-function

The H - function (see [11]) is defined, as a contour complex integral by

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1, A), & \ldots, & (a_n, A_n), & (a_{n+1}, A_{n+1}), & \ldots, & (a_p, A_p) \\ (b_1, B_1), & \ldots, & (b_m, B_m), & (b_{m+1}, B_{m+1}), & \ldots, & (b_q, B_q) \end{matrix}\right]$$
$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s) \prod_{j=n+1}^{p} \Gamma(a_j + A_j s)} z^{-s} ds, \tag{5}$$

where A_j and B_j are positive quantities and all the a_j and b_j may be complex. The contour L runs from $c - i\infty$ to $c + i\infty$ such that the poles of $\Gamma(b_j + B_j s)$, $j = 1, \ldots, m$ lie to the left of L and the poles of $\Gamma(1 - a_j - A_j s)$, $j = 1, \ldots, n$ lie to the right of L.

The Mellin transform of the H -function is given by

$$\int_0^\infty x^{s-1} H_{p,q}^{m,n}\left[cx \,\middle|\, \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix}\right] dx = \frac{c^{-s} \prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s) \prod_{j=n+1}^{p} \Gamma(a_j + A_j s)} z^{-s}. \tag{6}$$

In the next section, one shall proceed to obtain the p.d.f and c.d.f of the linear combination of independent GLD random variables by means of the H-function.

3. The Linear Combination of N GLDs

A brief description of the problem is presented and its solution is derived in this section.

3.1. Problem Statement

Let $X_i \sim GLD(\mu_i, \sigma_i, \alpha_i, \beta_i)$. Then, one seeks the probability density function of the random variable:

$$Z = \sum_{i=1}^{N} b_i X_i \tag{7}$$

where b_i, $i = 1, ..., N$ are real numbers.

At first, one may obtain the p.d.f. of the linearly scaled distributions $b_i X_i$. This can be easily done by considering the Jacobian rule as presented in [17]. Thus, when $X_i \sim GLD(\mu_i, \sigma_i, \alpha_i, \beta_i)$, the transformed random variable $b_i X_i$ has a p.d.f. mathematically represented as:

$$f_{b_i X_i}(x; \sigma_i, \mu_i, \alpha_i, \beta_i, b_i) = \frac{1}{|b_i|\sigma_i B(\alpha_i, \beta_i)} \frac{e^{-\beta_i(\frac{x-b_i\mu_i}{b_i\sigma_i})}}{[1 + e^{-(\frac{x-b_i\mu_i}{b_i\sigma_i})}]^{\alpha_i+\beta_i}}, \quad \forall x \in R, \tag{8}$$

This way, by using (7) and (8), the variable Z can be rewritten as the sum of the random variables Y_i, with $Y_i = b_i X_i$. To get a closed form exact probability density function of the random variable Z, one shall first proceed to obtain of the characteristic functions of the random variables Y_i, $i = 1, ..., N$.

3.2. The Characteristic Function of Y_i, $i = 1, ..., N$

The characteristic function of a given random variable can be obtained by calculating the Fourier transform of its probability density function. The characteristic function of a GLD random variable is widely known in the literature [12]. On the other hand, in order to better understand the scaling process performed to generate the variable $Y_i = b_i X_i$, the characteristic function of Y_i shall be obtained explicitly. This way, by means of (8), the characteristic function $\phi_i(t)$ (CF) for the random variables Y_i, $i = 1, ..., N$ is given by:

$$\phi_i(t) = \frac{1}{\sigma_i |b_i| B(\alpha_i, \beta_i)} \int_{-\infty}^{\infty} e^{jtx} \frac{e^{-\frac{(x-\mu_i b_i)}{\sigma_i b_i}}}{\left(1 + e^{-\frac{(x-\mu_i b_i)}{\sigma_i b_i}}\right)^{\alpha_i+\beta_i}} dx, \tag{9}$$

where $j = \sqrt{-1}$.

On substituting $r = e^{-\frac{(x-\mu_i b_i)}{\sigma_i b_i}} / \left(1 + e^{-\frac{(x-\mu_i b_i)}{\sigma_i b_i}}\right)$, the integral in (9) becomes:

$$\phi_i(t) = \frac{e^{jtb_i\mu_i}}{B(\alpha_i, \beta_i)} \int_0^1 r^{\beta_i - jt\sigma_i b_i - 1}(1-r)^{\alpha_i + jt\sigma_i b_i - 1} dr. \tag{10}$$

By considering the Beta function definition in (2), (10) becomes:

$$\phi_i(t) = \frac{e^{jtb_i\mu_i}}{B(\alpha_i, \beta_i)} B(\beta_i - jt\sigma_i b_i, \alpha_i + jt\sigma_i b_i). \tag{11}$$

Since the characteristic function of the sum of independent random variables is the product of the individual characteristic functions [17], the CF of the random variable Z, $\phi_Z(t)$, is given by:

$$\phi_Z(t) = \prod_{i=1}^{N} \left[\frac{e^{jtb_i\mu_i}}{B(\alpha_i, \beta_i)} B(\beta_i - jt\sigma_i b_i, \alpha_i + jt\sigma_i b_i) \right]. \tag{12}$$

The probability density function of the random variable Z is obtained by the Fourier transform as described in the next subsection.

3.3. The Probability Density Function of the Linear Combination of GLD Variables

Being the CF of the random variable Z known, by means of the inversion formula for Fourier transform, one shall get that the probability density function of Z, $f_Z(x)$. By considering the alternative representation of the Beta function in terms of Gamma functions presented in (2), $f_Z(x)$ can be expressed as:

$$f_Z(x; \overline{\sigma}, \overline{\mu}, \overline{\alpha}, \overline{\beta}, \overline{b}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jt\left(x - \sum\limits_{i=1}^{N} b_i\mu_i\right)} \prod_{i=1}^{N} \left[\frac{\Gamma(\beta_i - jt\sigma_i b_i)\Gamma(\alpha_i + jt\sigma_i b_i)}{\Gamma(\alpha_i)\Gamma(\beta_i)} \right] dt, \tag{13}$$

where $\overline{\sigma}$, $\overline{\mu}$, $\overline{\alpha}$,$\overline{\beta}$ and $\overline{b}$ represent the vectors of scale, mean and shape parameters and coefficients, respectively.

It is possible to transform the real integral in (13) into a contour integral by the variable change $jt = s$. In order to represent the equivalent contour integral in terms of the H-function, one has to split the coefficients b_i in positive and negative groups such that $b_i \geq 0$, for $1 \leq b_i \leq N_1$ and $b_i \leq 0$, for $N_1 + 1 \leq b_i \leq N$. This way, by means of the transformed complex integral and the definition of the H-function in (5), (13) can be rewritten as:

$$f_Z(x; \overline{\sigma}, \overline{\mu}, \overline{\alpha}, \overline{\beta}, \overline{b}) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)\Gamma(\beta_i)} \times \tag{14}$$

$$\times H_{N,N}^{N,N}\left[e^{x-\sum_{i=1}^{N}b_i\mu_i}\left|\begin{array}{c}(1-\beta_1,\sigma_1b_1),...,(1-\beta_{N_1},\sigma_{N_1}b_{N_1}),(1-\alpha_{N_1+1},\sigma_{N_1+1}|b_{N_1+1}|),...,(1-\alpha_N,\sigma_N|b_N|)\\(\alpha_1,\sigma_1b_1),...,(\alpha_{N_1},\sigma_{N_1}b_{N_1}),(\beta_{N_1+1},\sigma_{N_1+1}|b_{N_1+1}|),...,(\beta_N,\sigma_N|b_N|)\end{array}\right.\right].$$

Equation (14) provides a closed form exact representation for the probability density function of the linear combination of GLD random variables,valid for $\sigma_i, \alpha_i, \beta_i > 0$, $\mu_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$, $i = 1, ..., N$. The cumulative distribution function is given in the next subsection.

3.4. The Cumulative Distribution Function of the Linear Combination of GLD Random Variables

The cumulative distribution function of the random variable Z, F_Z, whose p.d.f. is in (14), is given by:

$$F_Z(x;\overline{\sigma},\overline{\mu},\overline{\alpha},\overline{\beta},\overline{b}) = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)\Gamma(\beta_i)} \times \quad (15)$$

$$\times \int_{-\infty}^{x} H_{N,N}^{N,N}\left[e^{x-\sum_{i=1}^{N}b_i\mu_i}\left|\begin{array}{c}(1-\beta_1,\sigma_1b_1),...,(1-\beta_{N_1},\sigma_{N_1}b_{N_1}),(1-\alpha_{N_1+1},\sigma_{N_1+1}|b_{N_1+1}|),...,(1-\alpha_N,\sigma_N|b_N|)\\(\alpha_1,\sigma_1b_1),...,(\alpha_{N_1},\sigma_{N_1}b_{N_1}),(\beta_{N_1+1},\sigma_{N_1+1}|b_{N_1+1}|),...,(\beta_N,\sigma_N|b_N|)\end{array}\right.\right]dx.$$

Equation (16) is nothing but nested real and contour integrals. By interchanging the order of integration, (16) becomes [11]:

$$F_Z(x;\overline{\sigma},\overline{\mu},\overline{\alpha},\overline{\beta},\overline{b}) = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)\Gamma(\beta_i)} \times \quad (16)$$

$$\times H_{N+1,N+1}^{N,N+1}\left[e^{x-\sum_{i=1}^{N}b_i\mu_i}\left|\begin{array}{c}(1-\beta_1,\sigma_1b_1),...,(1-\beta_{N_1},\sigma_{N_1}b_{N_1}),(1-\alpha_{N_1+1},\sigma_{N_1+1}|b_{N_1+1}|),...,(1-\alpha_N,\sigma_N|b_N|),(1,1)\\(\alpha_1,\sigma_1b_1),...,(\alpha_{N_1},\sigma_{N_1}b_{N_1}),(\beta_{N_1+1},\sigma_{N_1+1}|b_{N_1+1}|),...,(\beta_N,\sigma_N|b_N|),(0,1)\end{array}\right.\right].$$

Expression (17) is valid for the same values of parameters as that of (14).

4. Reliability $P(X < Y)$

The reliability measure $R = P(X < Y) = P(X - Y < 0)$ is of great interest to both pure and applied scientists. In the next sub-section, the exact value of R is provided in terms of the H-function by considering the difference of two generalized logistic random variables.

4.1. Exact Expression

Let $X \sim GLD(\mu_1, \sigma_1, \alpha_1, \beta_1)$ and $Y \sim GLD(\mu_2, \sigma_2, \alpha_2, \beta_2)$. Then, by means of (17), $R = P(X < Y) = P(X - Y < 0)$ can be exactly given as:

$$R = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)} H_{3,3}^{2,3}\left[e^{\mu_2-\mu_1}\left|\begin{array}{c}(1-\beta_1,\sigma_1),(1-\alpha_2,\sigma_2),(1,1)\\(\alpha_1,\sigma_1),(\beta_2,\sigma_2),(0,1)\end{array}\right.\right]. \quad (17)$$

Even though R is exactly expressed in (17), a mathematical software such as Mathematica is used to evaluate the H-function, as shown subsequently in the applications section. On the other hand, when out-of-computer quick calculations are needed, a simpler expression in terms of elementary functions is of great interest. In the next subsection, the series expansion of (17) is presented.

4.2. Series Expansion Expression

The H-function can be evaluated by means of the residue theorem [11]. This way, the contour integral can be calculated by summing the residues over the poles of the function. The series representation of the H-function may become even simpler when the poles of the Gamma functions in the numerator of fraction inside the contour integral are simple [11]. In order to guarantee that, the following restrictions should be applied to (17):

- $\alpha_1\sigma_2 - \sigma_1\beta_2 + \sigma_2 r - \sigma_1 w \neq 0 \quad \forall r, w \in \mathbb{N}$

and

- $\beta_1\sigma_2 - \alpha_2\sigma_1 + \sigma_2 u - \sigma_1 v \neq 0 \quad \forall u, v \in \mathbb{N}$

The restrictions above arise from considering that none of the poles of $\Gamma(\alpha_1 + s\sigma_1)$ coincide with the poles of $\Gamma(\beta_2 + s\sigma_2)$ and that none of the poles of $\Gamma(\beta_1 - s\sigma_1)$ coincide with the poles of $\Gamma(\alpha_2 - s\sigma_2)$. Thus, if the conditions above are satisfied, the reliability for two GLDs can be expressed as:

$$\begin{aligned} R &= \sum_{n=0}^{\infty} \frac{e^{\frac{(n+\alpha_1)(\mu_2-\mu_1)}{\sigma_1}}(-1)^n\Gamma(n+\alpha_1+\beta_1)\Gamma(-\frac{(n+\alpha_1)\sigma_2}{\sigma_1}+\beta_2)\Gamma(\frac{(n+\alpha_1)\sigma_2}{\sigma_1}+\alpha_2)}{n!(n+\alpha_1)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)} \\ &+ \sum_{n=0}^{\infty} \frac{e^{\frac{(n+\beta_2)(\mu_2-\mu_1)}{\sigma_2}}(-1)^n\Gamma(n+\alpha_2+\beta_2)\Gamma(-\frac{(n+\beta_2)\sigma_1}{\sigma_2}+\alpha_1)\Gamma(\frac{(n+\beta_2)\sigma_1}{\sigma_2}+\beta_1)}{n!(n+\beta_2)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)}, \end{aligned} \tag{18}$$

when $\mu_2 < \mu_1$, and

$$\begin{aligned} R &= 1 - \sum_{n=0}^{\infty} \frac{e^{\frac{(n+\beta_1)(\mu_1-\mu_2)}{\sigma_1}}(-1)^n\Gamma(n+\alpha_1+\beta_1)\Gamma(-\frac{(n+\beta_1)\sigma_2}{\sigma_1}+\alpha_2)\Gamma(\frac{(n+\beta_1)\sigma_2}{\sigma_1}+\beta_2)}{n!(n+\beta_1)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)} \\ &- \sum_{n=0}^{\infty} \frac{e^{\frac{(n+\alpha_2)(\mu_1-\mu_2)}{\sigma_2}}(-1)^n\Gamma(n+\alpha_2+\beta_2)\Gamma(\frac{(n+\alpha_2)\sigma_1}{\sigma_2}+\alpha_1)\Gamma(-\frac{(n+\alpha_2)\sigma_1}{\sigma_2}+\beta_1)}{n!(n+\alpha_2)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)}, \end{aligned} \tag{19}$$

otherwise.

5. Numerical Applications of the Results: Reliability of Generalized Logistic Distributions

The formulas developed in the present paper are numerically evaluated in order to show their applicability.

5.1. Reliability of the type $R = P(X < Y)$

The present paper provides both the exact and approximated formulas for evaluating the reliability measure R when two generalized logistic distributions are considered. A set of four generalized logistic random variables is considered to show the applicability of (17), (18)and (19). Such variables are shown in Table 1 and graphically in Figure 1. Using the generalized logistic variables considered in Table 1, the reliability measures are obtained numerically by means of a computational code in Mathematica and given in Table 2.

Table 1: Logistic Random Variables Considered

Random Variable	μ	σ	α	β
X_1	1.7	1.4	2.3	1.1
X_2	0.3	0.8	3.1	1.4
X_3	-1.3	2.1	0.9	2.3
X_4	-0.5	1.7	2.5	0.7

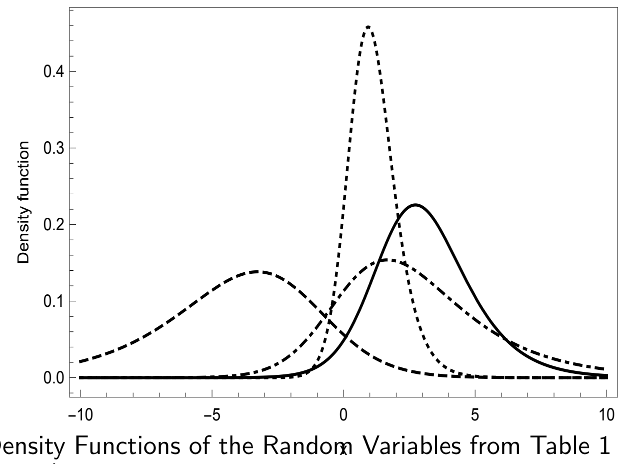

Figure 1: Probability Density Functions of the Random Variables from Table 1 (X_1 full, X_2 dotted, X_3 dashed and X_4 dot-dashed).

The values of R estimated from random data have been obtained by the procedure below:

- For the random variables X and Y generate random samples with 10^5 elements each, x_i and y_i, $i = 1, ..., 10^5$.

Table 2: Reliability Measures

R	Random Data	(17)	(18) or (19) w/ 20 term	(18) or (19) w/ 40 terms	(18) or (19) w/ 60 terms
$P(X_1 < X_2)$	0.16764	0.16748	0.16795	0.16748	0.16748
$P(X_2 < X_3)$	0.04099	0.04080	0.04529	0.04080	0.04080
$P(X_3 < X_4)$	0.95795	0.95807	-43.1999	0.83806	0.95799

- Consider the indicator function $I(x, y) = 1 - u(x - y)$, where $u(x) = 0$, $x < 0$ and $u(x) = 1$, otherwise. The value of R is estimated by $R_e = \sum_{i=1}^{10^5} I(x_i, y_i)$;
- Repeat the above process 1000 times and then take the mean value of the R_es generated. These values are shown in Table 2.

It is worth noticing that the variance of the estimator used above was of order 10^{-7} for all the cases. As can be seen from the analysis of Table 2, both the exact and approximate expressions obtained in the present paper correctly model the random data generated. Depending on the parameters of the distribution, the series presented may be slowly convergent. On the other hand, as the series themselves are quite simple to implement, taking more than 60 terms is not a hard task for any commercial computational software.

6. Conclusions

Skewness is present in most of the random data collected from nature. Even though mathematically simple, generalized logistic models have shown to be useful tools to model skewed data. The probability density and the cummulative distribution functions of the linear combination of N independent and not identically distributed generalized logistic random variables have been obtained in terms of the H-function.

The c.d.f. of the linear combination can be used to build reliability measures of the type $P(\sum_{i=1}^{N_1} X_i < \sum_{j=1}^{N_2} Y_j)$ when X_i, $i = 1, ..., N_1$ and Y_j, $j = 1, ..., N_2$ are generalized logistic random variables of type IV with different location, scale and shape parameters. Besides, a series expansion alternative expression has been derived for the case $N_1 = N_2 = 1$.

The applicability of the expressions developed has been illustrated by numerical experiments, indicating a very good accordance between the exact and the estimated reliability measures.

Acknowledgements

The authors acknowledge the support provided by the following institutions: The Brazilian National Council for Scientific and Technological Development (CNPq Grant 151778/2018-3) and the University of Brasilia (UnB).

References

[1] M. R. Alkasasbeh and M. Z. Raqab. Estimation of the generalized logistic distribution parameters: Comparative study. *Statistical Methodology*, 6(3):262–279, 2009.

[2] A. Asgharzadeh, R. Valiollahi, and M. Z. Raqab. Estimation of the stress-strength reliability for the generalized logistic distribution. *Statistical Methodology*, 15:73–94, 2013.

[3] N. Balakrishnan. Approximate maximum likelihood estimation for a generalized logistic distribution. *Journal of Statistical Planning and Inference*, 26(2):221–236, 1990.

[4] P. K. Bhunya, R. D. Singh, R. Berndtsson, and S. N. Panda. Flood analysis using generalized logistic models in partial duration series. *Journal of Hydrology*, 420-421:59–71, 2012.

[5] A. Chaturvedi, S. Kang, and Pathak A. Estimation and testing procedures for the reliability functions of generalized half logistic distribution. *Journal of the Korean Statistical Society*, 45(2):314–328, 2016.

[6] B. M. Das and K. Sobhan. *Principles of Geotechnical Engineering*. Cengage Learning, Stamford, US, 8th edition, 2014.

[7] A. I. Genç. Estimation of $p(x > y)$ with topp-leone distribution. *Journal of Statistical Computation and Simulation*, 83(2):326–339, 2013.

[8] R. Jacobs, A. A. Bekker, H. van der Voet, and C. J. Ter Braak. Parametric estimation of p(x ¿ y) for normal distributions in the context of probabilistic environmental risk assessment. *PeerJ*, 3:e1164, 2015.

[9] R. A. Johnson. Stress-strength model for reliability. In P. R. Krishnaiah and Rao C. R., editors, *Handbook of Statistics*, pages 27–54. Elsevier, Amsterdam, Netherlands, 1988.

[10] S. Kotz, S. Lumelskii, and M. Pensky. *The Stress − Strength Model and its Generalizations: Theory and Applications*. World Scientific, Singapore, 2003.

[11] A. M. Mathai, R. K. Saxena, and Haubold H. J. *The H-function: Theory and Applications*. Springer, New York (NY), 2010.

[12] M. M. Nassar and A. Elmasry. A study of generalized logistic distributions. *Journal of the Egyptian Mathematical Society*, 20:126–133, 2012.

[13] L. C. S. M. Ozelim, A. L. B. Cavalcante, A. P. Assis, and L. F. M. Ribeiro. Analytical slope stability analysis based on statistical characterization of soil primary properties. *International Journal of Geomechanics (ASCE)*, 15:2, 2015.

[14] A. Ragab. Estimation and predictive density for the generalized logistic distribution. *Microelectronics Reliability*, 31(1):91–95, 1991.

[15] Nadarajah S. and Kotz S. Linear combinations, products and ratios of t random variables. *Allgemeines Statistisches Archiv*, 89(3):263–280, 2005.

[16] H. M. Samawi, A. Helu, H. D. Rochani, J. Yina, and D. Lindera. Estimation of $p(x > y)$ when x and y are dependent random variables using different bivariate sampling schemes. *Communications for Statistical Applications and Methods*, 23(5):385–397, 2016.

[17] M. D. Springer. *The Algebra of Random Variables*. Wiley, New York (NY), 1979.

[18] L. Ventura and W. Racugno. Recent advances on Bayesian inference for $p(x < y)$. *Bayesian Analysis*, 6(3):411–428, 2011.

[19] D. Zelterman. Parameter estimation in the generalized logistic distribution. *omputational Statistics & Data Analysis*, 5(3):177–184, 1987.