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Supra b maps via topological ordered spaces

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Abstract. The authors utilize the notions of increasing, decreasing and balancing supra b-open sets to introduce and study several types of supra continuous, supra open, supra closed and supra homeomorphism maps in supra topological ordered spaces. They give the equivalent conditions for each one of these notions and illustrate the relationships among them with the help of examples. Apart from that, they investigate under which conditions these maps preserve some separation axioms between supra topological ordered spaces.

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1. Introduction

The concept of topological ordered spaces has been initiated by Nachbin [26] in 1965. In 1968, McCartan [25] carried out a detailed study about ordered separation axioms. He distinguished between two types of these axioms, one of the them depend on monotone supra neighborhoods and the other depend on monotone supra open neighborhoods. Arya and Gupta [16] introduced the concepts of semi T_1-ordered and semi T_2-ordered spaces. Kumar [23] studied the concepts of continuity, openness, closedness and homeomorphism between topological ordered spaces.

Mashhour et al. [24] introduced a notion of supra topological spaces and generalized some properties of topological spaces to supra topological spaces such as continuity and some separation axioms. Das [17] introduced ordered separation axioms in supra topological ordered spaces and discussed the validity of the results obtained by [25] on some

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ordered spaces. In 2016, Abo-Elhamayel and Al-shami [1] introduced and studied some new maps between supra topological spaces. El-Shafei et al. [20] formulated strong separation axioms via supra topological ordered spaces. Sayed and Noiri [27] introduced and studied supra b-open sets and supra b-continuous maps. In [2, 6, 8], Al-shami investigated the properties of new types of supra compact spaces, and in [3, 18], the authors presented two kinds of generalized supra open sets, namely supra semi open and supra R-open sets. In [5, 7, 15, 19, 21], the authors employed some generalizations of supra open sets to define new types of ordered maps. It is noteworthy that the study concerning soft topological ordered spaces was done in [10] and the studies concerning various types of soft ordered maps were done in [9, 11–13, 22].

The aim of the present paper is to establish some types of maps on supra topological spaces, namely x-supra b-continuous, x-supra b-open, x-supra b-closed and x-supra b-homeomorphism maps, where x$\in$ {I, D, B}. The equivalent conditions for these maps are investigated and the relationships among them are shown with the help of examples. Also, the sufficient conditions for these maps to preserve some separation axioms are given. It can be observed that many of the findings that raised at herein are generalizations of those findings obtained in [1].

2. Preliminaries

A topological ordered space is a triple $(X, \tau, \preceq)$, where (X, τ) is a topological space and $(X, \preceq)$ is a partially ordered set. From now on, the notations τ μ and $\triangle$ respectively refer to a topology, a supra topology and the diagonal relation on a non empty set X or Y.

We start this section with recalling some definitions and results which are necessary for the sequel of this study.

Definition 1. *[26] Let $a \in X$ and B be a subset of a partially ordered set $(X, \preceq)$. Then:*

(i) $i(a) = \{x \in X : a \preceq x\}$ *and* $d(a) = \{x \in X : x \preceq a\}$.

(ii) $i(B) = \bigcup\{i(b) : b \in B\}$ *and* $d(B) = \bigcup\{d(b) : b \in B\}$.

(iii) *A set B is called increasing (resp. decreasing), if $B = i(B)$ (resp. $B = d(B)$).*

Definition 2. *[23] A subset B of a partially ordered set $(X, \preceq)$ is called balancing if it is increasing and decreasing.*

Definition 3. *[1] A map $g : (X, \tau) \to (Y, \mu)$ is said to be supra open (resp. supra closed) if the image of any open (resp. closed) subset of X is a supra open (resp. supra closed) subset of Y.*

Definition 4. *[24]*

(i) *A map $g : (X, \mu) \to (Y, \tau)$ is said to be supra continuous if the inverse image of each open subset of Y is a supra open subset of X.*

(ii) *Let τ be a topology and μ be a supra topology on X. We say that μ is associated supra topology with τ if $\tau \subseteq \mu$.*

Definition 5. *[27] A subset E of (X, μ) is called supra b-open if $E \subseteq int(cl(E)) \bigcup cl(int(E))$ and its complement is called supra b-closed.*

Definition 6. *[27] A map $g : (X, \tau) \to (Y, \theta)$ is said to be:*

(i) *Supra b-continuous if the inverse image of each open subset of Y is a supra b-open subset of X.*

(ii) *Supra b-open (resp. supra b-closed) if the image of each open (resp. closed) subset of X is a supra b-open (resp. supra b-closed) subset of Y.*

Hereafter, we give a concept of supra b-homeomorphism maps.

Definition 7. *A map $g : (X, \tau) \to (Y, \theta)$ is said to be supra b-homeomorphism if it is bijective, supra b-continuous and supra b-open.*

Definition 8. *A map $f : (X, \preceq_1) \to (Y, \preceq_2)$ is called:*

(i) *Order preserving (or increasing) if $a \preceq_1 b$, then $f(a) \preceq_2 f(b)$ for each $a, b \in X$.*

(ii) *Order embedding provided that $a \preceq_1 b$ if and only if $f(a) \preceq_2 f(b)$ for each $a, b \in X$.*

Theorem 1. (i) *If $g : (X, \preceq_1) \to (Y, \preceq_2)$ is an increasing map, then the inverse image of each an increasing (resp. a decreasing) subset of Y is increasing (resp. decreasing).*

(ii) *If $g : (X, \preceq_1) \to (Y, \preceq_2)$ is a decreasing map, then the inverse image of each an increasing (resp. a decreasing) subset of Y is decreasing (resp. increasing).*

Definition 9. *[24, 27] Let E be a subset of a supra topological space (X, μ). Then:*

(i) *Supra interior of E, denoted by $sint(E)$, is the union of all supra open sets contained in E.*

(ii) *Supra closure of E, denoted by $scl(E)$, is the intersection of all supra closed sets containing E.*

(iii) *Supra b-interior of E, denoted by $sbint(E)$, is the union of all supra b-open sets contained in E.*

(iv) *Supra b-closure of E, denoted by $sbcl(E)$, is the intersection of all supra b-closed sets containing E.*

Definition 10. *[25] A topological ordered space $(X, \tau, \preceq)$ is called:*

(i) *Lower T_1-ordered if for each $a, b \in X$ such that $a \npreceq b$, there exists an increasing neighborhood G of a does not contain b.*

(ii) *Upper T_1-ordered if for each $a, b \in X$ such that $a \npreceq b$, there exists a decreasing neighborhood G of b does not contain a.*

(iii) *T_0-ordered if it is lower T_1-ordered or upper T_1-ordered.*

(iv) *T_1-ordered if it is both lower T_1-ordered and upper T_1-ordered.*

(v) *T_2-ordered if for every $a, b \in X$ such that $a \npreceq b$, there exist an increasing neighborhood W_1 of a and a decreasing W_2 of b such that $W_1 \bigcap W_2 = \emptyset$.*

Remark 1. *McCartan [25] named the axioms mentioned in the above definition, strong T_i-ordered spaces instead of T_i-ordered spaces when the word of a neighborhood is replaced by an open set.*

Definition 11. *[20] A supra topological ordered space $(X, \mu, \preceq)$ is called:*

(i) *Lower SST_1-ordered if for each $a, b \in X$ such that $a \npreceq b$, there exists an increasing supra open set G containing a does not contain b.*

(ii) *Upper SST_1-ordered if for each $a, b \in X$ such that $a \npreceq b$, there exists a decreasing supra open set G containing b does not contain a.*

(iii) *SST_0-ordered if it is lower SST_1-ordered or upper SST_1-ordered.*

(iv) *SST_1-ordered if it is both lower SST_1-ordered and upper T_1-ordered.*

(v) *SST_2-ordered if for every $a, b \in X$ such that $a \npreceq b$, there exist a supra open set W_1 containing a and a supra open set W_2 containing b such that $W_1 \bigcap W_2 = \emptyset$.*

3. Supra b-continuous maps

The concepts of I-supra b-continuous, D-supra b-continuous and B-supra b-continuous maps are presented and their main properties are investigated. The relationships among them are illustrated with the help of examples. The conditions under which such these types of supra b-continuous maps preserve some ordered supra b-separation axioms are studied.

Definition 12. *A subset E of $(X, \mu, \preceq)$ is said to be:*

(i) *I-supra (resp. D-supra, B-supra) b-open if it is supra b-open and increasing (resp. decreasing, balancing).*

(ii) *I-supra (resp. D-supra, B-supra) b-closed if it is supra b-closed and increasing (resp. decreasing, balancing).*

Definition 13. *A map $f : (X, \mu, \preceq) \to (Y, \tau)$ is said to be:*

(i) *I-supra (resp. D-supra, B-supra) b-continuous at $p \in X$ if for each open set H containing $f(p)$, there exists an I-supra (resp. a D-supra, a B-supra) b-open set G containing p such that $f(G) \subseteq H$.*

(ii) *I-supra (resp. D-supra, B-supra) b-continuous if it is I-supra (resp. D-supra, B-supra) b-continuous at each point $p \in X$.*

Theorem 2. *A map $f : (X, \mu, \preceq) \to (Y, \tau)$ is I-supra (resp. D-supra, B-supra) b-continuous if and only if the inverse image of each open subset of Y is an I-supra (resp. a D-supra, a B-supra) b-open subset of X.*

Proof. We only prove the theorem in the case of f is an I-supra b-continuous map and the other cases follow similar lines.
To prove the *necessary* part, let G be an open subset of Y, Then we have the following two cases:

(i) $f^{-1}(G) = \emptyset$ which is an I-supra b-open subset of X.

(ii) $f^{-1}(G) \neq \emptyset$. By choosing $p \in X$ such that $p \in f^{-1}(G)$, we obtain $f(p) \in G$. So there exists an I-supra b-open set H_p containing p such that $f(H_p) \subseteq G$. Since p is chosen randomly, then $f^{-1}(G) = \bigcup_{p \in f^{-1}(G)} H_p$. Thus $f^{-1}(G)$ is an I-supra b-open subset of X.

To prove the *sufficient* part, let G be an open subset of Y containing $f(p)$. Then $p \in f^{-1}(G)$. By hypothesis, $f^{-1}(G)$ is an I-supra b-open set. Since $f(f^{-1}(G)) \subseteq G$, then f is an I-supra b-continuous at $p \in X$ and since p is chosen randomly, then f is I-supra b-continuous.

Remark 2. **(i)** *Every I-supra (D-supra, B-supra) b-continuous map is supra b-continuous.*

(ii) *Every B-supra b-continuous map is I-supra (D-supra) b-continuous.*

The following two examples illustrate that the above remark cannot be reversed, in general.

Example 1. *Let a supra topology $\mu = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and a topology $\tau = \{\emptyset, Y, \{x\}\}$ on $X = \{a, b, c, d\}$ and $Y = \{x, y, z\}$, respectively. Let a partial order relation $\preceq = \triangle \bigcup \{(a, b), (b, d), (a, d)\}$ on X and let a map $f : (X, \mu, \preceq) \to (Y, \tau)$ be defined as follows, $f(a) = f(c) = f(d) = x$ and $f(b) = y$. Obviously, f is supra b-continuous. On the other hand, $f^{-1}(\{x\}) = \{a, c, d\}$ is neither a decreasing nor an increasing supra b-open subset of X. Then f is not I-supra (D-supra, B-supra) b-continuous.*

Example 2. *We replace only a partial order relation in Example (1) by $\preceq = \triangle \bigcup \{(b, d)\}$. Then a map f is I-supra b-continuous, but not B-supra b-continuous.*

Definition 14. *Let E be a subset of $(X, \mu, \preceq)$. Then*

(i) $E^{isbo} = \bigcup \{G : G$ *is an I-supra b-open set contained in* $E\}$.

(ii) $E^{dsbo} = \bigcup\{G : G \text{ is a D-supra } b\text{-open set contained in } E\}$.

(iii) $E^{bsbo} = \bigcup\{G : G \text{ is a B-supra } b\text{-open set contained in } E\}$.

(iv) $E^{isbcl} = \bigcap\{H : H \text{ is an I-supra } b\text{-closed set containing } E\}$.

(v) $E^{dsbcl} = \bigcap\{H : H \text{ is a D-supra } b\text{-closed set containing } E\}$.

(vi) $E^{bsbcl} = \bigcap\{H : H \text{ is a B-supra } b\text{-closed set containing } E\}$.

Lemma 1. *Let E be a subset of $(X, \mu, \preceq)$. Then:*

(i) $(E^{dsbcl})^c = (E^c)^{isbo}$.

(ii) $(E^{isbcl})^c = (E^c)^{dsbo}$.

(iii) $(E^{bsbcl})^c = (E^c)^{bsbo}$.

Proof.

(i) $(E^{dsbcl})^c = \{\bigcup F : F$ is a D-supra b-closed set containing $E\}^c = \bigcap\{F^c : F^c$ is an I-supra b-open set contained in $E^c\} = (E^c)^{isbo}$.

The proofs of (ii) and (iii) are similar to that of (i).

Theorem 3. *Let $g : (X, \mu, \preceq) \to (Y, \tau)$ be a map. Then the following five statements are equivalent:*

(i) *g is I-supra b-continuous;*

(ii) *The inverse image of each closed subset of Y is a D-supra b-closed subset of X;*

(iii) $(g^{-1}(H))^{dsbcl} \subseteq g^{-1}(cl(H))$ *for every* $H \subseteq Y$;

(iv) $g(A^{dsbcl}) \subseteq cl(g(A))$ *for every* $A \subseteq X$;

(v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{isbo}$ *for every* $H \subseteq Y$.

Proof. (i) $\Rightarrow$ (ii): Consider H is a closed subset of Y. Then H^c is open. Therefore $g^{-1}(H^c) = (g^{-1}(H))^c$ is an I-supra b-open subset of X. So $g^{-1}(H)$ is D-supra b-closed.
(ii) $\Rightarrow$ (iii): For any subset H of Y, we have $cl(H)$ is closed. Since $g^{-1}(cl(H))$ is a D-supra b-closed subset of X, then $(g^{-1}(H))^{dsbcl} \subseteq (g^{-1}(cl(H))^{dsbcl} = g^{-1}(cl(H))$.
(iii) $\Rightarrow$ (iv): Consider A is a subset of X. Then $A^{dsbcl} \subseteq (g^{-1}(g(A)))^{dsbcl} \subseteq g^{-1}(cl(g(A))$. Therefore $g(A^{dsbcl}) \subseteq g(g^{-1}(cl(g(A))) \subseteq cl(g(A))$.
(iv)$\Rightarrow$ (v): Let H be a subset of Y. By Lemma (1), we obtain $g(X - (g^{-1}(H))^{isbo}) = g(((g^{-1}(H))^c)^{dsbcl})$. By (iv) $g(((g^{-1}(H))^c)^{dsbcl}) \subseteq cl(g(g^{-1}(H))^c) = cl(g(g^{-1}(H^c))) \subseteq cl(Y - H) = Y - int(H)$. Therefore $(X - (g^{-1}(H))^{isbo}) \subseteq g^{-1}(Y - int(H)) = X - g^{-1}(int(H))$. Thus $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{isbo}$.
(v)$\Rightarrow$ (i): Consider H is an open subset of Y. Then $g^{-1}(H) = g^{-1}(int(H)) \subseteq (g^{-1}(H))^{isbo}$. From the fact that $(g^{-1}(H))^{isbo} \subseteq g^{-1}(H)$, we have $g^{-1}(H)$ is an I-supra b-open subset of X. Thus g is I-supra b-continuous.

Theorem 4. *Let $g : (X, \mu, \preceq) \to (Y, \tau)$ be a map. Then the following five statements are equivalent:*

(i) *g is D-supra b-continuous;*

(ii) *The inverse image of each closed subset of Y is an I-supra b-closed subset of X;*

(iii) $(g^{-1}(H))^{isbcl} \subseteq g^{-1}(cl(H))$ *for every* $H \subseteq Y$;

(iv) $g(A^{isbcl}) \subseteq cl(g(A))$ *for every* $A \subseteq X$;

(v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{dsbo}$ *for every* $H \subseteq Y$.

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *Let $g : (X, \mu, \preceq) \to (Y, \tau)$ be a map. Then the following five statements are equivalent:*

(i) *g is B-supra b-continuous;*

(ii) *The inverse image of each closed subset of Y is a B-supra b-closed subset of X;*

(iii) $(g^{-1}(H))^{bsbcl} \subseteq g^{-1}(cl(H))$ *for every* $H \subseteq Y$;

(iv) $g(A^{bsbcl}) \subseteq cl(g(A))$ *for every* $A \subseteq X$;

(v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{bsbo}$ *for every* $H \subseteq Y$.

Proof. The proof is similar to that of Theorem 3.

Definition 15. *A supra topological ordered space $(X, \mu, \preceq)$ is called:*

(i) *Lower strong supra bT_1-ordered (briefly, Lower $SSbT_1$-ordered) if for each $a, b \in X$ such that $a \npreceq b$, there exists an increasing supra b-open set G containing a does not contain b.*

(i) *Upper strong supra bT_1-ordered (briefly, Upper $SSbT_1$-ordered) if for each $a, b \in X$ such that $a \npreceq b$, there exists a decreasing supra b-open set G containing b does not contain a.*

(ii) *$SSbT_0$-ordered if it is lower $SSbT_1$-ordered or upper $SSbT_1$-ordered.*

(iii) *$SSbT_1$-ordered if it is lower $SSbT_1$-ordered and upper $SSbT_1$-ordered.*

(iv) *$SSbT_2$-ordered if for every $a, b \in X$ such that $a \npreceq b$, there exist disjoint supra b-open sets W_1 and W_2 containing a and b, respectively, such that W_1 is increasing and W_2 is decreasing.*

Theorem 6. *Let a bijective map $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be I-supra b-continuous and f^{-1} be an order preserving map. If $(Y, \tau, \preceq_2)$ is lower T_1-ordered, then $(X, \mu, \preceq_1)$ is lower $SSbT_1$-ordered.*

Proof. Let $a, b \in X$ such that $a \npreceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a), y = f(b)$. Since f^{-1} is order preserving, then $x \npreceq_2 y$ and since $(Y, \tau, \preceq_2)$ is lower T_1-ordered, then there exists an increasing neighborhood W of x in Y such that $y \notin W$. Therefore there exists an open set G such that $x \in G \subseteq W$. Since f is I-supra b-continuous, then $a \in f^{-1}(G)$ which is an I-supra b-open set and Since f is bijective, then $b \notin f^{-1}(G)$. Thus $(X, \mu, \preceq_1)$ is lower $SSbT_1$-ordered.

Theorem 7. *Let a bijective map $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be D-supra b-continuous and f^{-1} be an order preserving map. If $(Y, \tau, \preceq_2)$ is upper T_1-ordered, then $(X, \mu, \preceq_1)$ is upper $SSbT_1$-ordered.*

Proof. The proof is similar to that of Theorem (6).

Theorem 8. *Let a bijective map $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be B-supra b-continuous and f^{-1} be an order preserving map. If $(Y, \tau, \preceq_2)$ is T_i-ordered, then $(X, \mu, \preceq_1)$ is $SSbT_1$-ordered for $i = 0, 1, 2$.*

Proof. We only prove the theorem in the case of $i = 2$ and one can prove the theorem in the case of $i = 0, 1$ in a similar way.
Let $a, b \in X$ such that $a \npreceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a)$ and $y = f(b)$. Since f^{-1} is order preserving, then $x \npreceq_2 y$ and since $(Y, \tau, \preceq_2)$ is T_2-ordered, then there are disjoint neighborhoods W_1 and W_2 of x and y, respectively. Therefore there are disjoint open sets G and H containing x and y, respectively. Since f is B-supra b-continuous, then $a \in f^{-1}(G)$ and $b \in f^{-1}(H)$ which are B-supra b-open subsets of X. Obviously, $f^{-1}(G) \bigcap f^{-1}(H) = \emptyset$. Thus $(X, \mu, \preceq_1)$ is an $SSbT_2$-ordered space.

Theorem 9. *Consider a bijective soft map $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ is supra b-continuous such that f is ordered embedding. If $(Y, \tau, \preceq_2)$ is strong T_i-ordered, then $(X, \mu, \preceq_1)$ is $SSbT_i$-ordered for $i = 0, 1, 2$.*

Proof. We only prove the theorem in the case of $i = 2$ and one can prove the theorem in the case of $i = 0, 1$ in a similar way.
Let $a, b \in X$ such that $a \npreceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a)$ and $y = f(b)$. Since f is ordered embedding, then $x \npreceq_2 y$. Since $(Y, \tau, \preceq_2)$ is strong T_2-ordered, then there exist disjoint open sets G and H containing x and y, respectively, such that G is increasing and H is decreasing. Since f is supra b-continuous and order preserving, then $f^{-1}(G)$ is an I-supra b-open set containing a, $f^{-1}(H)$ is a D-supra b-open set containing b and $f^{-1}(G) \bigcap f^{-1}(H) = \emptyset$. Hence $(X, \mu, \preceq_1)$ is $SSbT_2$-ordered.

Theorem 10. *Consider an injective soft map $f : (X, \mu, \preceq) \to (Y, \tau)$ is B-supra b-continuous. If (Y, τ) is T_i-space, then $(X, \mu, \preceq)$ is $SSbT_i$-ordered for $i = 1, 2$.*

Proof. We prove the theorem in case of $i = 2$ and the other case is made similarly. Let $a, b \in X$ such that $a \not\preceq_1 b$. Then there exist $x, y \in Y$ such that $f(a) = x, f(b) = y$ and $x \neq y$. Since (Y, τ) is a T_2-space, then there exist disjoint open sets G and H such that $x \in G$ and $y \in H$. Therefore $a \in f^{-1}(G)$ and $b \in f^{-1}(H)$ which are B-supra b-open subsets of X. Obviously, $f^{-1}(G) \bigcap f^{-1}(H) = \emptyset$. Thus $(X, \mu, \preceq)$ is an $SSbT_2$-ordered space.

4. Supra b-open and supra b-closed maps

In this section, we introduce the concepts of I-supra b-open (I-supra b-closed), D-supra b-open (D-supra b-closed) and B-supra b-open (B-supra b-closed) maps. We demonstrate their main properties and illustrate the relationships among them with the help of examples.

Definition 16. *A map $g : (X, \tau) \to (Y, \mu, \preceq)$ is said to be:*

(i) *I-supra (resp. D-supra, B-supra) b-open if the image of any open subset of X is an I-supra (resp. a D-supra, a B-supra) b-open subset of Y.*

(ii) *I-supra (resp. D-supra, B-supra) b-closed if the image of any closed subset of X is an I-supra (resp. a D-supra, a B-supra) b-closed subset of Y.*

Remark 3. **(i)** *Every I-supra (D-supra, B-supra) b-open map is supra b-open.*

(ii) *Every I-supra (D-supra, B-supra) b-closed map is supra b-closed.*

(iii) *Every B-supra b-open map is I-supra (D-supra) b-open.*

(iv) *Every B-supra b-closed map is I-supra (D-supra) b-closed.*

The following two examples illustrate that the converse of the properties mentioned in the above remark need not be true in general.

Example 3. *Let a topology $\tau = \{\emptyset, X, \{1, 2\}\}$ and a partial order relation $\preceq = \triangle \bigcup \{(1, 3), (3, 2)(1, 2)\}$ on $X = \{1, 2, 3\}$. Let $\mu = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}\}$ be a associated supra topology with τ. The identity map $f : (X, \tau) \to (X, \mu, \preceq)$ is supra b-open and supra b-closed. On the other hand, $f(\{1, 2\}) = \{1, 2\}$ is neither an increasing nor a decreasing supra b-open subset of Y. So that f is not x-supra b-open map for $x \in \{I, D, B\}$. Also, $f(\{3\}) = \{3\}$ is neither an increasing nor a decreasing supra b-closed subset of Y. So that f is not x-supra b-closed map for $x \in \{I, D, B\}$.*

Example 4. *We replace only a partial order relation in Example* (3) *by $\preceq = \triangle \bigcup \{(1, 3), (1, 2)\}$. Then a map f is D-supra b-open, but is not B-supra b-open. Also, it is I-supra b-closed, but is not B-supra b-closed.*

Theorem 11. *The following statements are equivalent, for a map $f : (X, \tau) \to (Y, \mu, \preceq)$:*

(i) *f is I-supra b-open;*

(ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{isbo})$ *for every* $H \subseteq Y$*;*

(iii) $f(int(G)) \subseteq (f(G))^{isbo}$ *for every* $G \subseteq X$*.*

Proof. (i) ⇒ (ii): Since $int(f^{-1}(H))$ is an open subset of X, then $f(int(f^{-1}(H)))$ is an I-supra b-open subset of Y. Obviously, $f(int(f^{-1}(H))) \subseteq f(f^{-1}(H)) \subseteq H$. So $int(f^{-1}(H)) \subseteq f^{-1}(H^{isbo})$.
(ii) ⇒ (iii): Set $H = f(G)$ in (ii). Then $int(f^{-1}(f(G))) \subseteq f^{-1}((f(G))^{isbo})$. Since $int(G) \subseteq f^{-1}((f(G))^{isbo})$, then $f(int(G)) \subseteq (f(G))^{isbo}$.
(iii) ⇒ (i): Let G be an open subset of X. Then $f(int(G)) = f(G) \subseteq (f(G))^{isbo}$. So $f(G)$ is an I-supra b-open set. Thus f is an I-supra b-open map.

The following two results can be proved similarly.

Theorem 12. *The following statements are equivalent, for a map* $f : (X, \tau) \to (Y, \mu, \preceq)$*:*

(i) *f is D-supra b-open;*

(ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{dsbo})$ *for every* $H \subseteq Y$*;*

(iii) $f(int(G)) \subseteq (f(G))^{dsbo}$ *for every* $G \subseteq X$*.*

Theorem 13. *The following statements are equivalent, for a map* $f : (X, \tau) \to (Y, \mu, \preceq)$*:*

(i) *f is B-supra b-open;*

(ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{bsbo})$ *for every* $H \subseteq Y$*;*

(iii) $f(int(G)) \subseteq (f(G))^{bsbo}$ *for every* $G \subseteq X$*.*

Theorem 14. *We have the following results for a map* $f : (X, \tau) \to (Y, \mu, \preceq)$*.*

(i) *f is I-supra b-closed if and only if* $(f(G))^{isbcl} \subseteq f(cl(G))$ *for any* $G \subseteq X$*.*

(ii) *f is D-supra b-closed if and only if* $(f(G))^{dsbcl} \subseteq f(cl(G))$ *for any* $G \subseteq X$*.*

(iii) *f is B-supra b-closed if and only if* $(f(G))^{bsbcl} \subseteq f(cl(G))$ *for any* $G \subseteq X$*.*

Proof. (i) *Necessity*: Consider f is an I-supra b-closed map. Then $f(cl(G))$ is an I-supra b-closed subset of Y. Since $f(G) \subseteq f(cl(G))$, then $(f(G))^{isbcl} \subseteq f(cl(G))$.
Sufficiency: Consider B is a closed subset of X. Then $f(B) \subseteq (f(B))^{isbcl} \subseteq f(cl(B)) = f(B)$. Therefore $f(B) = (f(B))^{isbcl}$. Thus $f(B)$ is an I-supra b-closed set. Hence f is an I-supra b-closed map.
The proofs of (ii) and (iii) are similar to that of (i).

Theorem 15. *Let* $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ *be a bijective map. Then we have the following results.*

(i) *f is I-supra b-open if and only if it is D-supra b-closed.*

(ii) *f is D-supra b-open if and only if it is I-supra b-closed.*

(iii) *f is B-supra b-open if and only if it is B-supra b-closed.*

Proof. (i) *Necessity*: Let f be an I-supra b-open map and let G be a closed subset of X. Then G^c is open. Since f is bijective, then $f(G^c) = (f(G))^c$ is I-supra b-open. Therefore $f(G)$ is a D-supra b-closed subset of Y. Thus f is D-supra b-closed.
Sufficiency: Let f be a D-supra b-closed map and let B be an open subset of X. Then B^c is closed. Since f is bijective, then $f(B^c) = (f(B))^c$ is D-supra b-closed. Therefore $f(B)$ is I-supra b-open. Thus f is I-supra b-closed.
The proofs of (ii) and (iii) are similar to that of (i).

Theorem 16. *The following two statements hold.*

(i) *If the maps $f : (X, \tau) \to (Y, \theta)$ is open and $g : (Y, \theta) \to (Z, \nu, \preceq)$ is I-supra (resp. D-supra, B-supra) b-open, then a map $g \circ f$ is I-supra (resp. D-supra, B-supra) b-open.*

(ii) *If the maps $f : (X, \tau) \to (Y, \theta)$ is closed and $g : (Y, \theta) \to (Z, \nu, \preceq)$ is I-supra (resp. D-supra, B-supra) b-closed, then a map $g \circ f$ is I-supra (resp. D-supra, B-supra) b-closed.*

Proof. The proof is straightforward.

Theorem 17. *If the maps $g \circ f$ is I-supra (resp. D-supra, B-supra) b-open and $f : (X, \tau) \to (Y, \theta)$ is surjective and continuous, then a map $g : (Y, \theta) \to (Z, \nu, \preceq)$ is I-supra (resp. D-supra, B-supra) b-open.*

Proof. Consider f is a continuous map and let G be an open subset of Y. Then $f^{-1}(G)$ is an open subset of X. Since $g \circ f$ is I-supra b-open (resp. D-supra, B-supra) and f is surjective, then $(g \circ f)(f^{-1}(G)) = g(G)$ is an I-supra b-open (resp. a D-supra, a B-supra) subset of Z. Hence g is I-supra b-open (resp. D-supra, B-supra).

Proposition 1. *If the maps $g \circ f : (X, \tau, \preceq_1) \to (Z, \mu, \preceq_3)$ is closed and $g : (Y, \theta, \preceq_2) \to (Z, \mu, \preceq_3)$ is injective and I-supra (resp. D-supra, B-supra) b-continuous, then a map $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is D-supra (resp. I-supra, B-supra) b-closed.*

Proof. Consider $g \circ f$ is a closed map and let G be a closed subset of X. Then $(g \circ f)(G)$ is a closed subset of Z. Since g is injective and I-supra b-continuous, then $g^{-1}(g \circ f) = f(G)$ is a D-supra b-closed subset of Y. Hence f is D-supra b-closed.
A similar proof can be given for the cases between parentheses.

Proposition 2. *We have the following results for a bijective map $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$.*

(i) *f is I-supra (resp. D-supra, B-supra) b-open if and only if f^{-1} is I-supra (resp. D-supra, B-supra) b-continuous.*

(ii) *f is D-supra (resp. I-supra, B-supra) b-closed if and only if f^{-1} is I-supra (resp. D-supra, B-supra) b-continuous.*

Proof.

(i) We prove (i) when f is a B-supra b-open map, and the other cases follow similar lines.
$'\Rightarrow'$ Let f be a B-supra b-open map and let G be an open subset of X. Then $(f^{-1})^{-1}(G) = f(G)$ is a B-supra b-open subset of Y. Therefore f^{-1} is a B-supra b-continuous.
$'\Leftarrow'$ Let G be an open subset of X and f^{-1} be a B-supra b-continuous map. Then $f(G) = (f^{-1})^{-1}(G)$ is a B-supra b-open subset of Y. Therefore f is B-supra b-open.

(ii) Similarly, one can prove (ii).

Theorem 18. *Let a bijective map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ be I-supra b-open (D-supra b-closed) and order preserving. If $(X, \tau, \preceq_1)$ is lower T_1-ordered, then $(Y, \mu, \preceq_2)$ is lower $SSbT_1$-ordered.*

Proof. We prove the theorem when a map f is I-supra b-open.
Let $x, y \in Y$ such that $x \npreceq_2 y$. Since f is bijective, then there exist $a, b \in X$ such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$ and since f is order preserving, then $a \npreceq_1 b$. By hypothesis $(X, \tau, \preceq_1)$ is a lower T_1-ordered space, then there exists an increasing neighborhood W of a in X such that $b \notin W$. Therefore there exists an open set G such that $a \in G \subseteq W$. Thus $x \in f(G)$ which is an I-supra b-open subset of Y. Since f is bijective, then $y \notin f(G)$. Hence $(Y, \mu, \preceq_2)$ is a lower $SSbT_1$-ordered space.
Theorem 4.9 says that a bijective map is I-supra b-open if and only if it is D-supra b-closed. So the result holds for a D-supra b-closed map.

Theorem 19. *Let a bijective map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ be D-supra b-open (I-supra b-closed) and order preserving. If $(X, \tau, \preceq_1)$ is upper T_1-ordered, then $(Y, \mu, \preceq_2)$ is upper $SSbT_1$-ordered.*

Proof. The proof is similar to that of Theorem (18).

Theorem 20. *Let a bijective map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ be B-supra b-open (B-supra b-closed) and order preserving. If $(X, \tau, \preceq_1)$ is T_i-ordered, then $(Y, \mu, \preceq_2)$ is $SSbT_i$-ordered for $i = 0, 1, 2$.*

Proof. We only prove the theorem when a map f is B-supra b-open and in the case of $i = 2$. The other cases can be made similarly.
For all $x, y \in Y$ such that $x \npreceq_2 y$, there are $a, b \in X$ such that $a = f^{-1}(x), b = f^{-1}(y)$. Since f is order preserving, then $a \npreceq_1 b$. Since $(X, \tau, \preceq_1)$ is a T_2-ordered space, then

there exist disjoint neighborhoods W_1 and W_2 of a and b, respectively, such that W_1 is increasing and W_2 is decreasing. Therefore there are disjoint open sets G and H such that $a \in G \subseteq W_1$ and $b \in H \subseteq W_2$. Thus $x \in f(G)$ and $y \in f(H)$ which are B-supra b-open sets. Since f is bijective, then $f(G) \bigcap f(H) = \emptyset$. Hence $(Y, \mu, \preceq_2)$ is an $SSbT_2$-ordered space.

Theorem 21. *Consider a bijective map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ is supra b-open such that f and f^{-1} are order preserving. If $(X, \tau, \preceq_1)$ is strong T_i-ordered, then $(Y, \mu, \preceq_2)$ is $SSbT_i$-ordered for $i = 0, 1, 2$.*

Proof. We prove the theorem in case of $i = 2$. Let $x, y \in Y$ such that $x \not\preceq_2 y$. Then there exist $a, b \in X$ such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$. Since f is order preserving, then $a \not\preceq_1 b$. Since $(X, \tau, \preceq_1)$ is a strong T_2-ordered space, then there exist disjoint an increasing open set G containing a and a decreasing open set H containing b. By hypothesis, f is bijective and supra b-open, and f^{-1} is order preserving, then $f(G)$ is an I-supra b-open set containing x, $f(H)$ is a D-supra b-open set containing y and $f(G) \bigcap f(H) = \emptyset$. Therefore $(Y, \mu, \preceq_2)$ is $SSbT_2$-ordered.
Similarly, one can prove theorem in the cases of $i = 0, 1$.

Theorem 22. *Consider a bijective map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ is supra b-closed such that f and f^{-1} are order preserving. If $(X, \tau, \preceq_1)$ is strong T_i-ordered, then $(Y, \mu, \preceq_2)$ is $SSbT_i$-ordered for $i = 0, 1, 2$.*

Proof. The proof is similar to that of Theorem (21).

5. Supra b-homeomorphism maps

The concepts of I-supra b-homeomorphism, D-supra b-homeomorphism and B-supra b-homeomorphism maps are introduced and many of their properties are established. Some illustrative examples are provided.

Definition 17. *Let $\tau^\star$ and $\theta^\star$ be associated supra topologies with τ and θ, respectively. A bijective map $g : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is called I-supra (resp. D-supra, B-supra) b-homeomorphism if it is I-supra b-continuous and I-supra b-open (resp. D-supra b-continuous and D-supra b-open, B-supra b-continuous and B-supra b-open).*

Remark 4. **(i)** *Every I-supra (D-supra, B-supra) b-homeomorphism map is supra b-homeomorphism.*

(ii) *Every B-supra b-homeomorphism map is I-supra (D-supra) b-homeomorphism.*

The following two examples illustrate that the above remark cannot be reversed, in general.

Example 5. *Let a topology* $\tau = \{\emptyset, X, \{a, c\}\}$ *on* $X = \{a, b, c\}$*, a supra topology associated with* τ *be* $\{\emptyset, X, \{a\}, \{a, c\}\}$ *and a partial order relation* $\preceq_1 = \triangle \bigcup \{(c, a), (c, b)\}$*. Let a topology* $\theta = \{\emptyset, Y, \{y, z\}\}$ *on* $Y = \{x, y, z\}$*, a supra topology associated with* θ *be* $\{\emptyset, Y, \{y\}, \{y, z\}\}$ *and a partial order relation* $\preceq_2 = \triangle \bigcup \{(y, z)\}$ *on* Y*. A map* $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ *is defined as* $f(a) = y$*,* $f(b) = z$ *and* $f(c) = x$*. Now,* f *is supra b-homeomorphism, but is not x-supra b-homeomorphism for* $x \in \{I, D, B\}$*.*

Example 6. *We replace only a partial order relation* $\preceq_1$ *in Example* (5) *by* $\preceq = \triangle \bigcup \{(a, c)\}$*. Then a map* f *is D-supra b-homeomorphism, but not B-supra b-homeomorphism.*

Theorem 23. *Let a map* $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ *be bijective and I-supra b-continuous. Then the following statements are equivalent:*

(i) f *is I-supra b-homeomorphism;*

(ii) f^{-1} *is I-supra b-continuous;*

(iii) f *is D-supra b-closed.*

Proof. (i) ⇒ (ii): Let G be an open subset of X. Then $(f^{-1})^{-1}(G) = f(G)$ is an I-supra b-open set in Y. Therefore f^{-1} is I-supra b-continuous.
(ii) ⇒ (iii): It follows from (ii) of Theorem 4.13.
(iii) ⇒ (i): Let G be an open subset of X. Then G^c is a closed set and $f(G^c) = (f(G))^c$ is D-supra b-closed. Therefore $f(G)$ is an I-supra b-open subset of Y. Thus f is I-supra b-open. Hence f is an I-supra b-homeomorphism map.

The following two results can be proved similarly.

Theorem 24. *Let a map* $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ *be bijective and D-supra b-continuous. Then the following statements are equivalent:*

(i) f *is D-supra b-homeomorphism;*

(ii) f^{-1} *is D-supra b-continuous;*

(iii) f *is I-supra b-closed.*

Theorem 25. *Let a map* $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ *be bijective and B-supra b-continuous. Then the following statements are equivalent:*

(i) f *is B-supra b-homeomorphism;*

(ii) f^{-1} *is B-supra b-continuous;*

(iii) f *is B-supra b-closed.*

Theorem 26. *Let* $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ *be a supra b-homeomorphism map such that* f *and* f^{-1} *are order preserving. If* X *(resp.* Y*) is strong* T_i*-ordered, then* Y *(resp.* X*) is* $SSbT_i$*-ordered for* $i = 0, 1, 2$*.*

Proof.

(i) Let X be a strong T_i-ordered space, then by Theorem 21, Y is an $SSbT_i$-ordered space for $i = 0, 1, 2$.

(ii) Let Y be a strong T_i-ordered space, then by Theorem 8, X is an $SSbT_i$-ordered space for $i = 0, 1, 2$.

6. Conclusion

In the present paper, the concepts of I-supra b-continuous (I-supra b-open, I-supra b-closed, I-supra b-homeomorphism) maps, D-supra b-continuous (D-supra b-open, D-supra b-closed, D-supra b-homeomorphism) maps and B-supra b-continuous (B-supra b-open, B-supra b-closed, B-supra b-homeomorphism) maps are given and studied. The equivalent conditions for the concepts given herein are presented and the relationships among them are showed with the help of illustrative examples. The sufficient conditions for maps to preserve some separation axioms which introduced in [16, 20, 25] are demonstrated. In the next work, we plan to use a notion of somewhere dense sets [4, 14] to define various kinds of maps in topological ordered spaces.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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