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Simultaneous null controllability for two stroke nonlinear systems: Application to the sentinel of detection in population dynamics model with incomplete data

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Abstract. This paper deals with the simultaneous null controllability for some nonlinear two stroke systems. We shall solve this problem by transforming the simultaneous null controllability of uncoupled initial systems into a null controllability of a coupled system via a change of variables. This last problem is solved thanks to a global Carleman inequality, appropriates estimates adapted to the system and via some fixed point theorems. The obtained results are used to build a simultaneous sentinel of detection in a population dynamics model with incomplete data.

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \in \{1,2,3\}$ with boundary Γ of class C^2. Let $\omega \subset \Omega$ be an open nonempty subset. For a time $T > 0$ and the common life expectancy $A > 0$ of species, we set $U = (0,T) \times (0,A)$, $Q = U \times \Omega, Q_\omega = U \times \omega$, $Q_{\mathrm{T}} = (0,\mathrm{T}) \times \Omega$, $Q_{\mathrm{A}} = (0,\mathrm{A}) \times \Omega, \Sigma = U \times \Gamma$, $\Sigma_T = (0,T) \times \Gamma$ and we consider the following

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nonlinear two stroke systems :

$$\left\{\begin{array}{rcll} -\dfrac{\partial q_1}{\partial t}-\dfrac{\partial q_1}{\partial a}-\Delta q_1+\mu_1 q_1 &=& \beta_1 F\left(\int_0^A \beta_1 q_1 da\right) q_1(t,0,x) & \\ && +h+w\chi_\omega & \text{in} \quad Q, \\ -\dfrac{\partial q_2}{\partial t}-\dfrac{\partial q_2}{\partial a}-\Delta q_2+\mu_2 q_2 &=& \beta_2 G\left(\int_0^A \beta_2 q_2 da\right) q_2(t,0,x) & \\ && +h+w\chi_\omega & \text{in} \quad Q, \\ q_1(T,a,x)=q_2(T,a,x) &=& 0 & \text{in} \quad Q_A, \\ q_1(t,A,x)=q_2(t,A,x) &=& 0 & \text{in} \quad Q_T, \\ q_1=q_2 &=& 0 & \text{on} \quad \Sigma, \end{array}\right. \tag{1}$$

for some functions F, G defined on $\mathbb{R}$. We assume that

(H_0) the functions F, G belong to $L^\infty(\mathbb{R})$ and $F(0)=G(0)=0$.

The simultaneous null controllability problem can be stated as follows : *Given $h \in L^2(Q)$ find $w \in L^2(Q_\omega)$ such that the solution of (1) satisfies*

$$q_1(0,a,x)=q_2(0,a,x)=0 \ \ a.e \ \ (a,x) \text{ in } Q_A. \tag{2}$$

The null controllability problem for one two stroke system with one and only one control is well understood: it has been studied by several authors using different methods. We refer to B. Ainseba and M. Langlais [2], B. Ainseba and S. Anita [3]. We also refer to S. Sawadogo [9], O. Traoré [12], Y. Simporé and O. Traoré [10] and their bibliography for other related controllability problems. As far as we know, there is no results on simultaneous null controllability for nonlinear two stroke systems. In this paper we focus on the previous problem in order to applicate it to build a simultaneous sentinel of detection in population dynamics problem with incomplete data.

The remainder of this paper is organized as follows : In order to well pose our problem, in section 2 we make some assumptions, transform the system (1) into an equivalent cascade problem and we state the main result of this paper. In section 3, we state first some Carleman's inequalities that we had established in [11]. Afterwards, we study the controllability for a linear intermediate problem and for another nonlinear. The section 4 is devoted to the proof of the main result and in the last section we use the result obtained in section 4 to build a simultaneous sentinel.

2. Assumptions and main result

For the sequel, the following assumptions hold:

$$(H_1)\ \left\{\begin{array}{rcll} (\mu_i,\ \nabla\mu_i) & \in & (L^\infty(Q))^{N+1} & \text{for all} \quad i\in\{1;2\},\ N\in\{1,2,3\}, \\ \mu_i & \geq 0 & \text{in } Q & \text{for all} \quad i\in\{1;2\}, \\ \mu_1 & \neq \mu_2 & \text{in } Q_\omega. & \end{array}\right.$$

(H_2) $\begin{cases} \beta_i \in C^2(\overline{Q}) & \text{for all} \quad i \in \{1;2\}, \\ \beta_i \geq 0 \ \text{ in } \overline{Q} & \text{for all} \quad i \in \{1;2\}. \end{cases}$

(H_3) There exists positive constants non null a_0 and a_1 with $a_0 < a_1 < A$ such that for each $i \in \{1;\ 2\}$, $\beta_i(t,a,x) = 0$ a.e $(t,a,x) \in (0,T) \times ([0,a_0] \cup [a_1,A]) \times \Omega$.

Under the assumptions $(H_0) - (H_3)$, for all $h \in L^2(Q), w \in L^2(Q_\omega)$ the system (1) admits an unique solution (q_1,q_2) in $L^2(U,H_0^1(\Omega))^2$ such that $\frac{\partial q_i}{\partial t} + \frac{\partial q_i}{\partial a} \in L^2(U;H^{-1}(\Omega))$ where $H^{-1}(\Omega)$ is the dual of the Hilbert space $H_0^1(\Omega)$. Moreover (q_1,q_2) belong to $C((0,T);L^2(Q_A)) \cap C((0,A);L^2(Q_T)) \cap L^2(U,H_0^1(\Omega))^2$ (see Lemma 0 in [5]).

Remark 1. *Assume that* (H_1) *holds and set*

$$p_1 = q_1 + q_2 \quad ; \quad p_2 = q_1 - q_2. \tag{3}$$

Thus, the condition (2) *is equivalent to* $p_1(0,a,x) = p_2(0,a,x) = 0$ *a.e* (a,x) *in* Q_A. *The following changes are required :*

$$\begin{array}{rcl} \hat{\mu}_1 &=& \frac{1}{2}(\mu_1+\mu_2)\ ,\ \hat{\mu}_2 = \frac{1}{2}(\mu_1-\mu_2)\ , \quad f = 2h\ ,\ k = 2w, \\ \hat{\beta}_1(p_1,p_2) &=& \frac{1}{2}\left[\beta_1 F\left(\frac{1}{2}\int_0^A \beta_1(p_1+p_2)da\right) + \beta_2 G\left(\frac{1}{2}\int_0^A \beta_2(p_1-p_2)da\right)\right], \\ \hat{\beta}_2(p_1,p_2) &=& \frac{1}{2}\left[\beta_1 F\left(\frac{1}{2}\int_0^A \beta_1(p_1+p_2)da\right) - \beta_2 G\left(\frac{1}{2}\int_0^A \beta_2(p_1-p_2)da\right)\right]. \end{array}$$

Then, the null controllability problem (1)-(2) is equivalent to the problem : *for any* $\hat{\mu}_1, \hat{\mu}_2 \in L^\infty(Q)$ *and for* $f \in L^2(Q)$ *find a control*

$$k \in L^2(Q_\omega) \tag{4}$$

such that the pair $p = (p_1,p_2)$ *solution of the system*

$$\left\{ \begin{array}{rcll} -\dfrac{\partial p_1}{\partial t} - \dfrac{\partial p_1}{\partial a} - \Delta p_1 + \hat{\mu}_1 p_1 + \hat{\mu}_2 p_2 &=& \hat{\beta}_1(p)p_1(t,0,x) & \\ && +\, \hat{\beta}_2(p)p_2(t,0,x) & \\ && +\, f + k\chi_\omega & \text{in} \quad Q\ , \\ -\dfrac{\partial p_2}{\partial t} - \dfrac{\partial p_2}{\partial a} - \Delta p_2 + \hat{\mu}_1 p_2 + \hat{\mu}_2 p_1 &=& \hat{\beta}_2(p)p_1(t,0,x) & \\ && +\, \hat{\beta}_1(p)p_2(t,0,x) & \text{in} \quad Q\ , \\ p_1 = p_2 &=& 0 & \text{on} \quad \sum\ , \\ p_1(T,a,x) = p_2(T,a,x) &=& 0 & \text{in} \quad Q_A, \\ p_1(t,A,x) = p_2(t,A,x) &=& 0 & \text{in} \quad Q_T, \end{array} \right. \tag{5}$$

satisfies

$$p_1(0,a,x) = p_2(0,a,x) = 0 \ \text{ in } \ Q_A. \tag{6}$$

Notice that system (5) admits an unique solution (p_1,p_2) in $\big(C((0,T);L^2(Q_A)) \cap C((0,A);L^2(Q_T)) \cap L^2(U,H_0^1(\Omega))\big)^2$ for each control k verifying (4). The main goal of this paper is to prove the following result :

Theorem 1. *Let Ω be an open subset of $\mathbb{R}^N$ with boundary Γ of class C^2 and ω be a non empty subset of Ω. Assume that the hypothesis $(H_0)-(H_3)$ hold. There exists a positive real function θ (θ is defined by (13)) such that for any function $f \in L^2(Q)$ with $\theta f \in L^2(Q)$, there exists an unique control $\tilde{k}$, of minimal norm in $L^2(Q_\omega)$ such that $\left(\tilde{k}, \tilde{p_1}, \tilde{p_2}\right)$ is solution of the simultaneous null controllability problem (5)-(6). Moreover, the control $\tilde{k}$ is given by*

$$\tilde{k} = \tilde{\eta_1}\chi_\omega \tag{7}$$

and verifies

$$\|\tilde{k}\|_{L^2(Q_\omega)} \leq C\left(\|\theta f\|_{L^2(Q)} + \|f\|_{L^2(Q)}\right) \tag{8}$$

where $\tilde{\eta} = (\tilde{\eta_1}, \tilde{\eta_2})$ satisfies

$$\left\{\begin{array}{rcll}
\dfrac{\partial\tilde{\eta_1}}{\partial t} + \dfrac{\partial\tilde{\eta_1}}{\partial a} - \Delta\tilde{\eta_1} + \hat{\mu}_1\tilde{\eta_1} + \hat{\mu}_2\tilde{\eta_2} & = & 0 & in \quad Q, \\
\dfrac{\partial\tilde{\eta_2}}{\partial t} + \dfrac{\partial\tilde{\eta_2}}{\partial a} - \Delta\tilde{\eta_2} + \hat{\mu}_1\tilde{\eta_2} + \hat{\mu}_2\tilde{\eta_1} & = & 0 & in \quad Q, \\
\tilde{\eta_1} = \tilde{\eta_2} & = & 0 & on \quad \Sigma, \\
\tilde{\eta_1}(t,0,x) & = & \displaystyle\int_0^A \left(\hat{\beta}_1(p)\tilde{\eta_1} + \hat{\beta}_2(p)\tilde{\eta_2}\right) \mathrm{da} & in \quad Q_T, \\
\tilde{\eta_2}(t,0,x) & = & \displaystyle\int_0^A \left(\hat{\beta}_2(p)\tilde{\eta_1} + \hat{\beta}_1(p)\tilde{\eta_2}\right) \mathrm{da} & in \quad Q_T.
\end{array}\right. \tag{9}$$

with $\tilde{p} = (\tilde{p_1}, \tilde{p_2})$.

3. Null controllability result for some coupled models

Before tackling the controllability problem, we will state the following results.

3.1. Global Carleman's inequality and observability inequality result

For any positive parameters λ and τ, we define the positive functions:

$$\alpha(t,a,x) = \tau\frac{e^{\frac{4}{3}\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{at\,(T-t)} \quad \text{and} \quad \varphi(t,a,x) = \frac{e^{\lambda\psi(x)}}{at\,(T-t)}\,,\ \forall\,(t,a,x) \in Q.$$

Remark 2. *As a reminder (see [4]) the function $\psi \in C^2(\overline{\Omega})$ is such that :*

$$\forall x \in \Omega\,;\psi(x) > 0\ ;\ \ \forall x \in \Gamma, \psi(x) = 0 \ \textit{and}\ \ \forall x \in \overline{\Omega\setminus\omega_0}\ ,\ \nabla\psi(x) \neq 0$$

where ω_0 is an open set such that $\overline{\omega}_0 \subset \omega \subset \Omega$. In the sequel :

- *C represent different positive constants,*
- *we will use the following notations :*
$$\mathcal{V} = \left\{\rho \in C^\infty\left(\overline{Q}\right) \ \ \textit{such that}\ \ \rho_{|\Sigma} = 0\right\} \quad ; \quad \mathcal{W} = \mathcal{V}\times\mathcal{V}\,,$$

$$L\rho = -\frac{\partial \rho}{\partial t} - \frac{\partial \rho}{\partial a} - \Delta\rho \quad ; \quad L^*\rho = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} - \Delta\rho$$
$$M(\rho_1,\rho_2) = L^*\rho_1 + \hat{\mu}_1\rho_1 + \hat{\mu}_2\rho_2 \quad ; \quad N(\rho_1,\rho_2) = L^*\rho_2 + \hat{\mu}_1\rho_2 + \hat{\mu}_2\rho_1.$$
$$\| \hat{\mu}_1, \hat{\mu}_2 \|_\infty^2 = \| \hat{\mu}_1 \|_\infty^2 + \| \hat{\mu}_2 \|_\infty^2 \qquad \textit{and} \qquad dQ = dtdadx$$

Theorem 2. *[11] There exists $\lambda_0 > 0$, $\tau_0 > 0$ and a positive constant C such that for all $\lambda \geq \lambda_0$, $\tau \geq \tau_0$ and for all $s \geq -3$, the inequality*

$$\int_Q \left(\frac{1}{\lambda} \left| \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \right|^2 + \frac{1}{\lambda} |\Delta\rho|^2 + \lambda\tau^2\varphi^2 |\nabla\rho|^2 + \lambda^4\tau^4\varphi^4 |\rho|^2 \right) \varphi^{2s-1} e^{-2\alpha} \mathrm{dQ}$$
$$\leq C \left(\tau \int_Q \left| \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \pm \Delta\rho \right|^2 \varphi^{2s} e^{-2\alpha} \mathrm{dQ} + \lambda^4\tau^4 \int_0^{\mathrm{T}} \int_0^{\mathrm{A}} \int_\omega |\rho|^2 \varphi^{2\mathrm{s}+3} \mathrm{e}^{-2\alpha} \mathrm{dQ} \right) \tag{10}$$

holds for any function $\rho \in \mathcal{V}$ such that the member on the right hand side of the inequality (10) is finite.

Lemma 1. *[11] Let C be the constant given by the theorem 2. Assume that for $\lambda \geq \lambda_0$, $\tau \geq 1$ and $s \geq -3$, there exists a constant $b_0 > 0$ and a set ω_b such that*

$$\overline{\omega_b} \subset \omega \quad \textit{and} \quad |\hat{\mu}_2| \geq b_0 \quad \textit{in} \quad (0\,;\,T) \times (0\,;\,A) \times \omega_b. \tag{11}$$

Then, for all $r \in [0\,;\,2[$, there exists a constant $C = C(A, T, \| \hat{\mu}_1 , \hat{\mu}_2 \|_\infty, b_0, r)$ such that for all $\rho = (\rho_1, \rho_2) \in \mathcal{W}$, we have :

$$\int_0^T \int_0^A \int_{\omega'} \left(|\rho_1|^2 + |\rho_2|^2 \right) e^{-2\alpha} \mathrm{dQ} \leq C \Bigg(\int_Q \left[|M(\rho)|^2 + |N(\rho)|^2 \right] \varphi^{2s} e^{-2\alpha} \mathrm{dQ}$$
$$+ \int_{Q_\omega} |\rho_1|^2 e^{-r\alpha} \mathrm{dQ} \Bigg) \tag{12}$$

with $\overline{\omega'} \subset \overline{\omega}_b$.

Setting

$$\theta = e^\alpha \text{ and } \delta = \theta^{\frac{r}{2}-1}, \tag{13}$$

we have the following result

Lemma 2. *[11] Under the hypothesis of the lemma 1, for all $\rho = (\rho_1, \rho_2) \in \mathcal{W}$, there exists a positive constant $C = C(A, T, \| a_\mu , b_\mu \|_\infty, c_0, r)$ such that*

$$\int_Q \frac{1}{\theta^2} \left(|\rho_1|^2 + |\rho_2|^2 \right) \mathrm{dQ} \leq \mathrm{C} \left(\int_{\mathrm{Q}} \left(|\mathrm{M}(\rho)|^2 + |\mathrm{N}(\rho)|^2 \right) \mathrm{dQ} + \int_{\mathrm{Q}_\omega} \delta^2 |\rho_1|^2 \mathrm{dQ} \right). \tag{14}$$

At last, we deduct the following result.

Proposition 1. *[11] Under the hypothesis of the lemma 2 , there exists a positive constant C such that for all $\rho = (\rho_1 , \rho_2) \in \mathcal{W}$, we have*

$$\int_0^T \int_\Omega \left(|\rho_1(t,0,x)|^2 + |\rho_2(t,0,x)|^2 \right) dxdt + \int_0^A \int_\Omega \left(|\rho_1(0,a,x)|^2 + |\rho_2(0,a,x)|^2 \right) dxda$$
$$\leq C \left(\int_Q \left(|M(\rho)|^2 + |N(\rho)|^2 \right) \mathrm{dQ} + \int_{\mathrm{Q}_\omega} \delta^2 |\rho_1|^2 \mathrm{dQ} \right) \tag{15}$$

3.2. Study of the linear case :

In this paragraph, we study the following problem : *For given functions* $\tilde{\mu}_1, \tilde{\mu}_2,\ b_1, b_2 \in L^2(Q_T),\ \tilde{\beta}_1, \tilde{\beta}_2 \in C^2(\overline{Q})$ *and* $f \in L^2(Q)$ *find* $v \in L^2(Q_\omega)$ *such that the solution* (z_1, z_2) *of* :

$$\left\{\begin{array}{rcl} -\dfrac{\partial z_1}{\partial t} - \dfrac{\partial z_1}{\partial a} - \Delta z_1 + \tilde{\mu}_1 z_1 + \tilde{\mu}_2 z_2 & = & G_1(t,a,x) z_1(t,0,x) + f + v\chi_\omega \\ & & + G_2(t,a,x) z_2(t,0,x) \quad \text{in } Q \\ -\dfrac{\partial z_2}{\partial t} - \dfrac{\partial z_2}{\partial a} - \Delta z_2 + \tilde{\mu}_1 z_2 + \tilde{\mu}_2 z_1 & = & G_2(t,a,x) z_1(t,0,x) \\ & & + G_1(t,a,x) z_2(t,0,x) \quad \text{in } Q \\ z_i & = & 0 \quad \text{on} \quad \Sigma,\ i = 1,2 \\ z_i(T,a,x) & = & 0 \quad \text{in} \quad Q_A,\ \ i = 1,2 \\ z_i(t,A,x) & = & 0 \quad \text{in} \quad Q_T,\ \ i = 1,2 \end{array}\right. \tag{16}$$

verifies

$$z_i(0,a,x) = 0 \text{ in } \ Q_A, \ \ i = 1,2. \tag{17}$$

where,

$$\begin{aligned} G_1(t,a,x) &= \tilde{\beta}_1(t,a,x) b_1(t,x) + \tilde{\beta}_2(t,a,x) b_2(t,x) \\ G_2(t,a,x) &= \tilde{\beta}_1(t,a,x) b_1(t,x) - \tilde{\beta}_2(t,a,x) b_2(t,x) \end{aligned}$$

and for all $i \in \{1,2\}, \tilde{\mu}_i$ verifies $(H_1), \tilde{\beta}_i$ satisfies $(H_2) - (H_3)$.

We can state the following result:

Theorem 3. *Suppose that assumptions* $(H_1) - (H_3)$ *hold and* $b_1, b_2 \in L^2(Q_T)$. *For any function* $f \in L^2(Q)$ *such that* $\theta f \in L^2(Q)$, *there exists a control* $\tilde{v}$ *in* $L^2(Q_\omega)$ *such that* $(\tilde{v}, \tilde{z}_1, \tilde{z}_2)$ *is solution of simultaneous null controllability problem (16)-(17). Moreover,* $(\tilde{v}, \tilde{z}_1, \tilde{z}_2)$ *verifies*

$$\tilde{v} = \tilde{u}_1 \chi_\omega \tag{18}$$

$$\|\tilde{z}_1\|_{L^2(U;H^1(\Omega)} \leq C\left(\|\theta f\|_{L^2(Q)} + \|f\|_{L^2(Q)}\right) \tag{19}$$

$$\|\tilde{z}_2\|_{L^2(U;H^1(\Omega)} \leq C\left(\|\theta f\|_{L^2(Q)} + \|f\|_{L^2(Q)}\right) \tag{20}$$

where $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ *satisfies*

$$\left\{\begin{array}{rcll} \dfrac{\partial \tilde{u}_1}{\partial t} + \dfrac{\partial \tilde{u}_1}{\partial a} - \Delta \tilde{u}_1 + \tilde{\mu}_1 \tilde{u}_1 + \tilde{\mu}_2 \tilde{u}_2 & = & 0 & in \quad Q, \\ \dfrac{\partial \tilde{u}_2}{\partial t} + \dfrac{\partial \tilde{u}_2}{\partial a} - \Delta \tilde{u}_2 + \tilde{\mu}_1 \tilde{u}_2 + \tilde{\mu}_2 \tilde{u}_1 & = & 0 & in \quad Q, \\ \tilde{u}_1(0,a,x) = \tilde{u}_2(0,a,x) & = & 0 & in \quad Q_A, \\ \tilde{u}_1 = \tilde{u}_2 & = & 0 & on \quad \Sigma, \\ \tilde{u}_1(t,0,x) & = & \Upsilon_1(\tilde{u}) & in \quad Q_T, \\ \tilde{u}_2(t,0,x) & = & \Upsilon_2(\tilde{u}) & in \quad Q_T. \end{array}\right. \tag{21}$$

where

$$\Upsilon_1(\tilde{u}) = b_1 \int_0^A \tilde{\beta}_1(\tilde{u}_1 + \tilde{u}_2)da + b_2 \int_0^A \tilde{\beta}_2(\tilde{u}_1 - \tilde{u}_2)da$$
$$\Upsilon_2(\tilde{u}) = b_1 \int_0^A \tilde{\beta}_1(\tilde{u}_1 + \tilde{u}_2)da + b_2 \int_0^A \tilde{\beta}_2(\tilde{u}_2 - \tilde{u}_1)da$$

Proof. We will do it in two steps as follows :

Step 1: There exists a control v_ε that leads to extinction each distribution $z_{1_\varepsilon}, z_{2_\varepsilon}$. For any $\varepsilon > 0$, we consider the functional defined on $L^2(Q_\omega)$ by

$$J_\varepsilon(v) = \frac{1}{2}\|v\|^2_{L^2(Q_\omega)} + \frac{1}{2\varepsilon}\int_{Q_A} \left(z_1^2(0,a,x) + z_2^2(0,a,x)\right) dQ_A, \tag{22}$$

where $z = (z_1, z_2)$ is solution of (16). It is clear that J_ε is continuous, convex and coercive on $L^2(Q_\omega)$. Hence, the minimization problem of J_ε admits at least one solution v_ε associated to $(z_{1_\varepsilon}, z_{2_\varepsilon})$ solution of (16). From the maximum principle (see [10]), we get

$$v_\varepsilon = \eta_{1_\varepsilon}\chi_\omega \text{ in } Q \tag{23}$$

where $\eta_\varepsilon = (\eta_{1_\varepsilon}, \eta_{2_\varepsilon})$ verifies the system

$$\begin{cases} \dfrac{\partial \eta_{1_\varepsilon}}{\partial t} + \dfrac{\partial \eta_{1_\varepsilon}}{\partial a} - \Delta\eta_{1_\varepsilon} + \tilde{\mu}_1\eta_{1_\varepsilon} + \tilde{\mu}_2\eta_{2_\varepsilon} = 0 & \text{in } Q, \\ \dfrac{\partial \eta_{2_\varepsilon}}{\partial t} + \dfrac{\partial \eta_{2_\varepsilon}}{\partial a} - \Delta\eta_{2_\varepsilon} + \tilde{\mu}_1\eta_{2_\varepsilon} + \tilde{\mu}_2\eta_{1_\varepsilon} = 0 & \text{in } Q, \\ \eta_{1_\varepsilon} = \eta_{2_\varepsilon} = 0 & \text{on } \Sigma, \\ \eta_{1_\varepsilon}(0,a,x) = -\frac{1}{\varepsilon}z_{1_\varepsilon}(0,a,x) & \text{in } Q_A, \\ \eta_{2_\varepsilon}(0,a,x) = -\frac{1}{\varepsilon}z_{2_\varepsilon}(0,a,x) & \text{in } Q_A, \\ \eta_{1_\varepsilon}(t,0,x) = \Upsilon_1(\eta_\varepsilon) & \text{in } Q_T \\ \eta_{2_\varepsilon}(t,0,x) = \Upsilon_2(\eta_\varepsilon) & \text{in } Q_T, \end{cases} \tag{24}$$

herein $z_\varepsilon = (z_{1_\varepsilon}, z_{2_\varepsilon})$ is the solution of (16) associated to v_ε.
Let us multiply the first (with $v = v_\varepsilon$ and $z_1 = z_{1_\varepsilon}$) and the second (with $z_2 = z_{2_\varepsilon}$) equalities of (16) by η_{1_ε} and η_{2_ε} respectively, and integrate each equality by parts over Q. Using (24) we deduct that

$$\int_Q (-f)\eta_{1_\varepsilon} dQ = \|v_\varepsilon\|^2_{L^2(Q_\omega)} + \tfrac{1}{\varepsilon}\|z_{1_\varepsilon}(0,\cdot,\cdot)\|^2_{L^2(Q)} + \tfrac{1}{\varepsilon}\|z_{2_\varepsilon}(0,\cdot,\cdot)\|^2_{L^2(Q)}. \tag{25}$$

Elsewhere, Young's inequality gives: $\displaystyle\int_Q |f\eta_{1_\varepsilon}|\, dQ \leq 2C\|\theta f\|^2_{L^2(Q)} + \frac{1}{2C}\int_Q \frac{1}{\theta^2}\eta^2_{1_\varepsilon} dQ$ for any $C > 0$. Thus,

$$\int_Q (-f)\eta_{1_\varepsilon} \leq 2C\|\theta f\|^2_{L^2(Q)} + \frac{1}{2C}\int_Q \frac{1}{\theta^2}\left(\eta^2_{1_\varepsilon} + \eta^2_{2_\varepsilon}\right) dQ.$$

The lemma 2 allows, choosing C, the constant defined therein, to deduct that

$$\int_Q (-f)\eta_{1_\varepsilon} dQ \leq 2C\|\theta f\|^2_{L^2(Q)} + \frac{1}{2}\|v_\varepsilon\|^2_{L^2(G)}. \tag{26}$$

From (25) and (26) one obtains :

$$\|v_\varepsilon\|_{L^2(G)} \leq 2\sqrt{C}\|\theta f\|_{L^2(Q)} \tag{27}$$

$$\|z_{1_\varepsilon}(0,\cdot,\cdot)\|_{L^2(Q)} \leq \sqrt{2\varepsilon C}\|\theta f\|_{L^2(Q)} \tag{28}$$

$$\|z_{2_\varepsilon}(0,\cdot,\cdot)\|_{L^2(Q)} \leq \sqrt{2\varepsilon C}\|\theta f\|_{L^2(Q)} \tag{29}$$

We can extract subsequences denoted again $(v_\varepsilon)_\varepsilon$ and $(z_\varepsilon)_\varepsilon$ such that $v_\varepsilon \rightharpoonup \tilde{v}$ weakly in $L^2(Q_\omega)$ and $z_{i_\varepsilon} \rightharpoonup \tilde{z}_i\,, i = 1,2$ weakly in $L^2(U, H_0^1(\Omega))$. Note that $(\tilde{z}_1, \tilde{z}_2)$ is the unique couple solution of (16)-(17) associated to $\tilde{v}$. In the same ways, it follows that $(\eta_{1_\varepsilon}, \eta_{2_\varepsilon})$ converge weakly to $(\tilde{\eta}_1, \tilde{\eta}_2)$ and that $(\tilde{\eta}_1, \tilde{\eta}_2)$ satisfies (21). From (23) and (27) we obtain that $\tilde{v} = \tilde{\eta}_1\chi_\omega$ in Q.

Step 2 : Now we prove the inequalities (19)and (20).
Let set $\hat{z}_{i\varepsilon} = e^{-\lambda_0 t} z_{i\varepsilon}$, $i = 1,2$ where $(z_{1\varepsilon}, z_{2\varepsilon})$ verifies (16)-(17) and λ_0 is a positive real constant. Then $\hat{z}_{1\varepsilon}, \hat{z}_{2\varepsilon}$ verify the system

$$\begin{cases} -\dfrac{\partial \hat{z}_{1\varepsilon}}{\partial t} - \dfrac{\partial \hat{z}_{1\varepsilon}}{\partial a} - \Delta \hat{z}_{1\varepsilon} + \hat{\mu}_1 \hat{z}_{1\varepsilon} + \tilde{\mu}_2 \hat{z}_{2\varepsilon} & = \hat{G}_1(t,a,x) z_{1\varepsilon}(t,0,x) + \hat{f} + \hat{v}_\varepsilon \chi_\omega \\ & + \hat{G}_2(t,a,x) z_{2\varepsilon}(t,0,x) \quad \text{in } Q \\ -\dfrac{\partial \hat{z}_{2\varepsilon}}{\partial t} - \dfrac{\partial \hat{z}_{2\varepsilon}}{\partial a} - \Delta \hat{z}_{2\varepsilon} + \hat{\mu}_1 \hat{z}_{2\varepsilon} + \tilde{\mu}_2 \hat{z}_{1\varepsilon} & = \hat{G}_2(t,a,x) z_{1\varepsilon}(t,0,x) \\ & + \hat{G}_1(t,a,x) z_{2\varepsilon}(t,0,x) \quad \text{in } Q \\ \hat{z}_{i\varepsilon} & = 0 \quad \text{on} \quad \Sigma,\ i = 1,2 \\ \hat{z}_{i\varepsilon}(T,a,x) & = 0 \quad \text{in} \quad Q_A,\ \ i = 1,2 \\ \hat{z}_{i\varepsilon}(t,A,x) & = 0 \quad \text{in} \quad Q_T,\ \ i = 1,2 \end{cases} \tag{30}$$

where :

$$\hat{G}_i = e^{-\lambda_0 t} G_i,\ \hat{f} = e^{-\lambda_0 t} f,\ \hat{v}_\varepsilon = e^{-\lambda_0 t} v_\varepsilon \text{ and } \hat{\mu}_1 = \tilde{\mu}_1 + \lambda_0.$$

Multiplying the first and the second equations of (30) by $\hat{z}_{1\varepsilon}$ and $\hat{z}_{2\varepsilon}$ respectively, and integrating by parts over Q, we have thanks to Young's inequality :

$$\int_Q |\nabla \hat{z}_{1\varepsilon}|^2 dQ + \Gamma_1 \int_Q |\hat{z}_{1\varepsilon}|^2 dQ - \frac{\|\tilde{\mu}_2\|_\infty}{2C_1} \int_Q |\hat{z}_{2\varepsilon}|^2 dQ + \left(1 - \frac{A}{2C_2}\right) \int_{Q_T} \hat{z}^2_{1\varepsilon}(t,0,x) dQ_T$$

$$+ \int_{Q_A} \hat{z}^2_{1_\varepsilon}(0,a,x) dQ_A - \tfrac{A}{2C_3} \int_{Q_T} \hat{z}^2_{2\varepsilon}(t,0,x) dQ_T \leq \frac{1}{2C_4} \int_Q \left|\hat{f}\right| dQ + \tfrac{1}{2C_5} \int_G \hat{v}^2_\varepsilon dQ \tag{31}$$

and

$$\int_Q |\nabla \hat{z}_{2\varepsilon}|^2 dQ + \Gamma_2 \int_Q |\hat{z}_{2\varepsilon}|^2 dQ - \frac{\|\tilde{\mu}_2\|_\infty}{2K_1} \int_Q |\hat{z}_{1\varepsilon}|^2 dQ + \left(1 - \frac{A}{2K_3}\right) \int_{Q_T} \hat{z}^2_{2\varepsilon}(t,0,x) dQ_T$$

$$+\int_{Q_A}\hat{z}^2_{2_\varepsilon}(0,a,x)dQ_A-\frac{\|\tilde{\mu}_1\|_\infty}{2K_2}\|_\infty\int_{Q_T}\hat{z}^2_{1\varepsilon}(t,0,x)dQ_T\le 0 \tag{32}$$

where :
$\Gamma_1=\lambda_0-2C_1\|\tilde{\mu}_2\|_\infty-4A\left(C_2+C_3\right)\|\tilde{\beta}_1,\tilde{\beta}_2\|^2_\infty\|b_1,b_2\|^2_{Q_T}-\|\tilde{\mu}_1\|_\infty-2C_5,$

$\Gamma_2=\lambda_0-2K_1\|\tilde{\mu}_1\|_\infty-4A\left(K_2+K_3\right)\|\tilde{\beta}_1,\tilde{\beta}_2\|^2_\infty\|b_1,b_2\|^2_{Q_T}-\|\tilde{\mu}_1\|_\infty$ and the C_i,K_i are Young's constants for $i=1,2,3,5$.
Summing (31) and (32), one obtains :

$$\int_Q|\nabla\hat{z}_{1\varepsilon}|^2\,dQ+\Pi_1\int_Q|\hat{z}_{1\varepsilon}|^2dQ+\int_Q|\nabla\hat{z}_{2\varepsilon}|^2\,dQ+\Pi_2\int_Q|\hat{z}_{2\varepsilon}|^2dQ+$$

$$\left(1-\frac{A}{2C_2}-\frac{A}{2K_3}\right)\int_{Q_T}\hat{z}^2_{1\varepsilon}(t,0,x)dQ_T+\left(1-\frac{A}{2C_3}-\frac{A}{2K_2}\right)\int_{Q_T}\hat{z}^2_{2\varepsilon}(t,0,x)dQ_T$$

$$+\int_{Q_A}\hat{z}^2_{1_\varepsilon}(0,a,x)dQ_A+\int_{Q_A}\hat{z}^2_{2_\varepsilon}(0,a,x)dQ_A\le\frac{1}{2C_4}\int_Q\left|\hat{f}\right|dQ+\tfrac{1}{2C_5}\int_G\hat{v}^2_\varepsilon dQ \tag{33}$$

with :

$$\Pi_1=\Gamma_1-\frac{\|\tilde{\mu}_2\|_\infty}{2K_1}\ \text{ and }\ \Pi_2=\Gamma_2-\frac{\|\tilde{\mu}_2\|_\infty}{2C_1}.$$

Choosing λ_0 and the Young's constants such that:
$\lambda_0\ge\max\big\{2C_1\|\tilde{\mu}_2\|_\infty+4A\left(C_2+C_3\right)\|\tilde{\beta}_1,\tilde{\beta}_2\|^2_\infty\|b_1,b_2\|^2_{Q_T}+\|\tilde{\mu}_1\|_\infty+2C_5+\frac{\|\tilde{\mu}_2\|_\infty}{2K_1}+1;$
$2K_1\|\tilde{\mu}_1\|_\infty+4A\left(K_2+K_3\right)\|\tilde{\beta}_1,\tilde{\beta}_2\|^2_\infty\|b_1,b_2\|^2_{Q_T}+\|\tilde{\mu}_1+\frac{\|\tilde{\mu}_2\|_\infty}{2C_1}+1\big\}$
and $\min\left\{1-\frac{A}{2C_2}-\frac{A}{2K_3},1-\frac{A}{2C_3}-\frac{A}{2K_2}\right\}\ge 1$, One deducts from (27) and (33) that

$$\int_Q|\nabla\hat{z}_{1\varepsilon}|^2\,dQ+\int_Q|\hat{z}_{1\varepsilon}|^2dQ\ \le\ C\left(\|\hat{f}\|^2_{L^2(Q)}+\|\theta\hat{f}\|_{L^2(Q)}\right) \tag{34}$$

$$\int_Q|\nabla\hat{z}_{2\varepsilon}|^2\,dQ+\int_Q|\hat{z}_{2\varepsilon}|^2dQ\ \le\ C\left(\|\hat{f}\|^2_{L^2(Q)}+\|\theta\hat{f}\|_{L^2(Q)}\right) \tag{35}$$

$$\int_{Q_T}\hat{z}^2_{1_\varepsilon}(t,0,x)dQ_T\ \le\ C\left(\|\hat{f}\|^2_{L^2(Q)}+\|\theta\hat{f}\|_{L^2(Q)}\right) \tag{36}$$

$$\int_{Q_T}\hat{z}^2_{2_\varepsilon}(t,0,x)dQ_T\ \le\ C\left(\|\hat{f}\|^2_{L^2(Q)}+\|\theta\hat{f}\|_{L^2(Q)}\right) \tag{37}$$

Consequently, the sequences $(\hat{z}_{1_\varepsilon})_\varepsilon$, $(\hat{z}_{2_\varepsilon})_\varepsilon$, $(\hat{z}_{1_\varepsilon}(\cdot,0,\cdot))_\varepsilon$ and $(\hat{z}_{2_\varepsilon}(\cdot,0,\cdot))_\varepsilon$ are bounded respectively in $L^2(U,H^1_0(Q))$ and $L^2(Q_T)$. That ends this proof, thanks to limit's results obtained in the step 1. ∎

3.3. Study of the nonlinear case

Let $b_i(t,x) = T_i\left(\int_0^A \beta_i(t,a,x)z_i(t,a,x)da\right)$, $i = 1;2$ where $T_i \in L^\infty(\mathbb{R})$, β_i $i = 1,2$ verify $(H_2) - (H_3)$. we study here, the null controllability of the following system :

$$\begin{cases} -\dfrac{\partial z_1}{\partial t} - \dfrac{\partial z_1}{\partial a} - \Delta z_1 + \tilde{\mu}_1 z_1 + \tilde{\mu}_2 z_2 &= \beta_1 T_1(\xi_1) z_1(t,0,x) + f + v\chi_\omega \\ &\quad + \beta_2 T_2(\xi_2) z_2(t,0,x) \quad \text{in } Q \\ -\dfrac{\partial z_2}{\partial t} - \dfrac{\partial z_2}{\partial a} - \Delta z_2 + \tilde{\mu}_1 z_2 + \tilde{\mu}_2 z_1 &= \beta_1 T_2(\xi_2) z_1(t,0,x) \\ &\quad + \beta_2 T_1(\xi_1) z_2(t,0,x) \quad \text{in } Q \\ z_i &= 0 \quad \text{on} \quad \Sigma,\ i=1,2 \\ z_i(T,a,x) &= 0 \quad \text{in} \quad Q_A,\ \ i=1,2 \\ z_i(t,A,x) &= 0 \quad \text{in} \quad Q_T,\ \ i=1,2 \end{cases} \tag{38}$$

The system (38) is nonlinear. Let

$$\mathcal{A} = \left\{\tilde{v} \in L^2(Q_\omega) : (\tilde{z}_1, \tilde{z}_2) \text{ solves (38), verifies (17) and } \tilde{v} \text{ satifies (27)}\right\},$$

$$\mathcal{N} = L^2(Q_T) \times L^2(Q_T),$$

and define the multivalued mapping :

$$\Lambda : \mathcal{N} \longrightarrow 2^{\mathcal{N}},\ (\xi_1, \xi_2) \longmapsto \Lambda(\xi_1, \xi_2) \text{ by}$$

$$\Lambda(\xi_1,\xi_2) = \left\{\left(\int_0^A \beta_1 \tilde{z}_1 \mathrm{d}a, \int_0^A \beta_2 \tilde{z}_2 \mathrm{d}a\right) : (\tilde{z}_1, \tilde{z}_2) \text{ is associated to } \tilde{v} \in \mathcal{A}\right\}.$$

The null controllability problem of (38) is reduced to find a fixed point of Λ. In order to use the generalization of the Leray-Schauder's fixed point theorem, we set

$$N_\rho = \{(\xi_1,\xi_2) \in \mathcal{N} : \exists \rho \in (0,1), (\xi_1,\xi_2) \in \rho\Lambda(\xi_1,\xi_2)\}.$$

The following proposition is a direct consequence of the Leray-Schauder's fixed point theorem (see [1]).

Proposition 2. *Under the assumptions $(H_1) - (H_3)$, the multivalued mapping Λ admits at least one fixed point.*

Proof. We proceed in four steps :

Step 1: N_ρ is bounded in $\mathcal{N}$.
Let $(\xi_1,\xi_2) \in N_\rho$. Then, there exists $\rho \in (0,1), \tilde{z}_1, \tilde{z}_2$ such that $\xi_1 = \rho\int_0^A \beta_1 \tilde{z}_1 da$ and $\xi_2 = \rho\int_0^A \beta_2 \tilde{z}_2 da$. Then, $\int_{Q_T} |\xi_i|^2 dQ_T \le \|\beta_1,\beta_2\|_\infty^2 \int_Q \tilde{z}_i^2 dQ$, $i = 1;2$. So,

$$\|\xi_1\|_{L^2(Q_T)} + \|\xi_2\|_{L^2(Q_T)} \le \|\beta_1,\beta_2\|_\infty \left(\|\tilde{z}_1\|_{L^2(Q)} + \|\tilde{z}_2\|_{L^2(Q)}\right) \tag{39}$$

From the theorem 3, one deducts that there exists a positive constant C such that

$$\|\xi_1\|_{L^2(Q_T)} + \|\xi_2\|_{L)^2(Q_T)} \le 2C\|\beta_1,\beta_2\|_\infty \left(\|\theta f\|_{L^2(Q)} + \|f\|_{L^2(Q)}\right) \tag{40}$$

Hence, N_ρ is bounded in $\mathcal{N}$ since $L^2(U;H^1(\Omega)) \subset L^2(Q)$.

Step 2 : For all $(\xi_1, \xi_2) \in \mathcal{N}$, $\Lambda(\xi_1, \xi_2)$ is closed and convex subset of $\mathcal{N}$.
Let $(\xi_1, \xi_2) \in \Lambda(\xi_1, \xi_2)$. Under the assumptions $(H_1) - (H_3)$, the system (38) admits a solution and the corresponding control verifies (27). So, the set $\Lambda(\xi_1, \xi_2)$ is non empty. Elsewhere, like the mapping $(\xi_1, \xi_2) \longmapsto (\tilde{z}_1, \tilde{z}_2)$ is affine, then, the set $\Lambda(\xi_1, \xi_2)$ is convex. There rest to prove that this set is closed.
Let $(\eta_{1_n}, \eta_{2_n})_n \subset \Lambda(\xi_1, \xi_2)$ which converges strongly towards (η_1, η_2) in $\mathcal{N}$. Then, for each $n \in \mathbb{N}$, there exists a control $\tilde{v}_n \in \mathcal{A}$ and a corresponding solution $(\tilde{z}_{1_n}, \tilde{z}_{2_n})$ of (38) such that $\eta_{i_n} = \int_0^A \beta_i \tilde{z}_{i_n}$, $i = 1, 2$. From the inequalities (27), (34) and (35) one deduces that $(\tilde{z}_{1_n}, \tilde{z}_{2_n})$ and $\tilde{v}_n$ are bounded respectively in $(L^2(Q))^2$ and $L^2(Q_\omega)$. Thus, (η_{1_n}, η_{2_n}) is bounded in $\mathcal{N}$. Hence, we can extract subsequences denoted still $(\tilde{z}_{1_n}, \tilde{z}_{2_n})$, $\tilde{v}_n$ and (η_{1_n}, η_{2_n}) respectively such that $(\tilde{z}_{1_n}, \tilde{z}_{2_n})$, $\tilde{v}_n$ and (η_{1_n}, η_{2_n}) converge weakly towards $(\tilde{z}_1,\, \tilde{z}_2)$, $\tilde{v}$ and (η_1, η_2) respectively in $(L^2(Q))^2$, $L^2(Q_\omega)$ and $\mathcal{N}$ with $\eta_i = \int_0^A \beta_i \tilde{z}_i da$, $i = 1; 2$. Notice that $(\tilde{z}_1, \tilde{z}_2)$ is solution of (38) and $\tilde{v}$ verifies (27). So, $(\tilde{z}_1, \tilde{z}_2)$ satisfies (17).
As consequence, $(\eta_1, \eta_2) \in \Lambda(\xi_1, \xi_2)$.

Step 3 : Λ is a compact multivalued mapping.
Let $\mathcal{B}$ be a bounded subset of $\mathcal{N}$, $(\xi_1, \xi_2) \in \mathcal{B}$. Let $(\rho_{1_n}, \rho_{2_n}) \in \Lambda(\xi_1, \xi_2)$. Then, for all $n \in \mathbb{N}$, there exists $(\tilde{z}_{1_n}, \tilde{z}_{2_n})$, solution of (38), and $\tilde{v}_n$ in $\left(L^2(Q)\right)^2$ and $L^2(Q_\omega)$ respectively such that $\rho_{i_n} = \int_0^A \beta_i \tilde{z}_{i_n} da$, $i = 1; 2$ and $\tilde{v}_n$ satisfies (27). So, $(\tilde{v}_n)_n$ is bounded in $L^2(Q_\omega)$. Proceeding in the similar ways that the step 2 of the proof of the theorem 3, one deducts from (27), (34)-(37) and the fact that $H^1(\Omega) \subset L^2(\Omega)$ that $(\tilde{z}_{1_n}, \tilde{z}_{2_n})_n$ is bounded in $\left(L^2(Q)\right)^2$, and then, (ρ_{1_n}, ρ_{2_n}) is bounded in $\mathcal{N}$. Thus, there exists subsequences of $(\tilde{z}_{1_n}, \tilde{z}_{2_n})$ and $\tilde{v}_n$ also denoted by $(\tilde{z}_{1_n}, \tilde{z}_{2_n})$ which converges weakly in $\left(L^2(Q)\right)^2$ and $L^2(Q_\omega)$. Moreover, the subsequences $\rho_{i_n} = \int_0^A \beta_i \tilde{z}_{i_n} da$, $i = 1; 2$ of $(\rho_{i_n})_n$ verify the following system :

$$\left\{\begin{array}{rcll}
-\dfrac{\partial \rho_{1_n}}{\partial t} - \Delta\rho_{1_n} + \int_0^A \hat{\mu}_1 \beta_1 \tilde{z}_{1_n} da + \int_0^A \beta_1 \mu_2 \tilde{z}_{2_n} da & = & K_1(\xi_n) & \text{in } Q_T \\
-\dfrac{\partial \rho_{2_n}}{\partial t} - \Delta\rho_{2_n} + \int_0^A \hat{\mu}_1 \beta_2 \tilde{z}_{2_n} da + \int_0^A \beta_2 \mu_2 \tilde{z}_{1_n} da & = & K_2(\xi_n) & \text{in } Q_T \\
\rho_{1_n} = \rho_{2_n} & = & 0 & \text{on } \Sigma_T \\
\rho_{1_n}(0,x) = \rho_{2_n}(0,x) & = & 0 & \text{in } \Omega \\
\rho_{1_n}(T,x) = \rho_{2_n}(T,x) & = & 0 & \text{in } \Omega
\end{array}\right. \tag{41}$$

where $\Sigma_T = (0,T) \times \Gamma$, $\hat{\mu}_1 = \mu_1 + \lambda_0$ and for all $n \in \mathbb{N}$,

$$\begin{aligned}
K_{1_n}(\xi) = & -\int_0^A \left(\tfrac{\partial \beta_1}{\partial t} + \tfrac{\partial \beta_1}{\partial a} + \Delta\beta_1 + \mu_2\beta_2\right) \tilde{z}_{1_n} da + \int_0^A \beta_1 (f + \tilde{v}_n \chi_\omega) da \\
& + \int_0^A \beta_1^2 T_1(\xi_{1_n}) \tilde{z}_{1n}(t,0,x) da + \int_0^A \beta_1 \beta_2 T_2(\xi_{2_n}) \tilde{z}_{2_n}(t,0,x) da \\
& - 2 \sum_{i=1}^n \int_0^A \tfrac{\partial \beta_1}{\partial x_i} . \tfrac{\partial \tilde{z}_{1_n}}{\partial x_i} \, da
\end{aligned}$$

$$K_{2n}(\xi) = -\int_0^A \left(\frac{\partial \beta_2}{\partial t} + \frac{\partial \beta_2}{\partial a} + \Delta\beta_2 + \mu_2\beta_1\right)\tilde{z}_{2_n} da + \int_0^A \beta_2^2 T_1(\xi_{1_n})\tilde{z}_{2_n}(t,0,x)da$$
$$+ \int_0^A \beta_1\beta_2 T_2(\xi_{2_n})\tilde{z}_{1_n}(t,0,x)da - 2\sum_{i=1}^n \int_0^A \frac{\partial \beta_2}{\partial x_i}.\frac{\partial \tilde{z}_{2_n}}{\partial x_i} da$$

Under the assumptions $(H_1)-(H_3)$ the boundedness of $\mathcal{B}$ and of sequences $(\tilde{z}_{i_n})_n$ $i = 1; 2$, from (27), (34)-(37), one deducts that there exists positive constants C_i which depend on $\|\nabla\beta_i\|_\infty$, $\|\beta_1,\beta_2\|_\infty^2$, $\|T_1,T_2\|_\infty$ for $i = 1; 2$ such that

$$\|K_i(\xi_n)\|^2_{L^2(Q_T)} \le C_i\left(\|\theta f\|^2_{L^2(Q_\omega)} + \|f\|^2_{L^2(Q)}\right) \tag{42}$$

Now, multiplying the first and the second equations of (41) by ρ_{1_n} and ρ_{2_n} respectively and proceeding by integrations by parts over Q_T, one has

$$\int_{Q_T} |\nabla\rho_{1_n}|^2 dQ_T + \lambda_0 \int_{Q_T} \rho_{1_n}^2 dQ_T = \int_{Q_T}\left(K_1(\xi_n) - \int_0^A \beta_1(\tilde{\mu}_1\tilde{z}_{1_n} + \tilde{\mu}_2\tilde{z}_{2_n})da\right)\rho_{1_n} dQ_T$$

Since $\tilde{z}_{1_n}, \tilde{z}_{2_n}$ verify (35)-(36), one deducts that $K_1(\xi_n) - \int_0^A \beta_1(\tilde{\mu}_1\tilde{z}_{1_n} + \tilde{\mu}_2\tilde{z}_{2_n})da$ verifies (42). So, using Young inequality, one has

$$\int_{Q_T} |\nabla\rho_{1_n}|^2 dQ_T + (\lambda_0 - \frac{\lambda_1}{2})\int_{Q_T} \rho_{1_n}^2 dQ_T \le \frac{C_1}{2\lambda_1}\left(\|\theta f\|^2_{L^2(Q_\omega)} + \|f\|^2_{L^2(Q)}\right) \tag{43}$$

By analogy we show that

$$\int_{Q_T} |\nabla\rho_{2_n}|^2 dQ_T + (\lambda_0 - \frac{\lambda_2}{2})\int_{Q_T} \rho_{2_n}^2 dQ_T \le \frac{C_2}{2\lambda_2}\left(\|\theta f\|^2_{L^2(Q_\omega)} + \|f\|^2_{L^2(Q)}\right) \tag{44}$$

Taking $\lambda_0 - 1 \ge \max(\frac{\lambda_1}{2}, \frac{\lambda_2}{2})$, one deducts that $(\rho_{1_n})_n$ and $(\rho_{2_n})_n$ are bounded in $L^2((0,T); H^1(\Omega))$. Let remark that the system (41) is equivalent to the system

$$\left\{\begin{array}{rcll} -\dfrac{\partial\rho_{1_n}}{\partial t} - \Delta\rho_{1_n} + \lambda_0\rho_{1_n} &=& K_1'(\xi_n) & \text{in } Q_T \\ -\dfrac{\partial\rho_{2_n}}{\partial t} - \Delta\rho_{2_n} + \lambda_0\rho_{2_n} &=& K_2'(\xi_n) & \text{in } Q_T \\ \rho_{1_n} = \rho_{2_n} &=& 0 & \text{on } \Sigma_T \\ \rho_{1_n}(0,x) = \rho_{2_n}(0,x) &=& 0 & \text{in } \Omega \\ \rho_{1_n}(T,x) = \rho_{2_n}(T,x) &=& 0 & \text{in } \Omega \end{array}\right. \tag{45}$$

with $K_1' = K_1(\xi_n) - \int_0^A \beta_1(\mu_1\tilde{z}_{1_n} + \mu_2\tilde{z}_{2_n})da$, $K_2' = K_2(\xi_n) - \int_0^A \beta_2(\mu_1\tilde{z}_{2_n} + \mu_2\tilde{z}_{1_n})da$ and (45) is a system of retrograde heat equations which the source terms are bounded in $L^2(Q_T)$ and the distributions are bounded in $L^2((0,T); H^1(\Omega))$. So, the sequences $\left(\frac{\rho_{1n}}{\partial t}\right)_n$ and $\left(\frac{\rho_{2n}}{\partial t}\right)_n$ are bounded in $L^2((0,T); H^{-1}(\Omega))$. Thus, we deduct from Aubin-Lions lemma that there exists subsequences $(\rho_{1n_k})_k$ and $(\rho_{2n_k})_k$ of $(\rho_{1_n})_n$ and $(\rho_{2n})_n$ respectively that converge strongly towards ρ_1 and ρ_2 respectively in $L^2(Q_T)$. Hence, $(\rho_{1_n})_n$ and $(\rho_{2_n})_n$

converge weakly towards ρ_1 and ρ_2 respectively in $L^2(Q_T)$. Elsewhere, there exists subsequences $(\tilde{z}_{in_k})_k$ of $\tilde{z}_{i_n}$, $i = 1, 2$ associated to $(\rho_{in_k})_k$, $i = 1, 2$ respectively that converge weakly towards $\tilde{z}_i$, $i = 1, 2$ respectively in $L^2(U; H^1(\Omega))$, say us more precisely in $L^2(Q)$, since, $L^2(U; H^1(\Omega)) \subset L^2(Q)$. Thus, we have firsly

$$\rho_{in_k} \rightharpoonup \rho_i \text{ weakly in } L^2(Q_T) \;\; i = 1; 2 \tag{46}$$

and secondly

$$\rho_{in_k} \rightharpoonup \int_0^A \beta_i \tilde{z}_i da \text{ weakly in } L^2(Q_T) \;\; i = 1; 2, \tag{47}$$

then, from the uniqueness of the limit, for all $i \in \{1, 2\}$, one deducts that

$$\rho_i = \int_0^A \beta_i \tilde{z}_i da. \tag{48}$$

Similarly, we can prove that $(\tilde{v}_n)_n$ converges towards $\tilde{v} \in L^2(Q_\omega)$. Moreover, $(\tilde{z}_1, \tilde{z}_2)$ verifies (38) and $\tilde{v}$ satisfies (27). From the theorem 3, one deducts that $\tilde{z}_i$, $i = 1; 2$ satisfies (17).

Step 4 : Λ is upper semi-continuous on $\mathcal{N}$.

Let K be a closed subset of $\mathcal{N}$. Let $(k_{1_n}, k_{2_n})_n \subset \Lambda^{-1}(K)$ that converges strongly towards (k_1, k_2) in $\mathcal{N}$. Then, $(k_{1_n}, k_{2_n})_n$ is bounded in $\mathcal{N}$. Since $\Lambda^{-1}(K) = \{(k_1, k_2) \in K : \Lambda(k_1, k_2) \cap K \neq \emptyset\}$, there exists, a sequence $(\rho_{1_n}, \rho_{2_n})_n \in K$ that belongs to $\Lambda(k_{1_n}, k_{2_n})$. Now, proceeding as in the previous step with K instead of $\mathcal{B}$ and with $\Lambda^{-1}(k_{1_n}, k_{2_n})$ instead of $\Lambda^{-1}(\xi_1, \xi_2)$, one deduces that there exists subsequences still denoted by (ρ_{1_n}, ρ_{2_n}) and $(\tilde{v}_n)$ which converge weakly to (ρ_1, ρ_2) and $\tilde{v}$ respectively in $\mathcal{N}$ and $L^2(Q_\omega)$, and for all $i \in \{1, 2\}$, there exists $\tilde{z}_i \in L^2(U, H^2(\Omega))$ such that ρ_{i_n} verifies (47). So, for all $i \in \{1, 2\}$, ρ_i verifies (48). Let mention that $(\tilde{z}_1, \tilde{z}_2)$ solves (38), $\tilde{v}$ verifies (27) and $\tilde{z}_i$ $i = 1, 2$ satisfies (17). Consequently,

$$(\rho_1, \rho_2) \in \Lambda(k_1, k_2) \tag{49}$$

From (43), (44) and Lions-Aubin lemma one deduces that the subsequence (ρ_{1_n}, ρ_{2_n}) of the closed set K, converges strongly towards (ρ_1, ρ_2) in $\mathcal{N}$. Then,

$$(\rho_1, \rho_2) \in K. \tag{50}$$

(49) and (50) say that $(k_1, k_2) \in \Lambda^{-1}(K)$. ■

4. Proof of the main result

In this section, we study the controllability of the (8)-(9). In view of the above, let's set for any $\xi = (\xi_1, \xi_2) \in L^2(Q_T) \times L^2(Q_T)$

$$\begin{array}{lll} T_1(\xi) & = F(\xi_1 + \xi_2) \quad ; \quad T_2(\xi) = G(\xi_1 - \xi_2), \\ G_1(\xi) & = \beta_1(t, a, x) T_1(\xi) + \beta_2(t, a, x) T_2(\xi), \\ G_2(\xi) & = \beta_1(t, a, x) T_1(\xi) - \beta_2(t, a, x) T_2(\xi). \end{array} \tag{51}$$

Now, we consider the system that follows

$$\left\{\begin{array}{rcl} -\dfrac{\partial\widehat{p}_{1_\varepsilon}}{\partial t}-\dfrac{\partial\widehat{p}_{1_\varepsilon}}{\partial a}-\Delta\widehat{p}_{1_\varepsilon}+\tilde{\mu}_1\widehat{p}_{1_\varepsilon}+\mu_2\widehat{p}_{2\varepsilon} &=& G_1(\xi)\widehat{p}_{1_\varepsilon}(t,0,x)+\hat{f}+\widehat{v}_\varepsilon\chi_\omega \\ && +G_2(\xi)\widehat{p}_{2\varepsilon}(t,0,x) \qquad \text{in } \ Q, \\ -\dfrac{\partial\widehat{p}_{2\varepsilon}}{\partial t}-\dfrac{\partial\widehat{p}_{2\varepsilon}}{\partial a}-\Delta\widehat{p}_{2\varepsilon}+\tilde{\mu}_1\widehat{p}_{2\varepsilon}+\mu_2\widehat{p}_{1_\varepsilon} &=& G_2(\xi)\widehat{p}_{1_\varepsilon}(t,0,x) \\ && +G_1(\xi)\widehat{p}_{2\varepsilon}(t,0,x) \qquad \text{in } Q, \\ \widehat{p}_{1_\varepsilon}=\widehat{p}_{2\varepsilon} &=& 0 \quad \text{on} \quad \Sigma, \\ \widehat{p}_{1_\varepsilon}(T,a,x)=\widehat{p}_{2\varepsilon}(T,a,x) &=& 0 \quad \text{in} \quad Q_A, \\ \widehat{p}_{1_\varepsilon}(t,A,x)=\widehat{p}_{2\varepsilon}(t,A,x) &=& 0 \quad \text{in} \quad Q_T, \end{array}\right. \tag{52}$$

where : $\widehat{p}_{i_\varepsilon}=e^{-\lambda_0 t}p_{i_\varepsilon}, i=1;2, \widehat{f}=e^{-\lambda_0 t}f$, $\tilde{\mu}_1=\tilde{\mu}_1+\lambda_0$ and $\widehat{v}_\varepsilon=e^{-\lambda_0 t}v_\varepsilon$ for any $\lambda_0\geq 0$ with $(p_{1_\varepsilon},p_{2_\varepsilon})$ a solution of (8) associated to v_ε.

The controllability of the system (8) -(9) is summarized in the study of the null controllability of system (52). We consider the operator $\widehat{\Lambda}$ from $\mathcal{N}=L^2(Q_T)\times L^2(Q_T)$ into $2^{\mathcal{N}}$ defined by

$$(\xi_1,\xi_2)\longmapsto\widehat{\Lambda}(\xi_1,\xi_2)=\Lambda_{\xi_2}(\xi_1)\times\Lambda_{\xi_1}(\xi_2) \tag{53}$$

such that

$$\Lambda_{\xi_2}(\xi_1)=\left\{\int_0^A\beta_1\left(\widehat{p}_{1_\varepsilon}(\xi_1)+\widehat{p}_{2_\varepsilon}(\xi_2)\right)da\right\}$$
$$\Lambda_{\xi_1}(\xi_2)=\left\{\int_0^A\beta_2\left(\widehat{p}_{1_\varepsilon}(\xi_1)-\widehat{p}_{2_\varepsilon}(\xi_2)\right)da\right\}$$

where $(\widehat{p}_{1_\varepsilon}(\xi_1),\widehat{p}_{2_\varepsilon}(\xi_2))$ solves (52), verifies (28)-(29) and the associated control $\widehat{v}_\varepsilon$ satisfies (27).

The controllability of (52) is summarized to the study of the existence of a fixed point of the mapping $\widehat{\Lambda}$ [8]. We are going to show that $\widehat{\Lambda}$ admits a fixed point. To do that, we have to demonstrate that for each $(\xi_1,\xi_2)\in\mathcal{N}$, $\Lambda_{\xi_2}(\xi_1)$ and $\Lambda_{\xi_1}(\xi_2)$ are bounbed closed convex sets in $L^2(Q_T)$ and $\widehat{\Lambda}(\xi_1,\xi_2)$ is upper semicontinuous. Let set

$$Y_1(\xi)(t,x)=\int_0^A\beta_1\widehat{p}_{1_\varepsilon}(\xi_1)da+\int_0^A\beta_1\widehat{p}_{2\varepsilon}(\xi_2)da \tag{54}$$
$$Y_2(\xi)(t,x)=\int_0^A\beta_2\widehat{p}_{1_\varepsilon}(\xi_1)da-\int_0^A\beta_2\widehat{p}_{2\varepsilon}(\xi_2)da \tag{55}$$

Proceeding as in the step 2 of the proof of the Proposition 3, one deducts from (41)-(42) that $Y_i(\xi)$, $i=1;2$ verify for any positive real λ_0 the following system :

$$\left\{\begin{array}{rcll} -\dfrac{\partial Y_i(\xi)}{\partial t}-\Delta Y_i(\xi)+\lambda_0 Y_i &=& R_i(\xi) & \text{in} \quad Q_T \\ Y_i(\xi) &=& 0 & \text{on} \quad \Sigma_T \\ Y_i(\xi)(0,x) &=& 0 & \text{in} \quad \Omega \end{array}\right. \tag{56}$$

where

$$
\begin{aligned}
R_1(\xi) = & -\int_0^A \left(\frac{\partial\beta_1}{\partial t} + \frac{\partial\beta_1}{\partial a} + \Delta\beta_1 + (\mu_1+\mu_2)\beta_1\right)(\widehat{p}_{1_\varepsilon}(\xi_1) + \widehat{p}_{2\varepsilon}(\xi_2))da \\
& + \int_0^A \beta_1(G_1(\xi)\widehat{p}_{1_\varepsilon}(\xi_1)(t,0,x) + G_2(\xi)\widehat{p}_{2\varepsilon}(\xi_2)(t,0,x) + \widehat{f} + \widehat{v}_\varepsilon\chi_\omega)da \\
& + \int_0^A \beta_2\left(G_2(\xi)\widehat{p}_{1_\varepsilon}(\xi_1)(t,0,x) + G_1(\xi)\widehat{p}_{2\varepsilon}(\xi_2)(t,0,x)\right)da \\
& - 2\sum_{i=1}^n \int_0^A \frac{\partial\beta_1}{\partial x_i}.\left(\frac{\partial\widehat{p}_{1_\varepsilon}}{\partial x_i} + \frac{\partial\widehat{p}_{2\varepsilon}}{\partial x_i}\right)da \\
R_2(\xi) = & \int_0^A \left(\frac{\partial\beta_2}{\partial t} + \frac{\partial\beta_2}{\partial a} + \Delta\beta_2 + \beta_2(\mu_1-\mu_2)\right)(\widehat{p}_{1_\varepsilon}(\xi_1) - \widehat{p}_{2\varepsilon}(\xi_2))da \\
& + \int_0^A \beta_2(G_1(\xi)\widehat{p}_{1_\varepsilon}(t,0,x) + G_2(\xi)\widehat{p}_{2\varepsilon}(t,0,x) + \hat{f} + \widehat{v}_\varepsilon\chi_\omega)da \\
& - \int_0^A \beta_2\left(G_2(\xi)\widehat{p}_{1_\varepsilon}(t,0,x) + G_1(\xi)\widehat{p}_{2\varepsilon}(t,0,x)\right)da \\
& - 2\sum_{i=1}^n \int_0^A \frac{\partial\beta_2}{\partial x_i}.\left(\frac{\partial\widehat{p}_{1_\varepsilon}}{\partial x_i} - \frac{\partial\widehat{p}_{2\varepsilon}}{\partial x_i}\right)da.
\end{aligned}
$$

Under the hypothesis $(H_1)-(H_4)$, taking λ_0 as in the proof of the theorem 1, one deducts from (27), (34)-(37) that there exists a positive reals C_1, C_2 which depend on $\|\beta_1,\beta_2\|_\infty$, $\|F,G\|_\infty$ and $\|\mu_1,\mu_2\|_\infty$ such that

$$\|R_1(\xi)\|_\infty^2 \leq C_1\left(\|\theta f\|^2_{L^2(Q_\omega)} + \|f\|^2_Q\right) \tag{57}$$

$$\|R_2(\xi)\|_\infty^2 \leq C_2\left(\|\theta f\|^2_{L^2(Q_\omega)} + \|f\|^2_Q\right). \tag{58}$$

Multiplying respectively the first equation of (56) by $Y_i(\xi)$, $i = 1;2$ and by integrating by parts over Q_T, we show (using Young's inequality as in the step 2 of the proof of the Proposition 3) that Y_i, $i = 1;2$ are bounded in $L^2(0,T;H_0^1(\Omega))$. Thus, for each $i \in \{1;2\}$, the system (56) is a retrograde heat equation with the source term and the initial condition are bounded respectively in $L^2(Q_T)$ and $L^2(Q)$. Moreover, Y_i, $\frac{\partial Y_i(\xi)}{\partial t}$ $i = 1,2$ are bounded respectively in $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;H^{-1}(\Omega))$. Consequently, we conclude, thanks to Lions-Aubin Lemma, that Λ_{ξ_i} $i = 1,2$ are bounded and compact in $L^2(Q_T)$. Thus, $\widehat{\Lambda}$ is bounded and compact in $\mathcal{N}$.

Now, let K a closed subset of $\mathcal{N}$. Let $(\xi_{1_n},\xi_{2_n})_n \subset \widehat{\Lambda}^{-1}(K)$ that converges strongly towards (ξ_1,ξ_2) in $\mathcal{N}$. Then, $((\xi_{1_n},\xi_{2_n}))_n$ is bounded in $\mathcal{N}$. Let remember that $\widehat{\Lambda}^{-1}(K) = \{(\xi_1,\xi_2) \in K : \Lambda(\xi_1,\xi_2)\cap K \neq \emptyset\}$. So, there exists a sequence $(Y_{1_n},Y_{2_n})_n \in K$ that belongs to $\Lambda^{-1}_{\xi_2}(\xi_{1_n}) \times \Lambda^{-1}_{\xi_1}(\xi_{2_n}) = \widehat{\Lambda}^{-1}(\xi_{1_n},\xi_{2_n})$ such that Y_{1_n} and Y_{2_n} verifies respectively (54) and (55) with respectively ξ_{1_n} and ξ_{2_n} instead of ξ_1 and ξ_2, and moreover, the pair $(\widehat{p}_{1_\varepsilon}(\xi_{1_n}),\widehat{p}_{2\varepsilon}(\xi_{2_n}))$ satisfies (52) and the associated control $\widehat{v}_\varepsilon$ verifies (27). Using

(56) and the estimations (34)-(37), we show (as the step 4 in the section 4) that the sequel $(Y_{i_n})_n$, $i = 1, 2$ converge strongly to Y_i $i = 1, 2$. Since $\widehat{p}_{i_\varepsilon}(\xi_{i_n})$, $i = 1, 2$ and $\eta_{1_\varepsilon}(\xi_{1_n})$ are bounded independently to (ξ_{i_n}) , $i = 1; 2$, then, for all n, $R_i(\xi_n)$ $i = 1, 2$ are bounded in $L^2(Q_T)$. Consequently, one can extract a subsequence still denoted by $Y_{i_n}, R_i(\xi_n)$ $i = 1, 2$ such that

$$
\begin{aligned}
Y_{i_n} &\longrightarrow Y_i \text{ in } L^2(Q_T) \ i = 1, 2\,;\\
R_i(\xi_n) &\longrightarrow R_i(\xi) \ i = 1, 2\,;\\
\int_0^A \tilde{\mu}_1 \tilde{\beta}_i \widehat{p}_{i_\varepsilon}(\xi_{i_n}) &\longrightarrow \int_0^A \tilde{\mu}_1 \beta_i \widehat{p}_{i_\varepsilon}(\xi_i) da \text{ weakly in } L^2(Q_T) \ i = 1, 2\,;\\
\int_0^A \tilde{\mu}_1 \beta_1 \widehat{p}_{2_\varepsilon}(\xi_{2_n}) da &\longrightarrow \int_0^A \tilde{\mu}_1 \beta_1 \widehat{p}_{2_\varepsilon}(\xi_2) da \text{ weakly in } L^2(Q_T)\,;\\
\int_0^A \mu_2 \beta_2 \widehat{p}_{1_\varepsilon}(\xi_{1_n}) da &\longrightarrow \int_0^A \mu_2 \beta_2 \widehat{p}_{1_\varepsilon}(\xi_1) da \text{ weakly in } L^2(Q_T)\,;
\end{aligned}
$$

So, for each $i \in \{1; 2\}, Y_i(\xi)$ is solution of (56), $(\widehat{p}_{1_\varepsilon}(\xi_1), \widehat{p}_{2\varepsilon}(\xi_2))$ solves (52) and the associated control $\widehat{v}_\varepsilon = \eta_1(\xi_1)$ verifies (29). Hence, $(Y_1, Y_2) \in \Lambda^{-1}_{\xi_2}(\xi_1) \times \Lambda^{-1}_{\xi_1}(\xi_2)$ and so, $(\xi_1, \xi_2) \in \widehat{\Lambda}^{-1}(K)$. Endly, since $\xi_1 \longmapsto \widehat{p}_{1_\varepsilon}$ and $\xi_2 \longmapsto \widehat{p}_{2\varepsilon}$ are affine, then $\Lambda_{\xi_2}(\xi_1)$ and $\Lambda_{\xi_1}(\xi_2)$ are nonempty convex sets in $L^2(Q_T)$. Thus, the gragh $G_{\widehat{\Lambda}} = \{\langle(\xi_1, \xi_2), \widehat{\Lambda}(\xi_1, \xi_2)\rangle\}$ of $\widehat{\Lambda}$ is closed. Then, $\widehat{\Lambda}(\xi_1, \xi_2) = \Lambda_{\xi_2}(\xi_1) \times \Lambda_{\xi_1}(\xi_2)$ is upper semicontinuous, and from the Kakutani's fixed point theorem [8], we conclude that $\widehat{\Lambda}$ admits a fixed point. More precisely, there exists $\xi = (\xi_1, \xi_2) \in \mathcal{N}$ such that

$$
\widehat{\Lambda}(\xi) = \xi = \left(\int_0^A \beta_1(\widehat{p}_{1_\varepsilon}(\xi_1) + \widehat{p}_{2\varepsilon}(\xi_2)) da \ , \ \int_0^A \beta_2(\widehat{p}_{1_\varepsilon}(\xi_1) - \widehat{p}_{2\varepsilon}(\xi_2)) da \right)
$$

where $(\widehat{p}_{1_\varepsilon}, \widehat{p}_{2\varepsilon})$ is solution of the system (52) with

$$
\begin{aligned}
G_1 \left(\int_0^A \beta \widehat{p}_\varepsilon da \right) &= \beta_1(t, a, x) F \left(\int_0^A \beta_1(\widehat{p}_{1_\varepsilon} + \widehat{p}_{2\varepsilon}) da \right) + \beta_2(t, a, x) G \left(\int_0^A \beta_2(\widehat{p}_{1_\varepsilon} - \widehat{p}_{2\varepsilon}) da \right)\\
G_2 \left(\int_0^A \beta_2 \widehat{p}_\varepsilon da \right) &= \beta_1(t, a, x) F \left(\int_0^A \beta_1(\widehat{p}_{1_\varepsilon} + \widehat{p}_{2\varepsilon}) da \right) - \beta_2(t, a, x) G \left(\int_0^A \beta_2(\widehat{p}_{1_\varepsilon} - \widehat{p}_{2\varepsilon}) da \right).
\end{aligned}
$$

instead of $G_1(\xi)$ and $G_2(\xi)$ respectively.

5. Application to the sentinel of detection

We consider for given positive functions $G_i \; = 1;2$ the following systems :

$$\left\{\begin{array}{rcll} \dfrac{\partial y_i}{\partial t}+\dfrac{\partial y_i}{\partial a}-\Delta y_i+\mu_i y_i &=& 0 & \text{in} \quad Q, \\ y_i(0,a,x) &=& y_i^0+\tau_i\widehat{y}_i^0 & \text{in} \quad Q_A, \\ y_i(t,0,x) &=& G_i\left(\displaystyle\int_0^A \beta_i y_i da\right) & \text{in} \quad Q_T, \\ y_i &=& \left\{\begin{array}{ll} g_i+\lambda_i\widehat{g}_i & \text{on} \quad \Sigma_i, \\ 0 & \text{on} \quad \Sigma\setminus\Sigma_i. \end{array}\right. \end{array}\right. \tag{59}$$

where $\Sigma_i = (0,T)\times(0,A)\times\Gamma_i \; i = 1;2$, the $\Gamma_i, i = 1;2$ are such that $\Gamma_1\cup\Gamma_2 = \Gamma$ and $\Gamma = \partial\Omega$ is the smooth boundary of Ω, the functions μ_i, β_i and the reals T, A are defined respectively as in section 1. $y(t,a,x)$ is the distribution of individuals of age a at time t and location $x \in \Omega$. The expressions $\int_0^A \beta_i y_i da, \quad i = 1;2$ denote the distribution of newborn individuals at time t and location x. In an ovipare species it represents the total eggs hatch at time t and the position x and $G_i\left(\int_0^A \beta_i y_i da\right)$ denote the distribution of eggs that at time t and the position x. The functions $G_i \; i = 1;2$ are of class C^1, globally lipschitz and their derivate functions verify $G_i'(0) = 0$ and moreover $G_i' \in L^\infty(\mathbb{R})$ are globally lipschitz. The system (59) describes the evolution of the populations under the inhospitable boundary conditions when the flux of population takes the form $-\nabla y(t,a,x)$. As for the initial and boundary conditions of (59), y_i^0 and g_i are given respectively in $L^2(Q_A)\tau_i\widehat{y}_i^0, \lambda_i\widehat{g}_i \; i = 1;2$ are unknown where $\tau_i, \; \lambda_i \; i = 1;2$ are reals. As a matter of fact the terms $y_i^0+\tau_i\widehat{y}_i^0$ and $g_i+\lambda_i\widehat{g}_i$ are qualified as incomplete data. Suppose that :

(i) for i=1;2 $\widehat{g}_i \in L^2(\Sigma_i)$ and $\|\widehat{g}_i\|_{L^2(\Sigma_i)} \leq 1$,

(ii) for i=1;2 $\widehat{y}_i^0 \in L^2(Q_A)$ and $\|\widehat{y}_i^0\|_{L^2(Q_A)} \leq 1$,

(iii) for $i = 1;2$ the reals τ_i and λ_i are unknown and small enough.

It is now assumed that measures y_{iobs} , $i = 1;2$ are available on $Q_{\mathcal{O}} = U\times\mathcal{O}$ where $\mathcal{O}\subset\Omega$ is the observation set and $\mathcal{O}\cap\omega \neq \emptyset$. Assume moreover that

$$y_i = y_{iobs} = m_{0i} \;, \; i = 1;2 \;\text{ on }\; Q_{\mathcal{O}}. \tag{60}$$

where $m_{0i}, \; i = 1;2$ are known functions belonging to $L^2(Q_{\mathcal{O}})$. The aim is to calculate the pollution terms $\lambda_1\widehat{g}_1$ and $\lambda_2\widehat{g}_2$ independently from the missing terms $\tau_1\widehat{y}_1^0$ and $\tau_2\widehat{y}_2^0$ with one and only one sentinel. One of the methods to solve this problem is the least squares method. The sentinel concept was introduced by J.L. Lions [7] to study the systems with incomplete data. This concept relies on the following elements : the state y described by a equation or a partial differential equations system, an observation function y_{obs} defined on $U\times\mathcal{O}$ where $\mathcal{O}$ is the observation set and a control function v to be determined. Many papers use the definition of Lions in the theoretical aspect. As to applications, we quote S. Sawadogo in [9] who studied the detection of incomplete parameters for a linear population dynamic model. In [10] the author made the same study for a nonlinear population dynamic model. For the sentinel concept we refer to [9, 10] and the references therein. In this paragraph we study the simultaneous sentinel concept for a coupled nonlinear population dynamic model. We begin by the following proposition

Proposition 3. *For each $i = 1;2$, the functions $\lambda_i \longmapsto y_i(\lambda_i, \tau_i)$ and $\tau_i \longmapsto y_i(\lambda_i, \tau_i)$ are differentiable at the point 0.*

Proof. Let $\widehat{y}_i(t,a,x) = e^{-\lambda_0 t}\left(y_i(\lambda_i, \tau_i) - y_{0i}\right)$ $i = 1;2$ with $y_{0i} = y_i(\lambda_i, 0)$ and for each $i = 1;2$, $y_i(\lambda_i, \tau_i)$ and y_{0i} solve (59). Then $\widehat{y}_i$ $i = 1;2$ verify

$$\begin{cases} \dfrac{\partial \widehat{y}_i}{\partial t} + \dfrac{\partial \widehat{y}_i}{\partial a} - \Delta \widehat{y}_i + (\mu_i + \lambda_0)\widehat{y}_i &= 0 \quad \text{in} \quad Q, \\ \widehat{y}_i(0,a,x) &= \tau_i \widehat{y}_i^0 \quad \text{in} \quad Q_A, \\ \widehat{y}_i(t,0,x) &= e^{-\lambda_0 t}\left(G_i\left(\displaystyle\int_0^A \beta_i y_i da\right)\right. \\ &\quad \left. - G_i\left(\displaystyle\int_0^A \beta_i y_{0i} da\right)\right) \quad \text{in} \quad Q_T, \\ \widehat{y}_i &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{61}$$

System (61) is this one obtained in the proof of the Proposition 9 in [10] with here $\beta_i(t,a,x)$, G_i respectively in the place of $\beta(a)$, F and $\tau_i = \tau$, $\lambda_i = \lambda$ $i = 1;2$. Let multiply (61) by $\widehat{y}_i$ and integrate by parts over Q. Since G_i, $i = 1;2$ is globally lipschitz, proceeding as in [10], we have

$$\|\widehat{y}_i(\cdot,0,\cdot)\|_{L^2(Q_T)} \leq C\|\beta_i\|_\infty^2 \|\widehat{y}_i\|_{L^2(Q_T)}. \tag{62}$$

One deducts from (62) that

$$\|\nabla \widehat{y}_i\|_{L^2(Q_T)} + \|\widehat{y}_i\|_{L^2(Q_T)} \leq C\tau_i^2. \tag{63}$$

According to the expression of $\widehat{y}_i$ and the relation (61), we get y_i converges uniformly to y_{0i} on Q and $\displaystyle\int_0^A \beta_i y_i(\lambda_i, \tau_i) da$ converges uniformly to $\displaystyle\int_0^A \beta_i y_{0i} da$ on Q_T. Set now $z_{\tau_i} = \frac{\widehat{y}_i}{\tau_i}$ and $p_{\tau_i} = z_{\tau_i} - z_i$ for $i = 1;2$, where z_i verifies

$$\begin{cases} \dfrac{\partial z_i}{\partial t} + \dfrac{\partial z_i}{\partial a} - \Delta z_i + \mu_i z_i &= 0 \quad \text{in} \quad Q, \\ z_i(0,a,x) &= \widehat{y}_i^0 \quad \text{in} \quad Q_A, \\ z_i(t,0,x) &= G_i'\left(\displaystyle\int_0^A \beta_i y_{0i} da\right)\displaystyle\int_0^A \beta_i z_i da \quad \text{in} \quad Q_T, \\ y_i &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{64}$$

we show as in [10] that :
$p_{\tau_i} \longrightarrow 0, z_{\tau_i} \longrightarrow z_i$ $i = 1;2$ respectively in $L^2\left(U; H_0^1(\Omega)\right)$ as $\tau_i \to 0$.

Likewise let $\widehat{u}_i(t,a,x) = e^{-\lambda_0 t}\left(y_i(\lambda_i,\tau_i) - y_{i0}\right)$ $i = 1;2$ with $y_{i0} = y_i(0,\tau_i)$ and for each $i = 1;2$, $y_i(\lambda_i,\tau_i)$ and y_{i0} solve (59). Then $\widehat{u}_i$, $i = 1;2$ verify

$$\left\{\begin{array}{rcll} \dfrac{\partial \widehat{u}_i}{\partial t} + \dfrac{\partial \widehat{u}_i}{\partial a} - \Delta\widehat{u}_i + (\mu_i+\lambda_0)\widehat{u}_i & = & 0 & \text{in} \quad Q, \\ \widehat{u}_i(0,a,x) & = & 0 & \text{in} \quad Q_A, \\ \widehat{u}_i(t,0,x) & = & e^{-\lambda_0 t}\Bigg(G_i\left(\displaystyle\int_0^A \beta_i y_i da\right) & \\ & & \quad - G_i\left(\displaystyle\int_0^A \beta_i y_{0i} da\right)\Bigg) & \text{in} \quad Q_T, \\ \widehat{u}_i & = & \left\{\begin{array}{l} \lambda_i \widehat{g}_i \\ 0 \end{array}\right. & \begin{array}{l} \text{on} \quad \Sigma_i \\ \text{on} \quad \Sigma\setminus\Sigma_i. \end{array} \end{array}\right. \tag{65}$$

Multiplying (65) by $\widehat{u}_i$ and by integrating by parts over Q, we have

$$\begin{aligned} &\frac{1}{2}\int_{Q_A} \widehat{u}_i^2(T,a,x)dQ_A + \frac{1}{2}\int_{Q_T} \widehat{u}_i^2(t,A,x)dQ_T + \int_Q |\nabla \widehat{u}_i|^2 dQ \\ &+ \int_Q (\mu_i+\lambda_0)\widehat{u}_i^2 dQ = \tau_i \int_{\Sigma_i} \frac{\partial \widehat{u}_i}{\partial \sigma_i}\widehat{g}_i d\Sigma_i + \frac{1}{2}\int_{Q_T} \widehat{u}_i^2(t,0,x)dQ_T \end{aligned} \tag{66}$$

From (62), taking $\lambda_0 = 1 + C\|\beta_i\|_\infty^2$, one has

$$\|\widehat{u}_i\|^2_{L^2(Q)} + \|\nabla\widehat{u}_i\|^2_{(L^2(Q))^N} \le \lambda_i \int_{\Sigma_i} \nabla\widehat{u}_i \widehat{g}_i d\Sigma_i \tag{67}$$

Using Young inequality and according to hypothesis (i), there exists a positive constant C_Y such that

$$\|\widehat{u}_i\|^2_{L^2(Q)} + \|\nabla\widehat{u}_i\|^2_{(L^2(Q))^N} \le \frac{\lambda_i^2}{2C_Y}. \tag{68}$$

Then $\widehat{y}_i$ converges uniformly to y_{i0} on Q and from the regularity of G_i, $i = 1;2$ we proove that $\displaystyle\int_0^A \beta_i y_i(\lambda_i,\tau_i)da$ converges uniformly to $\displaystyle\int_0^A \beta_i y_{i0} da$ on Q_T.

One deducts from the proposition 9 in [10], that the functions $\lambda_i \longmapsto y(\lambda_i,\tau_i)$ $i = 1;2$ are differentiable. Set now $z_{\lambda_i} = \frac{\widehat{u}_i}{\lambda_i}$ and $p_{\lambda_i} = z_{\lambda_i} - z_i$ for $i = 1;2$, where z_i verifies

$$\left\{\begin{array}{rcll} \dfrac{\partial z_i}{\partial t} + \dfrac{\partial z_i}{\partial a} - \Delta z_i + \mu_i z_i & = & 0 & \text{in} \ Q, \\ z_i(0,a,x) & = & 0 & \text{in} \ Q_A, \\ z_i(t,0,x) & = & G_i'\left(\displaystyle\int_0^A \beta_i y_{0i} da\right)\displaystyle\int_0^A \beta_i z_i da & \text{in} \ Q_T, \\ y_i & = & \left\{\begin{array}{l} \widehat{g}_i \\ 0 \end{array}\right. & \begin{array}{l} \text{on} \quad \Sigma_i \\ \text{on} \quad \Sigma\setminus\Sigma_i. \end{array} \end{array}\right. \tag{69}$$

Then p_{λ_i} solves

$$\left\{\begin{array}{rcll}\dfrac{\partial p_{\lambda_i}}{\partial t}+\dfrac{\partial p_{\lambda_i}}{\partial a}-\Delta p_{\lambda_i}+\mu_i p_{\lambda_i} & = & 0 & \text{in} \quad Q,\\ p_{\lambda_i}(0,a,x) & = & 0 & \text{in} \quad Q_A,\\ p_{\lambda_i}(t,0,x) & = & e^{-\lambda_0 t}\Bigg[G_i\left(\displaystyle\int_0^A \beta_i y_i da\right) & \\ & & -\,G_i\left(\displaystyle\int_0^A \beta_i y_{0i} da\right)\Bigg] & \\ & & -\,G_i'\left(\displaystyle\int_0^A \beta_i y_{0i} da\right)\displaystyle\int_0^A \beta_i z_i da & \text{in} \quad Q_T,\\ p_{\lambda_i} & = & 0 & \text{on} \quad \Sigma.\end{array}\right. \tag{70}$$

We obtain the equality (66) when we multiply (70) by p_{λ_i} and integrate by parts over Q. From the fact that the functions G_i $i=1;2$ are globally lipschitz and $\lambda_i \longmapsto y_i(\lambda_i,\tau_i)$ converge uniformly, one deduces that the functions $\lambda_i \longmapsto y_i(\lambda_i,\tau_i)$ $i=1;2$ are differentiable (see Proposition 9 in [10]). ■

In the sequel, we consider for $h\in L^2(Q_{\mathcal{O}})$ and $w\in L^2(Q_\omega)$, the following functionals :

$$S_i(\lambda_i,\tau_i)=\int_{Q_{\mathcal{O}}} h y_i(\lambda_i,\tau_i)dQ+\int_{Q_\omega} w y_i(\lambda_i,\tau_i)dQ \quad i=1;2. \tag{71}$$

We obtain from the Proposition 3 the following result.

Corollary 1. *The functionals S_i $i=1;2$ are differentiable at the point $(0,0)$ and*

$$\frac{\partial S_i}{\partial \tau_i}(0,0)=\int_{Q_{\mathcal{O}}} h y_{\tau_i} dQ+\int_{Q_\omega} w y_{\tau_i} dQ \qquad i=1;2 \tag{72}$$

$$\frac{\partial S_i}{\partial \lambda_i}(0,0)=\int_{Q_{\mathcal{O}}} h y_{\lambda_i} dQ+\int_{Q_\omega} w y_{\lambda_i} dQ \qquad i=1;2 \tag{73}$$

where for each $i=1;2$, y_{τ_i} solves the system :

$$\left\{\begin{array}{rcll}\dfrac{\partial y_{\tau_i}}{\partial t}+\dfrac{\partial y_{\tau_i}}{\partial a}-\Delta y_{\tau_i}+\mu_i y_{\tau_i} & = & 0 & in \quad Q,\\ y_{\tau_i}(0,a,x) & = & \widehat{y}^0(a,x) & in \quad Q_A,\\ y_{\tau_i}(t,0,x) & = & G_i'\left(\displaystyle\int_0^A \beta_i y_{0i} da\right)\displaystyle\int_0^A \beta_i z_i da & in \quad Q_T,\\ y_{\tau_i} & = & 0 & on \quad \Sigma,\end{array}\right. \tag{74}$$

and y_{λ_i} solves the system

$$\left\{\begin{array}{rcll}\dfrac{\partial y_{\lambda_i}}{\partial t}+\dfrac{\partial y_{\lambda_i}}{\partial a}-\Delta y_{\lambda_i}+\mu_i y_{\lambda_i} & = & 0 & in \quad Q,\\ y_{\lambda_i}(0,a,x) & = & 0 & in \quad Q_A,\\ y_{\lambda_i}(t,0,x) & = & G_i'\left(\displaystyle\int_0^A \beta_i y_{0i} da\right)\displaystyle\int_0^A \beta_i y_{\lambda_i} da & in \quad Q_T,\\ y_{\lambda_i} & = & \left\{\begin{array}{ll}\widehat{g}_i & \\ 0 & \end{array}\right. & \begin{array}{l} on \quad \Sigma_i\\ on \quad \Sigma\setminus\Sigma_i.\end{array}\end{array}\right. \tag{75}$$

Moreover

$$y_{\lambda_i},\ y_{\tau_i} \in C((0,T);L^2(Q_A)) \cap C((0,A);L^2(Q_T)) \cap L^2(U,H_0^1(\Omega))\ \ i=1;2. \tag{76}$$

Proof. We know that for each pair $(\lambda_i,\tau_i)\in\mathbb{R}^2$, (59) admits an unique solution $y(\lambda_i,\tau_i)$ in $C((0,T);L^2(Q_A)) \cap C((0,A);L^2(Q_T)) \cap L^2(U,H_0^1(\Omega))^2$ (see [5]). We have $S_i(\lambda_i=0,\tau_i)=\displaystyle\int_{Q_{\mathcal{O}}} hy(\lambda_i=0,\tau_i)dQ+\int_{Q_\omega} wy(\lambda_i=0,\tau_i)dQ$. So

$$\begin{aligned}\frac{S_i(\lambda_i=0,\tau_i)-S_i(0,0)}{\tau_i} &= \int_{Q_{\mathcal{O}}} h\frac{y(\lambda_i=0,\tau_i)-y_i(0,0)}{\tau_i}dQ\\ &+\int_{Q_\omega} w\frac{y(\lambda_i=0,\tau_i)-y_i(0,0)}{\tau_i}dQ\end{aligned}$$

Passing to the limit as $\tau_i\to 0$ one obtain (72). Likewise, since $y(\lambda_i=0,\tau_i)-y_i(0,0)$ verifies (61) with $\lambda_0=0$, then from the regularities of the functions G_i 1; 2 and from the Proposition 3, one shows that $y_{\tau_i}=\displaystyle\lim_{\tau_i\to 0}\frac{y(\lambda_i=0,\tau_i)-y_i(0,0)}{\tau_i}$ solves (74) and verifies (76) for $i=1;2$. In the same ways setting $y_{\lambda_i}=\displaystyle\lim_{\lambda_i\to 0}\frac{y(\lambda_i,\tau_i=0)-y_i(0,0)}{\lambda_i}$, we proof that y_{λ_i} satisfies (73), (75) and (76). ■

Remark 3. *S_i $i=1;2$ is say to be a simultaneous sentinel if there exists a control $w\in L^2(Q_\omega)$ such that*

$$\frac{\partial S_i}{\partial \tau_i}(0,0)=0\ \ i=1;2 \tag{77}$$

and

$$\|w\|_{L^2(Q_\omega}=\min\left\{\|k\|_{L^2(Q_\omega}\ :\ k\in L^2(Q_\omega)\ \text{and } k \text{ verifies } (77)\ \right\} \tag{78}$$

Following [9, 10], we show that the simultaneous sentinel problem is equivalent to the following null controllability problem : find $w\in L^2(Q_\omega)$ with minimal norm such that (q_1,q_2) satisfies

$$\left\{\begin{array}{rcll} -\dfrac{\partial q_i}{\partial t}-\dfrac{\partial q_i}{\partial a}-\Delta q_i+\mu_i q_i &=& \beta_i G_i'\left(\displaystyle\int_0^A \beta_1 y_{0i}da\right) q_i(t,0,x) & \\ && +h\chi_{\mathcal{O}}+w\chi_\omega & \text{in}\quad Q,\\ q_i(T,a,x) &=& 0 & \text{in}\quad Q_A,\\ q_i(t,A,x) &=& 0 & \text{in}\quad Q_T,\\ q_i &=& 0 & \text{on}\quad \Sigma,\end{array}\right. \tag{79}$$

and

$$q_1(0,a,x)=q_2(0,a,x)=0 \text{ in } Q_A \tag{80}$$

$$\|w\|_{L^2(Q_\omega)}=\min_{k\in\mathcal{E}}\{\|k\|\} \tag{81}$$

where $\mathcal{E}=\left\{k\in L^2(Q_\omega) \text{ such that } (k,S_i) \text{ satisfies (71) and (77)}\right\}$.

Remark 4. *Setting* $G_1' = F$ *and* $G_2' = G$ *the problem* (79)-(80) *is exactly the problem* (1) *that we have solved. Since* $\mathcal{E}$ *is closed and convex subset of* $L^2(Q_\omega)$*, we can obtain* w *to be of minimal norm in* $L^2(Q_\omega)$ *by minimizing the norm of* k*, when* $k \in \mathcal{E}$.

6. Detection of the pollution term $\lambda_i \widehat{g}_i$ $i = 1;2$.

We know from the Corollary 1 that for each $i = 1;2$ the function

$$y_{\lambda_i} = \lim_{\lambda_i \to 0} \frac{y(\lambda_i, 0) - y_i(0,0)}{\lambda_i} \tag{82}$$

solve (75). Using the Taylor formula at the neighbourhood of $(0;0)$ we have :

$$S_i(\lambda_i, \tau_i) \approx S_i(0,0) + \lambda_i \frac{\partial S_i}{\partial \lambda_i}(0,0) + \tau_i \frac{\partial S_i}{\partial \tau_i}(0,0)\ ,\ i = 1;2. \tag{83}$$

According to (77), one deducts from (71), (73) and from the expression of $S_i(0,0)$ that (83) is equivalent to

$$\int_Q (h\chi_{\mathcal{O}} + w\chi_\omega) y_i(\lambda_i, \tau_i) dQ = \int_Q (h\chi_{\mathcal{O}} + w\chi_\omega) y_i(0,0) dQ + \lambda_i \int_Q (h\chi_{\mathcal{O}} + w\chi_\omega) y_{\lambda_i} dQ \tag{84}$$

Thanks to (60), the equality (84) becomes

$$\lambda_i \int_Q (h\chi_{\mathcal{O}} + w\chi_\omega) y_{\lambda_i} dQ = \int_Q (h\chi_{\mathcal{O}} + w\chi_\omega)(m_{0i} - y_i(0,0)) dQ\ ,\ i = 1;2. \tag{85}$$

Elsewhere, multiplying the first equation of (79) by y_{λ_i}, $i = 1;2$ and by integratings by parts over Q, we have thanks to (75) and (80)

$$\int_{\Sigma_i} \widehat{g}_i \frac{\partial q_i}{\partial \sigma} d\Sigma = \int_Q (h\chi_{\mathcal{O}} + w\chi_\omega) y_{\lambda_i} dQ \ \ i = 1;2. \tag{86}$$

where σ is the external unitary normal vector of Γ. Then (73) becames

$$\int_{\Sigma_i} \lambda_i \widehat{g}_i \frac{\partial q_i}{\partial \sigma} d\Sigma \approx \int_Q (h\chi_{\mathcal{O}} + w\chi_\omega)(m_{0i} - y_i(0,0)) dQ\ ,\ i = 1;2. \tag{87}$$

Since $q_i, h, w,$ and $y_i(0,0)$ $i = 1;2$ are known, (87) is a integral equation in $\lambda_i \widehat{g}_i$ that supply some informations on the terms $\lambda_i \widehat{g}_i$ $i = 1;2$.

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