



## Labelled Maximal Tubings on Paths and Graph Associahedra

S. Kaan Gürbüzler<sup>1</sup>, Bedia Akyar<sup>2,\*</sup>

<sup>1</sup> *Department of Mathematics, Dokuz Eylül University, İzmir, Turkey*

<sup>2</sup> *Department of Mathematical Sciences, Aalborg University, Aalborg Ø, Denmark*

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**Abstract.** We determine the triangulation of a cyclohedron compatible with the Tamari order on its faces. We define a name of a tubing on a path and a plumbing leading us to construct the dendriform algebra of the collection of maximal tubings on paths. Moreover, we give an operad structure of associahedra and a module structure of cyclohedra via tubings. We define labelled maximal tubings on paths and give to an application of tubings in homological algebra.

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### 1. Introduction

Stasheff [13] constructed the associahedron  $K^n$  as a space homeomorphic to an  $n$ -dimensional unit cube, where  $K^n$  has a convex, curvilinear form. In the nineties, Shnider and Sternberg [12] defined the associahedron  $K^n$  as a truncation of an  $n$ -simplex in  $\mathbb{R}^{n+1}$ . Carr and Devadoss [1] gave an alternative definition of  $K_n$  with respect to tubings to obtain a family of polytopes, namely graph associahedron. Given any simple finite graph, Devadoss [2] gave a realization of its corresponding graph associahedron and a geometric meaning of every collection of tubings on graphs for a chosen suitable algorithm.

Loday [6] gave a simple realization of an associahedron by taking the convex hull of the points corresponding to the set of planar binary trees. In addition, Loday [4] constructed some operations mainly addition and multiplication on the set of planar binary trees and also some algebraic structures such as Dendriform algebra of planar binary trees. On the other hand, Forcey and Springfield [3] constructed a module on the vertices of cyclohedron via tubings considering the relation between tubings and trees which enables to give a geometrical view point for graded algebras. The product on a graded algebra turns into the one on the vertices of a sequence of polytopes and this product is represented by the

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\*Corresponding author.

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*Email addresses:* [kaan.gurbuzer@deu.edu.tr](mailto:kaan.gurbuzer@deu.edu.tr) (S. Kaan Gürbüzler),  
[bedia@math.aau.dk](mailto:bedia@math.aau.dk) (Bedia Akyar)

sum of the vertices in higher dimensions. Since there exists a one to one correspondence between the set of planar binary trees and the vertices of associahedra, the dendriform algebra structure can also be considered on the vertices of associahedra.

Through the collection of planar rooted trees, Markl, Shnider and Stasheff [10] showed that the sequence  $\{K^n\}_{n \geq 0}$  has an operad which is as an abstraction of a family of composable functions of finitely many variables. Operads generalize various associativity properties that are already observed by modeling computational trees within algebra. Markl [8] constructed a module structure over an operad and in [9], he showed that the sequence  $\{W^n\}_{n \geq 0}$  of cyclohedra was not an operad but a module over the operad  $\{K^n\}_{n \geq 0}$ . Briefly, this module structure results from the indexes of the faces of the form  $W^{n-k} \times K^{k-1}$  for  $n \geq k \geq 1$  and it enables us to generalize many types of graph associahedra.

In this paper, Section 2 contains an introduction on graph associahedra from Devadoss [2] and the combinatorial properties of the faces of a graph associahedron from Forcey and Springfield [3]. We give the triangulation of a cyclohedron compatible with the Tamari order on its faces. In Section 3, we define a name of a tubing on a path as a sequence of positive integers and establish some operations on tubings on paths. Furthermore, we define a plumbing as a collection of maximal tubings with the sum and product operations. We construct the dendriform algebra of the collection of maximal tubings on paths together with the operations via names. After that, we determine the operad structure of the sequence  $\{K^n\}_{n \geq 0}$  and interpret the module structure of the sequence  $\{W^n\}_{n \geq 0}$  defining the comp maps and the cellular chain complex of an associahedron in terms of tubings. In last section, as an application, we construct the chain complex whose boundary map is defined on labelled maximal tubings on a path and compute its homology groups.

## 2. Graph Associahedron

For a finite simple graph  $G$ , Devadoss [2] defined a *tube* as a proper subset of nodes of  $G$  whose induced graph is a connected subgraph of  $G$ . The graph  $G$  itself is called the universal tube which is preferably not drawn. There are different positions of tubes on a graph with respect to each other. In particular, two tubes  $t_1, t_2$  are called *nested* if  $t_1 \subset t_2$  or  $t_2 \subset t_1$ ; *intersecting* if they are not nested and  $t_1 \cap t_2 \neq \emptyset$ ; *adjacent* if  $t_1, t_2$  do not intersect and  $t_1 \cup t_2$  is again a tube; *compatible* if they are neither adjacent nor intersect each other. A set of compatible tubes is called a *tubing* and it is assumed that each tubing always contains the universal tube. If a graph  $G$  is a disconnected simple graph with connected components  $G_1, \dots, G_k$  then it is also assumed that a tubing does not contain all the connected components. In general, a tubing on a graph with  $n$  nodes is called *k-tubing*,  $0 \leq k \leq n - 1$ , if it contains  $k$  tubes and universal tube. Especially,  $(n - 1)$ -tubings are called the maximal tubings whose collection is denoted by  $MG$ . The set of all tubings on a graph  $G$  is denoted by  $PG$ . This set becomes a partial ordered set ordered by inclusion such that if  $T_1, T_2 \in PG$  and  $T_1$  can be obtained by deleting one or more tubes in  $T_2$  then  $T_2 < T_1$ .

In [2], Devadoss proved that the poset  $PG$  admits a geometric realization  $\mathcal{KG}$ , that is, a convex polytope whose face poset is isomorphic to  $PG$ . The polytope  $\mathcal{KG}$  is called

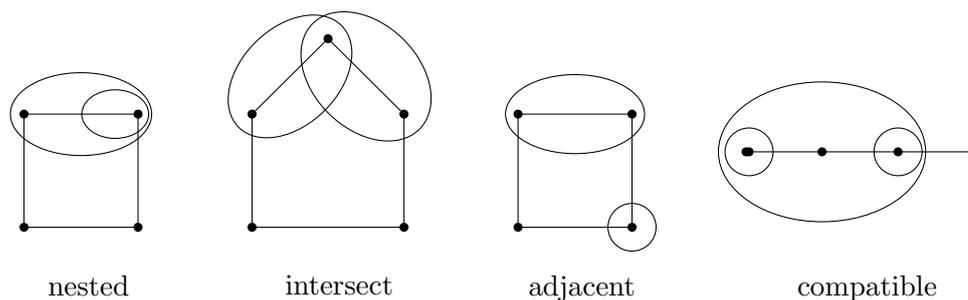


Figure 1: Types of tubes

the graph associahedra associated to  $G$ . In order to obtain the geometric realization  $\mathcal{KG}$ , Devadoss produced a combinatorial way which relates the tubings in  $MG$  with the points in space. He also proved that the convex hull of these points yields the graph associahedra  $\mathcal{KG}$ . For instance, the graph associahedra  $\mathcal{KG}$  associated to the graph  $G$  with  $n$  disconnected nodes is a simplex  $\Delta^{n-1}$  in  $\mathbb{R}^n$  given in terms of barycentric coordinates. From another point of view, his method creates a system or a set of affine hyperplanes to truncate the standard simplex. It is easily seen that these hyperplanes appear by adding a new edge between two nodes in the graph and it causes new truncations in which new faces appear.

A remarkable feature of a graph associahedra is that the faces of it can be obtained by a product of other two graph associahedra. Carr and Devadoss described the face structure of the graph associahedra  $\mathcal{KG}$  in [1]. Given a graph  $G$  and a tube  $t$  on  $G$ , they defined *the reconnected complement graph*  $G^*(t)$  of  $t$  in  $G$  as follows:

- Let  $V$  denote the set of nodes of  $G$ , then  $V \setminus t$  is the set of nodes of  $G^*(t)$ .
- Let  $v_1, v_2 \in V \setminus t$ . There is an edge between  $v_1$  and  $v_2$  if either  $\{v_1, v_2\}$  or  $\{v_1, v_2\} \cup t$  is connected in  $G$ .

**Theorem 1.** *All facets of  $\mathcal{KG}$  correspond to the set of 1-tubings. In particular, a facet associated to a 1-tubing  $T = \{G, t\}$  is combinatorially equivalent to  $\mathcal{KG}(t) \times \mathcal{KG}^*(t)$ , where  $\mathcal{KG}(t)$  and  $\mathcal{KG}^*(t)$  are graph associahedra corresponding to the tube  $t$  and the reconnected complement of  $t$  in  $G$ , respectively.*

One of the most well-known graph associahedra is the Stasheff polytope  $K^{n-1}$ , also called *associahedron*. It becomes the realization of the poset  $PP(n)$ , where  $\mathcal{P}(n)$  denotes an  $n$ -path which has  $n$  nodes labeled by the set  $\{1, 2, \dots, n\}$  with an increasing order. The classical definition of an associahedron  $K^n$  is that it is an  $n$ -dimensional cell complex whose cells are indexed by the meaningful bracketings of  $(n+2)$  variables  $1, \dots, n+2$ . This cell complex has a relation with the set of the planar rooted trees,  $\text{Tree}(n+1)$ , with  $(n+2)$  leaves. Each  $k$ -dimensional cell can be indexed by a rooted tree which has  $(n+2)$  leaves and  $(n-k+1)$  internal vertices. Moreover, the vertices of  $K^n$  can be indexed by the elements of the set  $Y_{n+1}$  of planar binary rooted trees. In [3], Forcey

and Springfield described a bijection between the set of planar rooted trees  $\text{Tree}(n)$  and the set  $\mathcal{PP}(n)$  of tubings on an  $n$ -path. The constructions of an associahedron given in Loday [6], Markl [9] and Devadoss [2] have similarities on specifying the coordinates of the vertices. Basically, Loday used the function  $f(n) = \frac{n(n+1)}{2}$  and Markl used an exponential function  $f(n) = 3^n$ . Devadoss [2] preferred to use an exponential function and took attention to the chosen function which can cause deep cuts during the truncation process of the simplex. In their methods, it can be easily seen that the boundary cells of  $K^n$  are of the form  $K^p \times K^q$ , where  $p + q = n - 1$ .

There is also another poset structure on the vertices of an associahedron. The partial order in this structure is called the *Tamari order* on  $\mathcal{MP}(n)$  and defined for two maximal tubings  $T_1, T_2$  in  $\mathcal{MP}(n)$  such that  $T_1 < T_2$  if  $T_2$  can be obtained from  $T_1$  by sliding a tube from left to right. By using this poset structure, Loday [7] proved that an associahedron  $K^n$  admits a triangulation by  $(n+1)^{n-1}$  simplices. This triangulation is compatible with the Tamari order and one can also give an orientation to the facets of an associahedron with respect to this order.

The second famous graph associahedra is so called cyclohedron  $W^n$  which appears in the study on compactifications of configuration spaces of  $n$  distinct points on a circle. With a view of the graph associahedra, an  $n$ -dimensional cyclohedron  $W^n$  is a geometric realization of the poset  $\mathcal{PC}(n+1)$ , where  $\mathcal{C}(n+1)$  denotes an oriented counterclockwise cycle with  $n+1$  nodes labelled by the set  $\{1, 2, \dots, n+1\}$ . A tube in a 1-tubing on an  $(n+1)$ -cycle can be seen as  $k$ -paths, where  $k = 1, \dots, n$ . This means that each 1-tubing represents a face of a cyclohedron of the form  $W^{n-k} \times K^{k-1}$  and there are exactly  $n(n+1)$  codimension one faces on  $W^n$ . In addition, there are exactly two types of tubes. The tubes in the first type contain consecutive nodes and the tubes in the second type contain the nodes of the form  $\{1, 2, \dots, i, j, \dots, n\}$ , where  $1 \leq i < i+1 < j \leq n$ . One can call the tubes in the second type as *exotic tubes* in the sense of the definition of exotic subintervals in [9] given by Markl. It is clear that  $W^n$  can be combinatorially obtained by taking the convex hull of  $n+1$  disjoint copies of  $K^{n-1}$  and also give an orientation on  $W^n$  using the Tamari order on these copies of  $K^{n-1}$ . Hence we can get a triangulation of cyclohedron similar to the once given by Loday for associahedron.

**Theorem 2.** *The  $n$ -dimensional cyclohedron  $W^n$  admits a triangulation by  $(n+1)^n$  simplices with respect to the orientation induced by the Tamari order on its faces.*

*Proof.* By using induction on  $n$ , we assume that all the associahedral components of the faces of cyclohedron are triangulated with respect to the Tamari order and take cones over all the simplices on the faces with a common vertex as the barycenter of the cyclohedron. Let  $d_n$  denote the number of the simplices in the triangulation of  $W^n$ . If  $n = 1$ , then it is clear that there are exactly 2 faces of the form  $W^0 \times K^0$  and there are only two simplices in the triangulation. Suppose that for  $k < n$ ,  $W^{n-k}$  admits a triangulation by  $\lambda_{n-k}$  simplices. Then we show that  $W^n$  admits a triangulation by

$$\lambda_n = \sum_{k=1}^n (n+1) \binom{n-1}{k-1} \lambda_{n-k} k^{k-2}$$

$$\begin{aligned}
 &= \sum_{k=1}^n (n+1) \binom{n-1}{k-1} (n-k+1)^{n-k} k^{k-2} \\
 &= (n+1) \sum_{k=0}^{n-1} \binom{n-1}{k} (n-k)^{n-k-1} (k+1)^{k-1} \\
 &= (n+1)(n+1)^{n-1} = (n+1)^n.
 \end{aligned}$$

simplices. The last row follows from the Abel’s equation for  $x^{-1}(x + y + n - 1)^{n-1}$  with  $x = 1, y = 1$ . For further details, see Riordan [11].

### 3. Operations on Tubes and Loday’s Dendriform Algebra

In this section, we define "name" for a tubing on a path as a sequence of positive integers for coding. We also define some new operations on the collection of maximal tubings on a path motivated by Loday’s work [4]. These operations can be visualized as binding or nesting two maximal tubings to get a bigger maximal tubing. We give the sum and the product of maximal tubings as a collection of maximal tubings. Furthermore, we interpret Loday’s dendriform algebra on the collection of maximal tubings.

**Definition 1.** *The name of a tubing on an  $n$ -path is a finite sequence of positive integers such that the  $j$ -th term of the sequence is the number of tubes including the universal ones containing the  $j$ -th node of an  $n$ -path. Especially, the name of the tubing  $\mathbf{0}$  on the empty graph is 0. From now on,  $T = a_1 \dots a_n$  denotes a tubing with the name representation.*



Figure 2: Example of the name of a tubing on 5-path

**Remark 1.** *A tube  $t$  can be slid only in the smallest tube containing it. Let  $T_1 = a_1 \dots a_n$  be a tubing and let  $t_1 \in T_1$  be a tube. If the tube  $t_2 \in T_2$  is obtained by sliding  $t_1 \in T_1$  then the name  $b_1 \dots b_n$  of  $T_2$  is*

$$b_i = \begin{cases} a_i, & i \notin t_1 \cup t_2; \\ a_i, & i \in t_1 \cap t_2; \\ a_i - 1, & i \in t_1 \setminus t_2; \\ a_i + 1, & i \in t_2 \setminus t_1 \end{cases} \quad \text{for } 1 \leq i \leq n.$$

Let  $T = a_1 \dots a_n$  be a maximal tubing on an  $n$ -path, where  $a_i = 1$ . The sequences  $(a_1 - 1) \dots (a_{i-1} - 1)$  and  $(a_{i+1} - 1) \dots (a_n - 1)$  are still possible names of maximal tubings on  $(i - 1)$ -path and  $(n - i)$ -path, respectively. Such a partition leads us an operation on maximal tubings on paths.

**Definition 2.** Let  $T_1 = a_1 \dots a_m$  and  $T_2 = b_1 \dots b_n$  be two tubings on an  $m$ -path and an  $n$ -path,  $m, n > 0$ , respectively. The transition operation " $\vee$ " between  $T_1$  and  $T_2$  is defined by

$$T_1 \vee T_2 = (a_1 + 1) \dots (a_m + 1)1(b_1 + 1) \dots (b_n + 1)$$

and  $\mathbf{0} \vee T_1 = 1(a_1 + 1) \dots (a_m + 1)$  and  $T_1 \vee \mathbf{0} = (a_1 + 1) \dots (a_m + 1)1$ . It is easily seen that  $T_1 \vee T_2$  is also a maximal tubing on an  $(m + n + 1)$ -path. This operation is neither associative nor commutative. A remarkable feature of the transition operation is that, any maximal tubing  $T$  can be divided into two parts, called left  $T^l$  and right  $T^r$  parts, from the combining node.

**Example 1.** Let  $T^l = 123 \in MP(3)$  and  $T^r = 2341 \in MP(4)$  be given as on the left hand side of Figure 3. The maximal tubing  $T = T^l \vee T^r = 23413452 \in MP(8)$  can be obtained by connecting these tubings via a new node.

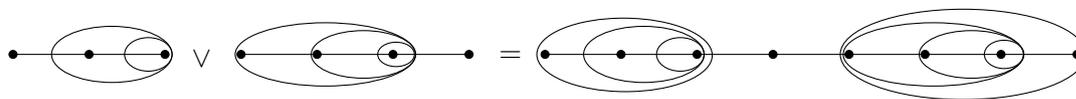


Figure 3: Example of the transition operation between the tubings

**Definition 3.** Let  $T_1 = a_1 \dots a_n \in MP(n)$  and  $T_2 = b_1 \dots b_m \in MP(m)$  be maximal tubings. The operations  $/, \setminus : MP(n) \times MP(m) \rightarrow MP(n + m)$  are called before and after operations and defined by

$$a_1 \dots a_n / b_1 \dots b_m = (a_1 + b_1) \dots (a_n + b_1) b_1 \dots b_m$$

$$a_1 \dots a_n \setminus b_1 \dots b_m = a_1 \dots a_n (b_1 + a_n) \dots (b_m + a_n)$$

**Example 2.** Let  $T_1 = 231 \in MP(3)$  and  $T_2 = 3212 \in MP(4)$  be given as on the left hand side in Figure 4. We illustrate the before and after operations for  $T_1$  and  $T_2$  in Figure 4.

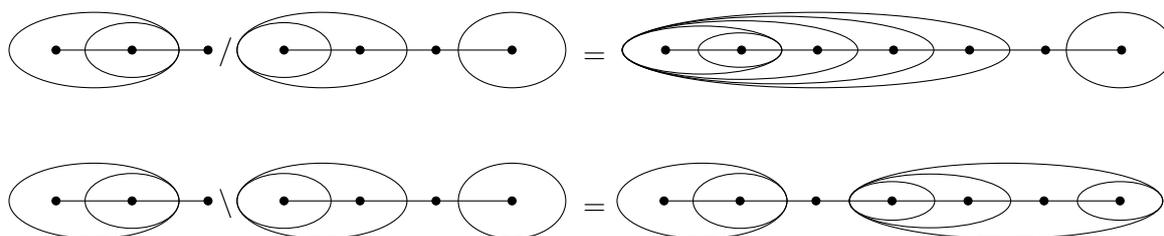


Figure 4: Before and after operations for the maximal tubings

**Definition 4.** The sum of two maximal tubings  $T_1$  and  $T_2$  is defined by

$$T_1 + T_2 := \bigcup_{T_1/T_2 \leq T \leq T_1 \vee T_2} T$$

as a set of maximal tubings. Let  $n$  denote the degree of  $T \in MP(n)$  which is the number of nodes in  $\mathcal{P}(n)$ . The degree of each maximal tubing in a sum of two maximal tubings is the sum of the degrees of the terms. Furthermore, since  $\mathbf{0}$  denotes the maximal tubing without any node, its degree is considered as zero and hence  $\mathbf{0}$  is the unit element with respect to “+”.

**Example 3.** Let  $T_1 = 123$  and  $T_2 = 212$  be two maximal tubings of degree 3. The sum of  $T_1$  and  $T_2$  is

$$T_1 + T_2 = \{345212, 245312, 235412, 234512, 134523, 124534, 123545\}.$$

**Definition 5.** A plumbing of degree  $n$  is a collection of maximal tubings in  $MP(n)$ . We denote the set of all plumbings of degree  $n$  by  $\mathbb{MP}(n)$  and  $\mathbb{MP}(\infty) := \bigcup_{n \geq 0} \mathbb{MP}(n)$ . The addition operation on  $\mathbb{MP}(\infty)$  is obtained from the one  $+ : \mathbb{MP}(n) \times \mathbb{MP}(m) \rightarrow \mathbb{MP}(n+m)$  which is defined by

$$\cup_i T_i + \cup_j T_j := \cup_{i,j} (T_i + T_j)$$

with the unit element  $\mathbf{0}$ .

We note that there is an involution of a tubing  $T$  in  $PP(n)$  denoted by  $\bar{T}$ . Let  $T = a_1 \dots a_n$  then  $\bar{T} = a_n \dots a_1$ . The idea of involution can be extended to  $\mathbb{MP}(\infty)$  and it makes  $\mathbb{MP}(\infty)$  an involutive graded monoid.

**Theorem 3.** Let  $T_1, T_2$  and  $T_3$  be three maximal tubings. We have the following equalities

$$(T_1/T_2) \setminus T_3 = T_1/(T_2 \setminus T_3), \quad T_1/(T_2 \vee T_3) = (T_1/T_2) \vee T_3, \quad (T_1 \vee T_2) \setminus T_3 = T_1 \vee (T_2 \setminus T_3)$$

and

$$\overline{T_1 \vee T_2} = \bar{T}_2 \vee \bar{T}_1, \quad \overline{T_1/T_2} = \bar{T}_2 \setminus \bar{T}_1, \quad \overline{T_1 \setminus T_2} = \bar{T}_2/\bar{T}_1, \quad \overline{T_1 + T_2} = \bar{T}_2 + \bar{T}_1$$

and also the inequalities

$$T \vee T_1 \leq T \vee T_2, \quad T_1 \vee T \leq T_2 \vee T, \quad T_1/T \leq T_2/T, \quad T \setminus T_1 \leq T \setminus T_2$$

hold. In addition, for all  $T, T' \in MP(n)$ , if  $T \leq T'$  then  $\bar{T}' \leq \bar{T}$ .

*Proof.* Let  $T_1 = a_1 \dots a_n$ ,  $T_2 = b_1 \dots b_m$  and  $T_3 = c_1 \dots c_l$  be maximal tubings. The first three equalities directly follow from definition. The next three equalities are obtained as follows:

$$\overline{T_1 \vee T_2} = \overline{a_1 \dots a_n \vee b_1 \dots b_m} = \overline{a_1 \dots a_n b_1 \dots b_m} = b_m \dots b_1 a_1 \dots a_n = \bar{T}_2 \vee \bar{T}_1,$$

$$\begin{aligned} \overline{T_1/T_2} &= \overline{a_1 \dots a_n / b_1 \dots b_m} = \overline{(a_1 + b_1) \dots (a_n + b_1) b_1 \dots b_m} \\ &= \overline{b_m \dots b_1 (a_n + b_1) \dots (a_1 + b_1)} = \overline{b_m \dots b_1 \setminus a_n \dots a_1} = \overline{T_2} \setminus \overline{T_1}, \\ \overline{T_1 \setminus T_2} &= \overline{a_1 \dots a_n \setminus b_1 \dots b_m} = \overline{a_1 \dots a_n (b_1 + a_n) \dots (b_m + a_n)} \\ &= \overline{(b_m + a_n) \dots (b_1 + a_n) a_n \dots a_1} = \overline{b_m \dots b_1 / a_n \dots a_n} = \overline{T_2} / \overline{T_1}. \end{aligned}$$

The inequalities also directly follow from Definitions 2 and 3 and Remark 1. For the last equality, we combine some of the equalities and the last two inequalities.

**Theorem 4.** *Let  $T_1$  and  $T_2$  be two maximal tubings different from  $\mathbf{0}$ . The sum of  $T_1$  and  $T_2$  is splitted into two parts as*

$$T_1 + T_2 = (T_1 + T_2^l) \vee T_2^r \cup T_1^l \vee (T_1^r + T_2),$$

for  $T_1 = T_1^l \vee T_1^r$  and  $T_2 = T_2^l \vee T_2^r$ , where  $T_i^l$  and  $T_i^r$  are left and right parts of the tubing respectively.

*Proof.* Let  $T_1 = a_1 \dots a_n \in MP(n)$  and  $T_2 = b_1 \dots b_m \in MP(m)$  be two maximal tubings such that  $a_i = 1$  and  $b_j = 1$ . Now we have

$$\begin{aligned} T_1 + T_2 &= \{T \in MP(n + m) \mid T_1/T_2 = (T_1/T_2^l) \vee T_2^r \leq T \leq T_1^l \vee (T_1^r \setminus T_2) = T_1 \setminus T_2\} \\ &= \{T = c_1 \dots c_{n+m} \in MP(n + m) \mid \\ &\quad (a_1 + b_1) \dots (a_n + b_1) b_1 \dots b_m \leq T \leq a_1 \dots a_n (b_1 + a_n) \dots (b_m + a_n)\}. \end{aligned}$$

From Definition 4, we directly observe that the tubes in  $T_1^l$  and  $T_2^r$  cannot be slid from left to right. We assume that there exist two tubes  $t_1$  and  $t_2$  different from the universal tube such that  $t_1 \notin T_1^l$  and  $t_2 \notin T_2^r$  contain all nodes in the tubes of  $T_1^l$  and  $T_2^r$ , respectively. According to the compatibility condition,  $i$ -th and  $(n + j)$ -th nodes must be contained in  $t_1$  and  $t_2$ , respectively. Since each tubing  $T$  in the sum  $(T_1 + T_2)$  is maximal, there must be a node contained only by the universal tube and this node must be between  $i$ -th and  $(n + j)$ -th nodes. This contradicts with  $a_k \geq 2$  for  $k = i + 1, \dots, n$  and  $b_k \geq 2$  for  $k = 1, \dots, j - 1$ . So there are exactly two types of tubings in  $(T_1 + T_2)$  in which the corresponding name has either  $c_i = 1$  or  $c_{n+j} = 1$ .

Now let us consider the first type of the tubings in  $T_1 + T_2$ , where  $c_i = 1$ . By definition, the maximum of these tubings is  $T_1 \setminus T_2 = a_1 \dots a_n (b_1 + a_n) \dots (b_m + a_n)$  but the minimum one can only be the tubing  $a_1 \dots a_i (a_{i+1} + b_1) \dots (a_n + b_1) (b_1 + 1) \dots (b_m + 1)$  and the rewriting the name of this tubing step by step as the following form

$$\begin{aligned} (a_1 - 1) \dots (a_{i-1} - 1) \vee ((a_{i+1} + b_1 - 1) \dots (a_n + b_1 - 1) b_1 \dots b_m) &= \\ (a_1 - 1) \dots (a_{i-1} - 1) \vee ((a_{i+1} - 1) \dots (a_n - 1) / b_1 \dots b_m) &= \\ T_1^l \vee T_1^r / T_2 & \end{aligned}$$

gives that the set of tubings of the first type is  $\{T \in MP(n + m) \mid T_1^l \vee (T_1^r / T_2) \leq T \leq T_1^l \vee (T_1^r \setminus T_2)\}$ . Hence the set of tubings of the first type is the plumbing  $T_1^l \vee (T_1^r + T_2)$ . Similarly

for the tubings of the second type, the minimum tubing is  $T_1/T_2$  and the maximum one is  $(T_1 \setminus T_2^l) \vee T_2^r$ . So the set of tubings of the second type is  $\{T \in MP(n+m) | (T_1/T_2^l) \vee T_2^r \leq T \leq (T_1 \setminus T_2^l) \vee T_2^r\}$  and clearly this set is the plumbing  $(T_1 + T_2^l) \vee T_2^r$ .

**Definition 6.** Let  $T_1$  and  $T_2$  be two maximal tubings. The left sum and the right sum of  $T_1$  and  $T_2$  are defined by  $T_1 \dashv T_2 := T_1^l \vee (T_1^r + T_2)$  and  $T_1 \vdash T_2 := (T_1 + T_2^l) \vee T_2^r$ , respectively. This can be extended over plumbings.

**Proposition 1.** The left and right sums of maximal tubings  $T_1, T_2$  and  $T_3$  satisfy the following relations

$$(T_1 \dashv T_2) \dashv T_3 = T_1 \dashv (T_2 + T_3), \tag{1}$$

$$(T_1 \vdash T_2) \dashv T_3 = T_1 \vdash (T_2 \dashv T_3), \tag{2}$$

$$T_1 \vdash (T_2 \vdash T_3) = (T_1 + T_2) \vdash T_3 \tag{3}$$

and  $\mathbf{0} \vdash T = T = T \dashv \mathbf{0}$  for all  $T$ .

*Proof.* To prove (1), we do the following computation

$$\begin{aligned} (T_1 \dashv T_2) \dashv T_3 &= (T_1 + T_2)^l \vee ((T_1 \dashv T_2)^r + T_3) = (T_1^l \vee (T_1^r + T_2))^l \vee ((T_1^l \vee (T_1^r + T_2))^r + T_3) \\ &= T_1^l \vee ((T_1^r + T_2) + T_3) = T_1^l \vee (T_1^r + (T_2 + T_3)) = T_1 \dashv (T_2 + T_3). \end{aligned}$$

For (2), we compute both sides and see that they are equal.

$$\begin{aligned} (T_1 \vdash T_2) \dashv T_3 &= (T_1 \vdash T_2)^l \vee ((T_1 \vdash T_2)^r + T_3) \text{ (by definition of } \dashv \text{)} \\ &= ((T_1 + T_2^l) \vee T_2^r)^l \vee (((T_1 + T_2^l) \vee T_2^r)^r + T_3) \text{ (by definition of } \vdash \text{)} \\ &= (T_1 + T_2^l) \vee (T_2^r + T_3) \\ T_1 \vdash (T_2 \dashv T_3) &= T_1 \vdash (T_2^l \vee (T_2^r + T_3)) \text{ (by definition of } \dashv \text{)} \\ &= (T_1 + (T_2^l \vee (T_2^r + T_3))^l) \vee (T_2^l \vee (T_2^r + T_3))^r \text{ (by definition of } \vdash \text{)} \\ &= (T_1 + T_2^l) \vee (T_2^r + T_3) \end{aligned}$$

Finally, for (3) we have the following equalities

$$\begin{aligned} T_1 \vdash (T_2 \vdash T_3) &= (T_1 + (T_2 \vdash T_3)^l) \vee (T_2 \vdash T_3)^r \text{ (by definition of } \vdash \text{)} \\ &= (T_1 + ((T_2 + T_3^l) \vee T_3^r)^l) \vee ((T_2 + T_3^l) \vee T_3^r)^r \text{ (by definition of } \vdash \text{)} \\ &= (T_1 + ((T_2 + T_3^l)) \vee T_3^r) = ((T_1 + T_2) + T_3^l) \vee T_3^r = (T_1 + T_2) \vdash T_3. \end{aligned}$$

**Corollary 1.** Let  $T_1 \in MP(n), T_2 \in MP(m)$  be two maximal tubings. The left and right sums satisfy

$$\overline{T_1 \vdash T_2} = \overline{T_2} \dashv \overline{T_1}, \quad \overline{T_1 \dashv T_2} = \overline{T_2} \vdash \overline{T_1}.$$

**Definition 7.** A unique way of writing  $T$  as a composition of  $n$  copies of the tubing  $\mathbf{1} \in MP(1)$  with the left and right sums modulo the relations given in Proposition 1 is called the universal expression of  $T \in MP(n)$  and denoted by  $w_T(\mathbf{1})$ .

**Example 4.** The universal expression of  $T = 2132$  is  $w_T(\mathbf{1}) = \mathbf{1} \vdash \mathbf{1} \dashv (\mathbf{1} \dashv \mathbf{1})$ .

**Definition 8.** Let  $T_1$  and  $T_2$  be any two maximal tubings. The product  $T_1 \times T_2$  of  $T_1$  and  $T_2$  is the ordered sum of the copies of  $T_2$  in the universal expression of  $T_1$ . The product is distributive from the left on each type of sums and it is not commutative. The product of two maximal tubings can be extended to the product of plumbings.

**Corollary 2.** Let  $T_1 = T_1^l \vee T_1^r$  be a maximal tubing. The product can be given by the formula

$$T_1 \times T_2 = (T_1^l \times T_2) \vdash T_2 \dashv (T_1^r \times T_2)$$

and  $\mathbf{0} \times T_2 = \mathbf{0}$  for all maximal tubings  $T_2$ . One also has  $\overline{T_1 \times T_2} = \overline{T_1} \times \overline{T_2}$  for any two maximal tubings  $T_1$  and  $T_2$ .

Now, we define the dendriform algebra on  $M\mathcal{P}(\infty)$ .

**Definition 9.** A dendriform algebra is a vector space  $A$  equipped with two binary operations  $\prec, \succ: A \otimes A \rightarrow A$  satisfying the following axioms

$$(a \prec b) \prec c = a \prec (b * c), \tag{4}$$

$$(a \succ b) \prec c = a \succ (b \prec c), \tag{5}$$

$$(a * b) \succ c = a \succ (b \succ c) \tag{6}$$

for all  $a, b, c \in A$ , where the operation  $*$  defined by  $a * b := a \prec b + a \succ b$  is associative.

**Definition 10.** Let  $F$  be a field and  $F[M\mathcal{P}(\infty)']$  be the vector space generated by the elements  $X^T$ , for  $T \in M\mathcal{P}(n)$  and  $n \geq 1$ , that is, we do not consider the elements of the form  $X^{\mathbf{0}} = 1$ . Operations on  $F[M\mathcal{P}(\infty)']$  are defined by

$$X^T \prec X^{T'} = X^{T \dashv T'}, \tag{7}$$

$$X^T \succ X^{T'} = X^{T \vdash T'} \tag{8}$$

for any two maximal tubings  $T, T'$  and  $X^{T \cup T'} := X^T + X^{T'}$ .

**Proposition 2.** The vector space  $F[M\mathcal{P}(\infty)']$  equipped with the two operations  $\prec$  and  $\succ$  becomes a dendriform algebra by defining  $X^T * X^{T'} = X^{T \dashv T'}$ . The operations  $\prec$  and  $\succ$  can be partially extended to  $F[M\mathcal{P}(\infty)]$  as  $X^{\mathbf{0}} \succ X^T = X^T = X^T \prec X^{\mathbf{0}}$  for all  $T$  and then  $F[M\mathcal{P}(\infty)] = F[M\mathcal{P}(\infty)'] \oplus F \cdot 1$  becomes an augmented unital associative algebra.

### 4. An Operad on Paths via Tubings

In [9], Markl gives a description of the cellular operad structure of associahedron. Here we reconstruct it by using tubings on paths. In order to do that, first we define the comp or composition maps on the collection  $\{PP(n)\}$ . Let  $T_1 = a_1 \dots a_n \in PP(n)$  and  $T_2 = b_1 b_2 \dots b_m \in PP(m)$ . For  $n, m \geq 1$  and  $0 \leq i \leq n$ , the comp map  $\circ_i: PP(n) \times PP(m) \rightarrow PP(n+m)$  is given by

$$T_1 \circ_i T_2 = a_1 a_2 \dots a_i (b_1 + \tilde{a}_i) (b_2 + \tilde{a}_i) \dots (b_m + \tilde{a}_i) a_{i+1} \dots a_n \tag{9}$$

where  $\tilde{a}_i$  is the maximum of the integers  $a_i, a_{i+1}$ , that is,  $\tilde{a}_i = \max(a_i, a_{i+1})$ , for  $0 < i < n$  and  $\tilde{a}_0 = a_1$  and  $\tilde{a}_n = a_n$ . It can be easily checked that for any three tubings  $T_1, T_2$  and  $T_3$ , the comp map satisfies the following equations.

$$T_1 \circ_i (T_2 \circ_j T_3) = (T_1 \circ_i T_2) \circ_{j+i} T_3, \quad 0 \leq j \leq m, \tag{10}$$

$$(T_1 \circ_i T_2) \circ_j T_3 = (T_1 \circ_{j-m} T_3) \circ_i T_2, \quad i + m + 1 < j \leq n + m. \tag{11}$$

Since we shift the indices 1, one can think that they are different from the usual definition of comp maps in a non symmetric operad. But the operad structure of  $PP(\infty) := \{PP(m)\}_{m \geq 1}$  with these comp maps has a one to one correspondence with the operad structure of  $\{K^n\}_{n \geq 0}$ . If we let the comp maps to be distributive over the union, then the collection  $MP(\infty)$  is closed under the comp maps  $\circ_i$ . This property leads us to give the dendriform operad  $\{Dend(n)\}_{n \geq 1}$  in terms of tubings on paths. The sequence  $\{Dend(n)\}_{n \geq 1}$  forms an operad whose  $i$ -th comp map  $\circ_i : Dend(n) \times Dend(m) \rightarrow Dend(n + m - 1)$  is defined by  $X^T \circ_i X^{T'} = X^{T \circ_i T'}$ , where  $Dend(n) = F[MP(n - 1)]$  for any field  $F$ .

Now, we construct the cellular chain complex of an associahedron in terms of tubings on paths. Let  $T \in PP(n)$  be a  $k$ -tubing whose tubes are labelled by an order  $e_1, \dots, e_k$  except the universal one. Two orderings  $e_{i_1}, \dots, e_{i_k}$  and  $e_{j_1}, \dots, e_{j_k}$  are equivalent if they are related by an even permutation. The equivalence class corresponding to an ordering  $e_{i_1}, \dots, e_{i_k}$  is called the orientation and denoted by  $e_{i_1} \wedge \dots \wedge e_{i_k} = \omega$ . An oriented  $k$ -tubing  $T \in PP(n)$  with its orientation  $\omega$  is a pair  $(T, \omega)$ . Let  $CC_{n-k}(K^n)$  be a vector space spanned by the oriented  $(n - k)$ -cells in  $K^n$  which are related by the oriented  $k$ -tubings in  $PP(n + 1)$  modulo the relation  $(T, \omega_1) = -(T, \omega_2)$ , where  $\omega_1$  and  $\omega_2$  are distinct orientations. The boundary operator on  $CC_*(K^n)$  is defined by

$$\partial(T, \omega) := \sum_{T' = \{1, \dots, n+1\}, t_1, \dots, t_{k+1}} (T', e' \wedge \omega) \tag{12}$$

for oriented  $k$ -tubings on  $\mathcal{P}(n + 1)$ . This sum is taken over all  $(k + 1)$ -tubings such that  $T = T' \setminus \{t_i\}$  for some  $i = 1, \dots, k + 1$  and  $e'$  is the label of  $t_i$ . The boundary map  $\partial$  satisfies  $\partial^2 = 0$ . Moreover, the comp maps in the operad  $\{PP(n)\}_{n \geq 1}$  can also be extended on the oriented  $k$ -tubings with a sign convention. For  $0 \leq i \leq n$ , the comp map  $\circ_i : PP(n) \times PP(m) \rightarrow PP(n + m)$  is given by

$$(T_1, \omega) \circ_i (T_2, \omega') := (-1)^{(n+1)l+m(i+1)} (T_1 \circ_i T_2, \omega \wedge \omega' \wedge e),$$

where  $T_1, T_2$  are  $(n - k)$  and  $(m - l)$ -tubings and  $e$  denotes the label of the universal tube of  $T_2$  and  $\omega \wedge \omega' \wedge e$  is the consecutive composition of orientations.

Finally, we define a name of a tubing on a cycle as a finite sequence of positive integers such that the  $j$ -th term of the sequence is the number of tubes containing the  $j$ -th node of  $\mathcal{C}(n)$ . This leads us to construct the module structure of the collection  $\{W^n\}_{n \geq 0}$  via tubings. The  $i$ -th (right) comp map  $\circ_i^r : PC(n) \times PP(m) \rightarrow PC(n + m)$  is defined by

$$T_2 \circ_i^r T_1 = b_1 \dots b_i (a_1 + \tilde{b}_i) \dots (a_m + \tilde{b}_i) b_{i+1} \dots b_n,$$

where  $T_1 = a_1 \dots a_m \in PP(m)$ ,  $T_2 = b_1 \dots b_n \in PC(n)$  and  $\tilde{b}_i = \max(b_i, b_{i+1})$  for  $1 \leq i \leq n - 1$  and  $\tilde{b}_i = \max\{b_1, b_n\}$  for  $i = 0$  and  $i = n$ .

**Proposition 3.** *The collection  $PC(\infty) := \{PC(n)\}_{n \geq 1}$  is a (right) module over the operad  $PP(\infty) := \{PP(m)\}_{m \geq 1}$ . Clearly, this module structure is the same as the module structure of  $\{W^n\}_{n \geq 0}$  over  $\{K^m\}_{m \geq 0}$  in the sense of given in Markl [9].*

### 5. On Integral Sequences via Tubings

In this section, we give the definition of a labelled maximal tubing on a path. We work on a sequence in order to construct a chain complex and define the boundary map. Finally, we compute the corresponding homology groups.

**Definition 11.** *A labelled maximal tubing on a path is a maximal tubing such that each tube is labeled by an element of a finite set  $I$  of indices.*

Note that these elements are not necessarily distinct. A labelled maximal tubing can be considered as a pair  $(T, f)$  such that  $f : T \rightarrow I$  maps a tube to its label. Clearly, there is a bijection between the set of labelled maximal tubings on  $\mathcal{P}(n)$  and  $M\mathcal{P}(n) \times I^n$ . In particular,  $M\mathcal{P}(2)$  has only two elements whose names are 21 and 12 and there are exactly two types of labelled maximal tubings in  $M\mathcal{P}(2) \times I^2$  such that either of the form  $(21, i_1, i_2)$  or  $(12, i_1, i_2)$ .

**Definition 12.** *The nested tubes  $t_1$  and  $t_2$  in a maximal tubing  $T \in M\mathcal{P}(n)$  such that  $t_1 \subset t_2$  are called local pattern in the sense of Loday [5] if there is no any tube  $t$  such that  $t_1 \subset t \subset t_2$ .*

In order to construct a chain complex, we define a sequence  $S = \{S_n \times I^n\}_{n \geq 0}$  by an induction on  $n$  as follows: By convention, we assume that  $S_0 \times I^0 = M\mathcal{P}(0) \times I^0, S_1 \times I^1 = M\mathcal{P}(1) \times I, S_2 \times I^2 \subset M\mathcal{P}(2) \times I^2$ . A labelled maximal tubing  $T$  is in  $S_n \times I^n$  if and only if all local patterns of  $T$  illustrated either by the form 21 or 12 are in  $S_2$ . The alternating series of the sequence  $S$  is determined by the numbers of elements in  $S_n \times I^n$  as follows:

$$f(S, t) = \sum_{n \geq 0} (-1)^{n+1} (\#(S_n \times I^n)) t^{n+1}.$$

Let  $Z = \{Z_n \times I^n\}$  be another sequence such that  $Z_2 \times I^2$  is the complement of  $S_2 \times I^2$  in  $M\mathcal{P}(2) \times I^2$  which implies that a labelled maximal tubing  $T$  is in  $Z_n \times I^n$  if it has a local pattern which is not contained in  $S_2 \times I^2$ .

Now, we define a chain complex  $C_* = (C_n, \partial)_{n \geq 0}$  over a field  $F$ . The space of  $n$  chains is defined by

$$C_n = \bigoplus_{i_0, \dots, i_n} F[Z_n \times S_{i_0} \times \dots \times S_{i_n}]$$

where  $i_j \geq 0$ . A vector  $\omega := (z_n, T_{i_0}, \dots, T_{i_n})$  in  $F[Z_n \times S_{i_0} \times \dots \times S_{i_n}]$  is of the form

$$\omega = \left( \dots ((z_n \circ_0 T_{i_0}) \circ_{i_0+1} T_{i_1}) \dots \right) \circ_{n-1} \begin{matrix} T_{i_n} \\ (\sum_{k=0}^{n-1} i_k + 1) \end{matrix}$$

The boundary map of the complex is defined by

$$\partial = \sum_{k=1}^n (-1)^k d_k, \tag{13}$$

where  $d_k \omega = d_k(z_n; T_0, \dots, T_n) = 0$  if the minimal tube containing the  $k$ -th node in  $z_n$  is not a cup, that is, it contains only one node and otherwise  $d_k \omega = (\varepsilon_k(z_n), T_0, \dots, T_{k-1} \vee T_k, \dots, T_n)$  where  $\varepsilon_k$  deletes the  $k$ -th node and the cup containing it in  $z_n$ . Note that the label of the tube covering  $T_{k-1} \vee T_k$  is the label of the deleted cup in  $z_n$ . If  $T_{k-1} \vee T_k$  contains a local pattern in  $Z$  then  $d_k(z_n, T_0, \dots, T_n) = 0$ . The complex  $C_*$  is called the Koszul complex of  $S$ .

**Proposition 4.** *The boundary map  $\partial$  satisfies the boundary condition  $\partial^2 = 0$ .*

*Proof.* Let  $\omega = (z_n, T_0, \dots, T_n)$ . We need to prove that  $d_k d_j = d_{j-1} d_k$  for  $k < j$ . If  $k < j - 1$ , we have the following three cases.

- If  $z_n$  does not have a cup at the  $k$ -th node then  $d_k d_j = d_{j-1} d_k = 0$ .
- If  $z_n$  does not have a cup at the  $j$ -th node then  $d_k d_j = 0$  and by renumbering the maximal tubings we get  $d_{j-1} d_k = 0$ .
- If  $z_n$  does not have a cup neither at the  $k$ -th nor  $j$ -th nodes then

$$\begin{aligned} d_k d_j(z_n, T_0, \dots, T_n) &= d_k(\varepsilon_j(z_n), T_0, \dots, T_{j-1} \vee T_j, \dots, T_n) \\ &= (\varepsilon_k \varepsilon_j(z_n), T_0, \dots, T_{k-1} \vee T_k, \dots, T_{j-1} \vee T_j, \dots, T_n) \\ &= (\varepsilon_{j-1} \varepsilon_k(z_n), T_0, \dots, T_{k-1} \vee T_k, \dots, T_{j-1} \vee T_j, \dots, T_n) \\ &= d_{j-1}(\varepsilon_k(z_n), T_0, \dots, T_{k-1} \vee T_k, \dots, T_n) \\ &= d_{j-1} d_k(z_n, T_0, \dots, T_n) \end{aligned}$$

For  $k = j - 1$ , we examine  $d_k d_{k+1}$  and  $d_k d_k$ . It is clear that  $d_k d_k = 0$  because  $\varepsilon_k(z_n)$  cannot have a cup at the  $k$ -th node. On the other hand, if  $z_n$  does not have a cup at the  $(k + 1)$ -st node then  $d_k d_{k+1} = 0$ , otherwise

$$d_k d_{k+1}(\omega) = (\varepsilon_k \varepsilon_{k+1}(z_n), T_0, \dots, T_{k-1} \vee (T_k \vee T_{k+1}), \dots, T_n)$$

since  $\varepsilon_{k+1}(z_n)$  have a cup at the  $k$ -th node. The universal tubes of  $T_k \vee T_{k+1}$  and  $T_{k-1} \vee (T_k \vee T_{k+1})$  constitute a local pattern with the labels of the cups in  $z_n$ . Hence  $T_{k-1} \vee (T_k \vee T_{k+1})$  is in  $Z$  which gives that  $d_k d_{k+1}(\omega) = 0$ .

**Definition 13.** *The basis vector  $\omega = (z_n, T_0, \dots, T_n)$  is called an extremal vector with  $i$  cups in  $z_n$  for each  $n \geq 0$  and  $0 < i < n$  if there is no any element  $\omega'$  such that  $d_k(\omega') = \omega$  for some  $k \geq 0$ .*

**Proposition 5.** *For each extremal vector  $\omega$  with  $k$  cups, the subcomplex  $C_\omega$  spanned by the basis vectors  $d_{i_1} \dots d_{i_r} \omega$ ,  $r \leq k$ , is isomorphic to the augmented chain complex of the standard simplex  $\Delta^{k-1}$ .*

*Proof.* The graded subvector space of  $C_*$  spanned by the elements  $d_{i_1} \dots d_{i_r} \omega$  is stable by  $\partial$  and forms a subcomplex. The bijection between the cells of  $\Delta^{k-1}$  and the set of non-zero vectors  $\{d_{i_1} \dots d_{i_r} \omega\}$  is constructed by sending the  $(j - 1)$ -st vertex of  $\Delta^{k-1}$  to the vector  $d_{i_1} \dots d_{i_j} \dots d_{i_k} \omega$  where  $i_j$  is the  $j$ -th cup of  $z_n$ . Here the vertices of  $\Delta^{k-1}$  are enumerated by 0 to  $(k - 1)$ . This leads us that the boundary map on the chain complex of the standart simplex corresponds to the boundary map  $\partial$  defined in (13).

**Proposition 6.** *The chain complex  $C_*$  is isomorphic to  $\bigoplus_{\omega} C_{\omega}$  for all extremal vectors  $\omega$ .*

*Proof.* If a vector  $\omega \in C_*$  is an extremal vector, then it is clear that  $\omega \in C_{\omega}$ . Otherwise, there exists an extremal vector  $\omega_1$  such that  $d_{i_1} d_{i_2} \dots d_{i_r}(\omega_1) = \omega$  for some  $i_j$ , where  $1 \leq j \leq r$  and it gives that  $\omega \in C_{\omega_1}$ .

Now, we want to prove that any basis vector belongs to one and only one subcomplex of the form  $C_{\omega}$ , that is, if a basis vector belongs to both  $C_{\omega}$  and  $C_{\omega_1}$  then we want to show that  $\omega = \omega_1$ . Let  $\omega = (z_n, T_0, \dots, T_n)$  and  $\omega_1 = (z'_n, T'_0, \dots, T'_n)$  be given. If  $d_i(\omega) = d_j(\omega_1) \neq 0$ , then the  $i$ -th node is contained by a cup in  $z_n$  and the  $j$ -th node is contained by a cup in  $z'_n$  such that  $\varepsilon_i(z_n) = \varepsilon_j(z'_n)$ . If  $i < j$ , then there exists  $\hat{\omega}$  such that  $d_j(\hat{\omega}) = \omega$  and  $d_i(\hat{\omega}) = \omega_1$ . Therefore  $\omega$  and  $\omega_1$  are not extremal which follows that  $i = j$ . In other words,

$$d_i(\omega) = d_i(\omega_1) = (\varepsilon_i(z), T_0, \dots, T_{i-1} \vee T_i, \dots, T_n). \tag{14}$$

The only vector is  $(z_n, T_0, \dots, T_n)$  satisfying (14) and therefore  $\omega = \omega_1$ .

**Proposition 7.** *The complex  $C_*$  is acylic for any choice of a sequence  $S$ , that is,  $H_n(C_*) = 0$  for all  $n > 0$  and  $H_0(C_*) = F$ .*

*Proof.* By Propositions 5 and 6 the homology of the complex is trivial because the standart simplex is contractible. There is only one exception in dimension 0 because the subcomplex corresponding to the element  $\omega = (\mathbf{0}, \mathbf{0})$  is  $F$  in dimension 0. So we have  $H_n(C_*) = 0$  for all  $n > 0$  and  $H_0(C_*) = F$ .

**Proposition 8.** *The Poincare series of the complex  $C_*$  is equal to  $f(Z, f(S, t))$ . This also gives that if  $Z_2 \times I^2$  is the complement of  $S_2 \times I^2$  in  $MP(2) \times I^2$  then  $f(Z, f(S, t)) = t$ .*

*Proof.* By the construction of the complex  $C_*$ , it is clear that the Poincare series of the complex  $C_*$  is equal to  $f(Z, f(S, t))$ . Since the Poincare series of a complex is the same as the Poincare series of its homology and using Proposition 7,  $f(Z, f(S, t))$  becomes an identity polynomial on the variable  $t$ .

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