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Comparison of SBA numerical method and method of separation of variables (Fourier) on wave equations.

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Abstract. In this paper, our aim is to use the SBA numerical method (combination of Adomian method and Picard successive approximations) and Fourier method or method of separation of variables to construct the solution of some wave equations. We compare the two methods and apply them to some wave equations.

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1. Introduction

Many problems are governed by partial differential equations, or by systems of partial differential equations. It is difficult to find their exact solutions. In this work, the SBA numerical method, [3, 9] and Fourier method permitted us to find the exact solution of some wave equations.

2. Description of the methods

2.1. Description of the SBA numerical method

Let's consider the following functional equation

$$Au = f \tag{1}$$

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Where $A : H \to H$ is an operator not necessarily linear and H is a Hilbert space adequately chosen given the operator A.

Let :

$$A = L - R - N \tag{2}$$

Where L is an invertible operator in the Adomian sense, R the linear remainder and N a nonlinear operator.

Equation (2) therefore becomes:

$$Lu - Ru - Nu = f \Longleftrightarrow u = \theta + L^{-1}(f) + L^{-1}(Ru) + L^{-1}(Nu) \tag{3}$$

Where θ is such that $L\theta = 0$

Equation (3) is the Adomian canonical form [2, 6–8]

Using the successive approximations [1, 4], we get:

$$u^k = \theta^k + L^{-1}(f^k) + L^{-1}(R(u^k)) + L^{-1}(N(u^{k-1})); \; k \geq 1 \tag{4}$$

This yields the following Adomian algorithm [5]

$$\begin{cases} u_0^k = \theta^k + L^{-1}(f^k) + L^{-1}(N(u^{k-1})); \; k \geq 1 \\ u_n^k = L^{-1}(R(u_{n-1}^k)); \; n \geq 1 \end{cases} \tag{5}$$

The Picard principle is then applied to equation (5) : let u^0 be such that $N(u^0) = 0$ for $k = 1$, we get :

$$\begin{cases} u_0^1 = \theta^1 + L^{-1}(f^1) + L^{-1}(N(u^0)) \\ u_n^1 = L^{-1}(R(u_{n-1}^1)); \; n \geq 1 \end{cases}$$

If the series $\left(\sum_{n\geq 0} u_n^1\right)$ converges, then $u^1 = \sum_{n\geq 0} u_n^1$

For $k = 2$,we get:

$$\begin{cases} u_0^2 = \theta^2 + L^{-1}(f^2) + L^{-1}(N(u^1)) \\ u_n^2 = L^{-1}(R(u_{n-1}^2)); \; n \geq 1 \end{cases}$$

If the series $\left(\sum_{n\geq 0} u_n^2\right)$converges, then $u^2 = \sum_{n\geq 0} u_n^2$. This process is repeated to k.

If the series $\left(\sum_{n\geq 0} u_n^k\right)$converges, then $u^k = \sum_{n\geq 0} u_n^k$.

Therefore $u = \lim\limits_{k\to+\infty} u^k$ is the solution of the problem.

Thus, given the problem

$$(p) : Au = f,$$

we combine ideas from the classical techniques to derive the following appropriate approximate scheme.

$$\begin{cases} u_0^k = \theta^k + L^{-1}(f^k) + L^{-1}(N(u^{k-1})); \quad k \geq 1 \\ u_n^k = L^{-1}(R(u_{n-1}^k)); \; n \geq 1 \end{cases} \tag{6}$$

called SBA algorithm.

2.2. Description of the Fourier method or method of separation of variables

The Method of separation of variable (also known as Fourier method) is one of several methods for solving ordinary and partial differential equations.

This method cannot always be used, even when it can be used it will not always be possible to get the solution of the problem. However, it can be used to easily in the $(1-D)$ heat equation with no sources, in the $(1-D)$ wave equation and in the $(2-D)$ heat and wave equations.

Let's consider the following general functional equation

$$Au = f \tag{7}$$

The method of separation of variables relies upon the assumption that a function of the form:

$$u(x,t) = X(x)T(t) \tag{8}$$

if we are in the case of one -dimension x of space$(1-D)$

and

$$u(x,y,t) = U(x,y)T(t) = X(x)Y(y)T(t) \tag{9}$$

if we are in the case of two dimension x and y of space $(2-D)$ will be a solution to linear homogeneous partial differential equation in x and t when when we are in $(1-D)$ and x, y and t when when we are in $(2.-D)$.

This is called a product solution and provided the boundary conditions are also linear and homogeneous this will also satisfy the boundary conditions.

However , as noted above this will only rarely satisfy the initial condition, but that is something for us to worry about in the next section.

3. Applications

3.1. Problem 1

Let's consider the following Wave's model of one-dimension of space:

$$(P_1)\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = c^2 \triangle u(x,t), c > 0 \\ u(x,0) = \varphi(x) = \sin\left(\dfrac{\pi x}{L}\right) \\ u_t(x,0) = \phi(x) = \pi \sin\left(\dfrac{\pi x}{L}\right) \\ u(0,t) = 0 \\ u(L,t) = 0, L > 0 \end{cases} \tag{10}$$

Solving yhe wave equation involves identifyinf the functions $u(x,t)$ that solve the partial differential equation that represent the amplitude of the wave at any position x at any time t.

• **Solving by SBA method**

By Integrating (10) we get the Adomian canonical form [2, 9] of the problem (10) :

$$u(x,t) = u(0,x) + t\frac{\partial u(0,x)}{\partial t} + c^2 L_{tt}^{-1}(\triangle u(x,t)) \tag{11}$$

Applying successive approximations method to (11), we get :

$$u^k(x,t) = u^k(0,x) + t\frac{\partial u^k(0,x)}{\partial t} + c^2 L_{tt}^{-1}(\triangle u^k(x,t)) + \tilde{N}(u^{k-1}(x,t)), k \geq 1 \tag{12}$$

where $\tilde{N}(u^k(x,t)) = 0 \ \forall \ \ k \in \mathbb{N}$

Applying the SBA algorithm to (12), we get :

$$(P_{SBA}^k)\begin{cases} u_0^k(x,t) = u^k(x,0) + t\dfrac{\partial u^k(0,x)}{\partial t}, \ k \geq 1 \\ u_n^k(x,t) = c^2 L_{tt}^{-1}(\triangle u_{n-1}^k(x,t)), \ n \geq 1 \end{cases} \tag{13}$$

For $k = 1$, We have:

$$(P_{SBA}^1)\begin{cases} u_0^1(x,t) = u^1(x,0) \\ u_n^1(x,t) = c^2 L_{tt}^{-1}(\triangle u_{n-1}^k(x,t)), \ n \geq 1 \end{cases} \tag{14}$$

We obtain :

$$\left\{\begin{array}{c} u_0^1(x,t)=\varphi(x)+t\phi(x) \\ u_1^1(x,t)=c^2(\varphi^2(x)\dfrac{t^2}{2!}+\phi^2(x)\dfrac{t^3}{3!}) \\ u_2^1(x,t)=c^4(\varphi^4(x)\dfrac{t^4}{4!}+\phi^4(x)\dfrac{t^5}{5!}) \\ u_3^1(x,t)=c^6(\varphi^6(x)\dfrac{t^6}{6!}+\phi^6(x)\dfrac{t^7}{7!}) \\ \dots \\ u_n^1(x,t)=c^{2n}(\varphi^{2n}(x)\dfrac{t^{2n}}{(2n)!}+\phi^{2n}(x)\dfrac{t^{2n+1}}{(2n+1)!}) \end{array}\right.$$

We have

$$\left\{\begin{array}{c} \varphi(x)=\sin\left(\dfrac{\pi x}{L}\right)\Rightarrow\varphi^{2n}(x)=(-1)^n\left(\dfrac{\pi}{L}\right)^{2n}\sin\left(\dfrac{\pi x}{L}\right) \\ \phi(x)=\pi\sin\left(\dfrac{\pi x}{L}\right)\Rightarrow\phi^{2n}(x)=\pi(-1)^n\left(\dfrac{\pi}{L}\right)^{2n}\sin\left(\dfrac{\pi x}{L}\right) \end{array}\right.$$

Then

$$u_n^1(x,t)=\sin\left(\frac{\pi x}{L}\right)\left((-1)^n\frac{\left(\dfrac{c\pi t}{L}\right)^{2n}}{(2n)!}+(-1)^n\frac{L}{c}\frac{\left(\dfrac{c\pi t}{L}\right)^{2n+1}}{(2n+1)!}\right)$$

$$\varphi_m^1(x,t)=\sum_{n=0}^{m-1}u_n^1(x,t)$$

Let's put

Then the approached solution at the the first step is:

$$\begin{aligned} u^1(x,t) &= \lim_{m\to+\infty}\varphi_m^1(x,t) \\ &= \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{c\pi t}{L}\right)+\frac{L}{c}\sin\left(\frac{c\pi t}{L}\right) \end{aligned}$$

such us $\tilde{N}(u^k(x,t))=0$, $\forall k\geq 0$ at the step k ,we have :

$$(P_{SBA}^k)\left\{\begin{array}{c} u_0^k(x,t)=u^k(x,0)+t\dfrac{\partial u^k(0,x)}{\partial t},\ k\geq 1 \\ u_n^k(x,t)=c^2L_{tt}^{-1}(\triangle u_{n-1}^k(x,t)),\ n\geq 1 \end{array}\right. \tag{15}$$

By unfolding :

$$\begin{cases} u_0^k(x,t) = \varphi(x) + t\phi(x) \\ u_1^k(x,t) = c^2(\varphi^2(x)\dfrac{t^2}{2!} + \phi^2(x)\dfrac{t^3}{3!}) \\ u_2^k(x,t) = c^4(\varphi^4(x)\dfrac{t^4}{4!} + \phi^4(x)\dfrac{t^5}{5!}) \\ u_3^k(x,t) = c^6(\varphi^6(x)\dfrac{t^6}{6!} + \phi^6(x)\dfrac{t^7}{7!}) \\ \dots \\ u_n^k(x,t) = c^{2n}(\varphi^{2n}(x)\dfrac{t^{2n}}{(2n)!} + \phi^{2n}(x)\dfrac{t^{2n+1}}{(2n+1)!}) \end{cases}$$

Then

$$u_n^k(x,t) = \sin(\frac{\pi x}{L})((-1)^n\frac{(\frac{c\pi t}{L})^{2n}}{(2n)!} + (-1)^n\frac{L}{c}\frac{(\frac{c\pi t}{L})^{2n+1}}{(2n+1)!}$$

$$\varphi_m^k(x,t) = \sum_{n=0}^{m-1} u_n^k(x,t)$$

Then the approached solution at the the first step is:

$$\begin{aligned} u^k(x,t) &= \lim_{m\to+\infty}\varphi_m^k(t) \\ &= \sin(\frac{\pi x}{L})(\cos(\frac{c\pi t}{L}) + \frac{L}{c}\sin(\frac{c\pi t}{L}) \end{aligned}$$

Therefore, we obtain the exact solution of the problem (P_1) :

$$\begin{aligned} u(x,t) &= \lim_{k\to+\infty} u^k(x,t) \\ &= (\cos(\frac{c\pi t}{L}) + \frac{L}{c}\sin(\frac{c\pi t}{L}))\sin(\frac{\pi x}{L}) \end{aligned}$$

• Solving by Fourier method

We find all solutions of the wave equation (P_1) with the general form:

$$u(x,t) = T(t)X(x) \tag{16}$$

for some function $X(x)$ that depends on x but not t and some function $T(t)$ that depends only on t but not x.

Substitute equation (16) into the one-dimensional equation (10), we get

$$X(x)T^{''}(t) = c^2T(t)X^{''}(x) \Longleftrightarrow \frac{T^{''}(t)}{c^2T(t)} = \frac{X^{''}(x)}{X(t)} \tag{17}$$

Such $\frac{T^{''}(t)}{c^2T(t)}$ depends on t and $\frac{X^{''}(x)}{X(t)}$ depends on x, we can put:

$$\frac{T^{''}(t)}{c^2T(t)} = \frac{X^{''}(x)}{X(t)} = -k^2 \tag{18}$$

Then

$$\left\{\begin{array}{c} T^{''}(t) = -(ck)^2T(t) \\ \text{and} \\ X^{''}(x) = -k^2X(x) \end{array}\right. \Longleftrightarrow \left\{\begin{array}{c} T^{''}(t) + (ck)^2T(t) = 0 \\ \text{and} \\ X^{''}(x) + k^2X(x) = 0 \end{array}\right. \tag{19}$$

Then

$$\left\{\begin{array}{c} T(t) = A\cos(ckt) + B\sin(ckt) \\ \text{and} \\ X(x) = C\cos(kx) + D\sin(kx) \end{array}\right. \tag{20}$$

where A, B, C and D are arbibrairy constantes.
We have

$$u(x,t) = (A\cos(ckt) + B\sin(ckt))\,(C\cos(kx) + D\sin(kx)) \tag{21}$$

Let's calculate the constantes

$$\left\{\begin{array}{c} u(t,0) = 0 \\ \text{and} \\ u(t,L) = 0 \end{array}\right. \Longrightarrow \left\{\begin{array}{c} C = 0 \\ D\sin(kL) = 0 \end{array}\right.$$

$$D\sin(kL) = 0 \Longrightarrow \left\{\begin{array}{c} D \neq 0 \\ \sin(kL) = 0 \end{array}\right.$$

$$\sin(kL) = 0 \Longleftrightarrow k = \frac{n\pi}{L}(n \in \mathbb{Z})$$

Then
$\forall n \geq 1$, we have

$$u_n(x,t) = (A_n \cos(\frac{cn\pi t}{L}) + B_n \sin(\frac{cn\pi t}{L})) \sin(\frac{n\pi x}{L}) \quad (22)$$

Then

$$u(x,t) = \sum_{n=1}^{+\infty} (A_n \cos(\frac{cn\pi t}{L}) + B_n \sin(\frac{cn\pi t}{L})) \sin(\frac{n\pi x}{L}) \quad (23)$$

We can noticed that if we choose k^2 instead of $-k^2$ the solution of the equation

$$X^{''}(x) + k^2 X(x) = 0 \quad (24)$$

which is

$$X(x) = Ae^{kx} + Be^{-kx}$$

don't verify the initial condition:

$$u(t,0) = u(t,L) = 0$$

So choosing k^2 is impossible.

We have

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{+\infty} \left(-\frac{cn\pi}{L} A_n \cos(\frac{cn\pi t}{L}) + \frac{cn\pi}{L} B_n \sin(\frac{cn\pi t}{L}) \right) \sin(\frac{n\pi x}{L}) \quad (25)$$

For $t = 0$, we have

$$\sum_{n=1}^{+\infty} \frac{cn\pi}{L} B_n \sin(\frac{n\pi x}{L}) = \phi(x) \quad (26)$$

$$u(0,x) = \varphi(x) \Longleftrightarrow \varphi(x) = \sum_{n=1}^{+\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \quad (27)$$

We have

$$\begin{cases} A_n = \dfrac{2}{L} \displaystyle\int_0^L \varphi(z) \sin\left(\frac{n\pi z}{L}\right) dz \\ \quad \text{and} \\ B_n = \dfrac{2}{cn\pi} \displaystyle\int_0^L \phi(z) \sin\left(\frac{n\pi z}{L}\right) dz \end{cases} \quad (28)$$

3.2. Problem 2

Let's consider the following wave's model :

$$(P_2)\begin{cases} \dfrac{\partial^2 u(x,y,t)}{\partial t^2} = c^2 \bigtriangleup u(x,y,t), c > 0 \\ u(x,y,0) = f_1(x,y) \\ u_t(x,y,0) = f_2(x,y) \\ u(0,y,t)) = h_1(0,y,t) \\ u(L,y,t)) = h_2(L,y,t) \\ u(x,0,t)) = g_1(x,0,t) \\ u(x,l,t)) = g_2(x,l,t) \end{cases} \tag{29}$$

Where

$$\begin{cases} \bigtriangleup u(x,y,t) = \dfrac{\partial^2 u(x,y,t)}{\partial x^2} + \dfrac{\partial^2 u(x,y,t)}{\partial y^2} \\ f_1(x,t) = \sin\left(\dfrac{\pi x}{L}\right)\sin\left(\dfrac{\pi y}{l}\right) \\ f_2(x,t) = 0 \\ h_1(x,t) = 0 \\ h_2(x,t) = 0 \\ g_1(x,t) = 0 \\ g_2(x,t) = 0 \end{cases} \tag{30}$$

Solving yhe wave equation involves identifyinf the functions $u(x,y,t)$ that solve the partial differential equation that represent the amplitude of the wave at any position x and y at any time t.

• **Solving by SBA method**

By Integrating (29) we get the Adomian canonical form [2, 9] of the problem (29) :

$$u(x,y,t) = u(x,y,0) + t\frac{\partial u(x,y,0)}{\partial t} + c^2 L_{tt}^{-1}(\bigtriangleup u(x,y,t)) \tag{31}$$

Applying successive approximations method to (31), we get :

$$u^k(x,y,t) = u^k(x,y,0) + t\frac{\partial u^k(x,y,0)}{\partial t} + c^2 L_{tt}^{-1}(\bigtriangleup u^k(x,y,t)) + \tilde{N}(u^{k-1}(x,y,t)), k \geq 1 \tag{32}$$

where $N^{\sim}(u^k(x,y,t)) = 0 \ \forall k \in \mathbb{N}$

Applying the SBA algorithm to (32), we get :

$$(P_{SBA}^k)\begin{cases} u_0^k(x,y,t) = u^k(x,y,0) + t\dfrac{\partial u^k(x,y,0)}{\partial t} + \tilde{N}(u^{k-1}(x,y,t)), k \geq 1,\ k \geq 1 \\ u_n^k(x,t) = c^2 L_{tt}^{-1}(\triangle u_{n-1}^k(x,y,t)),\ n \geq 1 \end{cases} \tag{33}$$

For $k = 1$, We have:

$$(P_{SBA}^1)\begin{cases} u_0^1(x,y,t) = u^1(x,,y,0) + t\dfrac{\partial u^1(x,y,0)}{\partial t} \\ u_n^1(x,t) = c^2 L_{tt}^{-1}(\triangle u_{n-1}^k(x,y,t)),\ n \geq 1 \end{cases} \tag{34}$$

We obtain:

$$\begin{cases} u_0^1(x,t) = \sin(\frac{\pi x}{L})\sin(\frac{\pi y}{L}) \\ u_1^1(x,y,t) = -c^2(\frac{\pi^2}{L^2} + \frac{\pi^2}{l^2})\sin(\frac{\pi x}{L})\sin(\frac{\pi y}{l})\dfrac{t^2}{2!} \\ u_2^1(x,y,t) = c^4(\frac{\pi^2}{L^2} + \frac{\pi^2}{l^2})^2\sin(\frac{\pi x}{L})\sin(\frac{\pi y}{l})\dfrac{t^4}{4!} \\ u_3^1(x,y,t) = -c^6(\frac{\pi^2}{L^2} + \frac{\pi^2}{l^2})^3\sin(\frac{\pi x}{L})\sin(\frac{\pi y}{l})\dfrac{t^6}{6!} \\ \cdots \\ u_n^1(x,y,t) = (-1)^n c^{2n6}(\frac{\pi^2}{L^2} + \frac{\pi^2}{l^2})^n\sin(\frac{\pi x}{L})\sin(\frac{\pi y}{l})\dfrac{t^{2n}}{(2n)!} \end{cases}$$

Let's put

$$\varphi_m^1(x,y,t) = \sum_{n=0}^{m-1} u_n^1(x,y,t)$$

$$\varphi_m^1(x,y,t) = \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{l}\right)\sum_{n=0}^{m-1}(-1)^n\frac{\left(\dfrac{c\pi t}{Ll}\sqrt{L^2+l^2}\right)^{2n}}{(2n)!}$$

Let's put

Then the approached solution at the the first step is:

$$\begin{aligned} u^1(x,y,t) &= \lim_{m\to+\infty}\varphi_m^1(x,y,t) \\ &= \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{l}\right)\cos\left(\frac{c\pi t}{Ll}\sqrt{L^2+l^2}\right) \end{aligned}$$

.
such us $N^{\sim}(u^k(x,y,t)) = 0 \; \forall k \geq 0$ at the step k ,we have :

$$(P^k_{SBA}) \begin{cases} u^k_0(x,y,t) = u^k(x,,y,0) + t\dfrac{\partial u^k(x,y,0)}{\partial t} \\ u^1_n(x,y,t) = c^2 L^{-1}_{tt}(\triangle u^k_{n-1}(x,y,t)), \; n \geq 1 \end{cases} \tag{35}$$

By unfolding :

$$\begin{cases} u^k_0(x,t) = \sin\left(\dfrac{\pi x}{L}\right)\sin\left(\dfrac{\pi y}{L}\right) \\ u^k_1(x,y,t) = -c^2\left(\dfrac{\pi^2}{L^2}+\dfrac{\pi^2}{l^2}\right)\sin\left(\dfrac{\pi x}{L}\right)\sin\left(\dfrac{\pi y}{l}\right)\dfrac{t^2}{2!} \\ u^k_2(x,y,t) = c^4\left(\dfrac{\pi^2}{L^2}+\dfrac{\pi^2}{l^2}\right)^2\sin\left(\dfrac{\pi x}{L}\right)\sin\left(\dfrac{\pi y}{l}\right)\dfrac{t'^4}{4!} \\ u^k_3(x,y,t) = -c^6\left(\dfrac{\pi^2}{L^2}+\dfrac{\pi^2}{l^2}\right)^3\sin\left(\dfrac{\pi x}{L}\right)\sin\left(\dfrac{\pi y}{l}\right)\dfrac{t'^6}{6!} \\ \ldots \\ u^k_n(x,y,t) = (-1)^n c^{2n6}\left(\dfrac{\pi^2}{L^2}+\dfrac{\pi^2}{l^2}\right)^n\sin\left(\dfrac{\pi x}{L}\right)\sin\left(\dfrac{\pi y}{l}\right)\dfrac{t'^{2n}}{(2n)!} \end{cases}$$

Then
Let's put

$$\varphi^k_m(x,y,t) = \sum_{n=0}^{m-1} u^1_n(x,y,t)$$

$$\varphi^k_m(x,y,t) = \sin(\frac{\pi x}{L})\sin(\frac{\pi y}{l})\sum_{n=0}^{m-1}(-1)^n\frac{(\frac{c\pi t}{Ll}\sqrt{L^2+l^2})^{2n}}{(2n)!}$$

Then the approached solution at the the first step is:

$$\begin{aligned} u^k(x,y,t) &= \lim_{m\to+\infty}\varphi^k_m(x,y,t) \\ &= \sin(\frac{\pi x}{L})\sin(\frac{\pi y}{l})\cos(\frac{c\pi t}{Ll}\sqrt{L^2+l^2}) \end{aligned}$$

Therefore, we obtain the exact solution of the problem (P_2) :

$$u(x,y,t) \;\; = \;\; \lim_{k\to+\infty} u^k(x,y,t) \tag{36}$$

$$= \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{l}\right)\cos\left(\frac{c\pi t}{Ll}\sqrt{L^2+l^2}\right)$$

• **Solving by Fourier method or separation of variables Method**

We find all solutions of the wave equation (P_2) with the general form:

$$u(x,y,t) = T(t)U(x,y) \tag{37}$$

where U depends on x and y :

$$U(x,y) = X(x)Y(y) \tag{38}$$

for some function $X(x)$ that depends on x, some function (y) that depends on y and some function $T(t)$ that depends only on t but not x and y.

Substitute equation (37) and (38) into the two-dimensional equation (31) we get:

$$\begin{cases} \dfrac{\partial^2 u(x,y,t)}{\partial t^2} = T^{''}(t)U(x,y) \\ \dfrac{\partial^2 u(x,y,t)}{\partial x^2} = T(t)\dfrac{\partial^2 U(x,y)}{\partial x^2} \\ \dfrac{\partial^2 u(x,y,t)}{\partial y^2} = T(t)\dfrac{\partial^2 U(x,y)}{\partial y^2} \\ \text{and} \\ T^{''}(t)U(x,y) = c^2T(t)(\dfrac{\partial^2 U(x,y)}{\partial x^2} + \dfrac{\partial^2 U(x,y)}{\partial y^2}) \end{cases}$$

Then

$$\frac{T^{''}(t)}{c^2T(t)} = \frac{\dfrac{\partial^2 U(x,y)}{\partial x^2} + \dfrac{\partial^2 U(x,y)}{\partial y^2}}{U(x,y)} = -\lambda^2 \tag{39}$$

Then we have

$$T^{''}(t) + (\lambda c)^2 T(t) = 0 \Longrightarrow T(t) = \alpha\cos(\lambda ct) + \beta\sin(\lambda ct) \tag{40}$$

where α and β are arbibrairy constantes.
And we get

$$u(x,y,t) = (\alpha\cos(\lambda ct) + \beta\sin(\lambda ct))U(x,y) \tag{41}$$

Then we have

$$\frac{\dfrac{\partial^2 U(x,y)}{\partial x^2} + \dfrac{\partial^2 U(x,y)}{\partial y^2}}{U(x,y)} = -\lambda^2 \Longrightarrow \frac{X^{''}Y(y) + X(x)Y^{''}(y)}{X(x)Y(y)} = -\lambda^2 \tag{42}$$

Then

$$X^{''}Y(y) + X(x)Y^{''}(y) + \lambda^2 X(x)Y(y) = 0 \tag{43}$$

Let's put

$$\frac{X^{''}(x)}{X(x)} = \frac{Y^{''}(y) + \lambda^2 Y(y)}{Y(y)} = -\mu^2 \tag{44}$$

We obtain

$$\begin{cases} X^{''}(x) + \mu^2 X(x) = 0 \\ \text{and} \\ Y^{''}(y) + \varsigma^2 X(y) = 0;\ \varsigma^2 = \lambda^2 + \mu^2 \end{cases} \tag{45}$$

Then

$$\begin{cases} X(x) = A\cos(\mu x) + B\sin(\mu x) \\ \text{and} \\ Y(y) = C\cos(\varsigma y) + D\sin(\varsigma y) \end{cases} \tag{46}$$

where A, B, C and D are arbibrairy constantes.
We have

$$u(x,y,t) = ((\alpha\cos(\lambda ct) + \beta\sin(\lambda ct))X(x)Y(y) \tag{47}$$

Let's calculate the constantes

With the initial condition

$$\begin{cases} u(x,y,0) = \sin\left(\dfrac{\pi x}{L}\right)\sin\left(\dfrac{\pi y}{l}\right) \\ u_t(x,y,0) = 0 \\ u(0,y,t)) = 0 \\ u(L,y,t)) = 0 \\ u(x,0,t)) = 0 \\ u(x,l,t)) = 0 \end{cases}$$

we get

$$\left\{\begin{array}{c} \alpha X(x)Y(y)=\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{l}\right) \\ X(0)=0 \\ X(L)=0 \\ Y(0)=0 \\ Y(l)=0 \\ \beta c\lambda=0 \end{array}\right. \Longrightarrow \left\{\begin{array}{c} \alpha X(x)Y(y)=\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{l}\right) \\ X(0)=0 \\ X(L)=0 \\ Y(0)=0 \\ Y(l)=0 \\ \beta c\lambda=0 \end{array}\right.$$

we obtain by unfoilding

$$\left\{\begin{array}{c} A=0 \\ \sin(\mu L)=0 \\ C=0 \\ \sin(\varsigma l)=0 \\ \beta=0 \end{array}\right. \Longrightarrow \left\{\begin{array}{c} A=0 \\ \mu=\frac{n\pi}{L};\forall n\geq 1 \\ C=0 \\ \varsigma=\frac{n\pi}{l};\forall n\geq 1 \\ \beta \end{array}\right.$$

Then we get

$$\left\{\begin{array}{c} X(x)=\sin(\frac{n\pi x}{L}) \\ \text{and} \\ Y(y)=\sin(\frac{n\pi y}{l}) \end{array}\right. \tag{48}$$

Then
$\forall n\geq 1$, we have

$$u_n(x,y;t)=((\alpha_n\cos(\lambda_n ct)+\beta_n\sin(\lambda_n ct))\sin(\frac{n\pi x}{L})\sin(\frac{n\pi y}{l});\forall n\geq 1 \tag{49}$$

and

$$\lambda_n^2=(\frac{n\pi}{Ll})^2(L^2+l^2) \tag{50}$$

Let's put

$$T_n(t)=((\alpha_n\cos(\lambda_n ct)+\beta_n\sin(\lambda_n ct));\forall n\geq 1 \tag{51}$$

We obtain

$$u(x,y,t)=\sum_{n=1}^{+\infty}T_n(t)\sin(\frac{n\pi x}{L})\sin(\frac{n\pi y}{l}) \tag{52}$$

Such

$$\begin{cases} u(x;y,0) = f_1(x,y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{l}\right) \\ \text{and} \\ u_t(x,y,0) = f_2(x,y) = 0 \end{cases}$$

We have

$$\begin{cases} f_1(x,y) = \sum_{n=1}^{+\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{l}\right) \\ \text{and} \\ \alpha_n = \int_{0.}^{L}\int_{0}^{l} f_1(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{l}\right)dxdy \end{cases} \tag{53}$$

Then by unfolding ,we get

$$\begin{cases} \alpha_1 = \int_{0.}^{L} \sin^2\left(\frac{\pi x}{L}\right)dx \int_{0}^{l}\sin^2\left(\frac{\pi y}{l}\right)dy = 1 \\ and \\ \alpha_n = \int_{0.}^{L}\int_{0}^{l} f_1(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{l}\right)dxdy = 0; \forall n \neq 0 \end{cases}$$

Therefore, we obtain the exact solution of the problem $(P2)$:

$$\begin{aligned} u(x,y,t) &= \sum_{n=1}^{+\infty} u_n(x,y,t) \\ &= \cos(\lambda_1 ct)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{l}\right) \end{aligned}$$

where

$$\lambda_n^2 = \left(\frac{n\pi}{Ll}\right)^2(L^2+l^2) \tag{54}$$

$$\lambda_1 = \frac{n\pi\sqrt{L^2+l^2}}{Ll} \tag{55}$$

Therefore, we obtain the exact solution of the problem $(P2)$:

$$u(x,y,t) = \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{l}\right)\cos\left(\frac{c\pi t\sqrt{L^2+l^2}}{Ll}\right) \tag{56}$$

We can noticed that if we choose λ^2 and μ^2 instead of $-\lambda^2$ and $-\mu^2$ the solution of the equation

don't verify the initial condition:

$$u(0,y,t) = u(L,y,0) = u(x,0,t) = u(x,l,0) = 0$$

So choosing λ^2 and μ^2 is impossible.

Where

$$\begin{cases} A_1 = \dfrac{2}{L}\displaystyle\int_0^L \sin^2\left(\dfrac{\pi z}{L}\right) dz = 1 \\ A_n = \dfrac{2}{L}\displaystyle\int_0^L \varphi(z)\sin\left(\dfrac{n\pi z}{L}\right) dz = 0 \text{ if } n \neq 1 \\ \text{and} \\ B_n = \dfrac{2}{c}\displaystyle\int_0^L \sin^2\left(\dfrac{\pi z}{L}\right) dz = \dfrac{L}{c} \\ B_n = \dfrac{2}{cn\pi}\displaystyle\int_0^L \phi(z)\sin\left(\dfrac{n\pi z}{L}\right) dz = 0 \quad \text{if} \quad n \neq 1 \end{cases}$$

We obtain

$$\begin{cases} A_1 = 1 \\ A_n = 0 \;\forall\; n \neq 1 \\ \text{and} \\ B_n = \dfrac{L}{c} \\ B_n = 0 \;\forall\; n \neq 1 \end{cases}$$

Therefore, we obtain the exact solution of the problem (P_1) :

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{+\infty}\left(A_n\cos(\frac{cn\pi t}{L}) + B_n\sin(\frac{cn\pi t}{L})\right)\sin\left(\frac{n\pi x}{L}\right) \\ &= \left(\cos(\frac{c\pi t}{L}) + \frac{L}{c}\sin(\frac{c\pi t}{L})\right)\sin\left(\frac{\pi x}{L}\right) \end{aligned} \tag{57}$$

3.3. Comparison of the solution

Method	SBA method
Problem 1	$u(x,t) = \left(\cos\left(\frac{c\pi t}{L}\right) + \frac{L}{c}\sin\left(\frac{c\pi t}{L}\right)\right)\sin\left(\frac{\pi x}{L}\right)$
Problem 2	$u(x,y,t) = \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{l}\right)\cos\left(\frac{c\pi t\sqrt{L^2+l^2}}{Ll}\right)$

Method	Fourier method
Problem 1	$u(x,t) = \left(\cos\left(\frac{c\pi t}{L}\right) + \frac{L}{c}\sin\left(\frac{c\pi t}{L}\right)\right)\sin\left(\frac{\pi x}{L}\right)$
Problem 2	$u(x,y,t) = \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{l}\right)\cos\left(\frac{c\pi t\sqrt{L^2+l^2}}{Ll}\right)$

4. Conclusion

The numerical method SBA and Fourier method or method of separation of variable permitted us to resolve some partial differential equations in this paper.

In this paper, we showed that using the both methods, we get the same solution.

There are then some very powerful numerical tools of analysis for the resolution of partial differential equations.

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