EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 12, No. 4, 2019, 1779-1786
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Forcing Subsets for γ_c-sets and γ_t-sets in the Lexicographic Product of Graphs

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Abstract. In this paper, the connected dominating sets and total dominating sets in the lexicographic product of two graphs are characterized. Further, the connected domination, total domination, forcing connected domination and forcing total domination numbers of these graphs are determined.

2010 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Connected domination, total domination, forcing subset, lexicographic product

1. Introduction

Let $G = (V(G), E(G))$ be a connected graph. A set $D \subseteq V(G)$ is a *dominating set* of G if every vertex in $V(G)\backslash D$ is adjacent to at least one vertex in D. A set $S \subseteq V(G)$ is a *total dominating set* (resp. *connected dominating set*) of G if each vertex in $V(G)$ is adjacent to some vertex in S (resp. S is a dominating set and the subgraph $\langle S\rangle$ induced by S is connected in G). The *total domination number* $\gamma_t(G)$ (resp. *connected domination number* $\gamma_c(G)$) of G is the minimum cardinality of a *total dominating set* (resp. *connected dominating set*). If S is a *total dominating set* (resp. *connected dominating set*) with $|S| = \gamma_t(G)$ (resp. $|S| = \gamma_c(G)$), then we call S a *minimum total dominating set* (resp. *minimum connected dominating set*) of G or a γ_t-set (resp. γ_c-set) in G.

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DOI: https://doi.org/10.29020/nybg.ejpam.v12i4.3485

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Let T be a γ_t-set of a graph G. A subset S of T is said to be a *forcing subset for T* if T is the unique γ_t-set containing S. The *forcing total domination number of T* is given by $f\gamma_t(T) = \min\{|S| : S$ is a forcing subset for $T\}$. The *forcing total domination number of G* is given by

$$f\gamma_t(G) = \min\{f\gamma_t(T) : T \text{ is a } \gamma_t\text{-set of } G\}.$$

Let C be a γ_c-set of a graph G. A subset D of C is said to be a *forcing subset for C* if C is the unique γ_c-set containing D. The *forcing connected domination number of C* is given by $f\gamma_c(C) = \min\{|D| : D$ is a forcing subset for $C\}$. The *forcing connected domination number of G* is given by

$$f\gamma_c(G) = \min\{f\gamma_c(C) : C \text{ is a } \gamma_c\text{-set of } G\}.$$

Chartrand et. al [2] initiated the investigation on the relation between forcing and domination concepts in 1997 and used the term "forcing domination number". In 2017, John et. al [3] investigated the forcing connected domination of a graph. In 2018, Canoy et. al [1] investigated the forcing domination number of graphs under some binary operations.

The *lexicographic product (composition)* $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$, and $(u, u')(v, v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

For each $\varnothing \neq C \subseteq V(G) \times V(H)$, the G-projection and H-projection of C are, respectively, the sets $C_G = \{x \in V(G) : (x, a) \in C$ for some $a \in V(H)\}$ and $C_H = \{a \in V(H) : (y, a) \in C$ for some $y \in V(G)\}$. Observe that any non-empty subset C of $V(G) \times V(H)$ can be written as $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x = \{a \in C_H : (x, a) \in C\}$ for all $x \in S$.

2. Total Domination in the Lexicographic Product of Graphs

We shall use the following well-known result.

Lemma 2.1. [1] *Let G be a connected graph and S a dominating set of G. Then $\gamma_t(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|$. In particular, $\gamma_t(G) \leq 2\gamma(G)$.*

Theorem 2.2. *Let G and H be both nontrivial connected graphs. Then $C = \cup_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is a total dominating set of $G[H]$ if and only if either*
(i) S is a total dominating set of G or
(ii) S is a dominating set of G and T_x is a total dominating set of H for every $x \in S\backslash N_G(S)$.

Proof. Suppose that $C = \cup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a total dominating set of $G[H]$. Let $u \in V(G)\backslash S$ and pick any $b \in V(H)$. Since $(u, b) \in V(G[H]) \setminus C$ and C is a dominating set of $G[H]$, there exists $(y, c) \in C$ such that $(y, c)(u, b) \in E(G[H])$. This implies that $y \in S$ and $u \in N_G(y)$. This shows that S is a dominating set of G. If S is a total dominating set of G, then we are done. So suppose S is not a total dominating set of G. Then $S \setminus N_G(S) \neq \emptyset$. Let $x \in S \setminus N_G(S)$. Suppose there exists $y \in V(H) \setminus N_H(T_x)$. Then $yz \notin E(H)$ for all $z \in T_x$. This implies that $(x, y) \notin N_{G[H]}(C)$, contrary to our assumption that C is a total dominating set of $G[H]$. Therefore, $N_H(T_x) = V(H)$, i.e., T_x is a total dominating set of H.

For the converse, let $C = \cup_{x\in S}(\{x\} \times T_x)$ and $(u, t) \in V(G[H])$. Assume first that S is a total dominating set of G. Then there exists $x \in S \setminus \{u\}$ such that $u \in N_G(x)$. Choose $d \in T_x$. Then $(x, d) \in C$ and $(u, t)(x, d) \in E(G[H])$. Hence, $(u, t) \in N_{G[H]}(C)$.

Suppose now that (ii) holds. If $u \in V(G) \setminus S$, then because S is a dominating set of G, there exists $y \in S$ such that $u \in N_G(y)$. Pick $a \in T_y$. Then $(y, a) \in C$ and $(u, t)(y, a) \in E(G[H])$. Suppose that $u \in S$. If $u \in N_G(z)$ for some $z \in S \setminus \{u\}$, then there exists $(z, b) \in C$ such that $(u, t)(z, b) \in E(G[H])$. If $u \notin N_G(z)$ for all $z \in S \setminus \{u\}$, then by assumption, T_u is a total dominating set of H. Since $(u, t) \notin C$, $t \notin T_u$. This implies that there exists $s \in T_u$ such that $ts \in E(H)$. It follows that $(u, s) \in C$ and $(u, t)(u, s) \in E(G[H])$. Thus, $(u, t) \in N_{G[H]}(C)$. In both cases, we have shown that $(u, t) \in N_{G[H]}(C)$. Therefore, $N_{G[H]}(C) = V(G[H])$, i.e., C is a total dominating set of $G[H]$. □

Corollary 2.3. *Let G and H be nontrivial connected graphs with $\gamma_t(H) = 2$. Then $C = \cup_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ $\forall x \in S$, is a γ_t-set of $G[H]$ if and only if either*
(i) S is a γ_t-set of G and $|T_x| = 1$ for all $x \in S$; or
(ii) S is a dominating set of G such that $|S \cap N_G(S)| + 2|S\backslash N_G(S)| = \gamma_t(G)$, $|T_x| = 1$ for all $x \in S \cap N_G(S)$, and T_x is a γ_t-set of H (hence $|T_x| = 2$) for every $x \in S\backslash N_G(S)$.

Proof. Suppose $C = \cup_{x\in S}(\{x\} \times T_x)$ is a γ_t-set of $G[H]$. By Theorem 2.2, S is a total dominating set of G or S is a dominating set of G and T_x is a total dominating set of H for every $x \in S \setminus N_G(S)$. Suppose first that S is total dominating set. Suppose further that that $|T_z| \geq 2$ for some $z \in S$. Let $a \in T_z$ and define $T^*_z = \{a\}$. Then $C^* = [\cup_{x\in S\setminus\{z\}}(\{x\} \times T_x)] \cup (\{z\} \times T^*_z)$ is a total dominating set by Theorem 2.2(i). This, however, is impossible because $|C^*| < |C|$. Thus, $|T_x| = 1$ for all $x \in S$. Thus, (i) holds.

Suppose now that S is a dominating (not a total dominating) set of G. Suppose first that $\gamma_t(G) < |S \cap N_G(S)| + 2|S \setminus N_G(S)| \leq |C|$. Choose a γ_t-set R in G and set $S_x = \{v\}$

for every $x \in R$, where $v \in V(H)$. Then $Y = \cup_{x\in R}(\{x\} \times S_x)$ is a total dominating set by Theorem 2.2(i). It follows that $\gamma_t(G) = |R| = |Y| < |C|$, contrary to our assumption of C. Thus, by Lemma 2.1, $\gamma_t(G) = |S \cap N_G(S)| + 2|S \setminus N_G(S)|$.

Next, suppose that there exists $z \in S \cap N_G(S)$ with $|T_z| \geq 2$. Let $a \in T_z$ and define $T_z^* = \{a\}$. Then $C^* = [\cup_{x\in S\setminus\{z\}}(\{x\} \times T_x)] \cup (\{z\} \times T_z^*)$ is a total dominating set by Theorem 2.2(ii). This is not possible because $|C^*| < |C|$ Therefore $|T_x| = 1$ for all $x \in S \cap N_G(S)$. Finally, suppose there exists $w \in S \setminus N_G(S)$ such that T_w is not a γ_t-set of H. Since T_w is a dominating set and $\gamma_t(H) = 2$, $|T_w| > 2$. Let $L_w = \{a, b\}$ be a γ_t-set of H. Then $C_1 = [\cup_{x\in S\setminus\{w\}}(\{x\} \times T_x)] \cup (\{w\} \times L_w)$ is a total dominating set by Theorem 2.2(ii). Again, this is not possible because $|C_1| < |C|$. Therefore, T_x is a γ_t-set of H for every $x \in S \setminus N_G(S)$.
The converse is easy. □

Corollary 2.4. *Let G and H be nontrivial connected graphs with $\gamma_t(H) \neq 2$. Then a subset $C = \cup_{x\in S}(\{x\} \times T_x)$ of $V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is a γ_t-set of $G[H]$ if and only if S is a γ_t-set of G and $|T_x| = 1$ for all $x \in S$.*

Proof. Suppose $C = \cup_{x\in S}(\{x\} \times T_x)$ is a γ_t-set of $G[H]$. Suppose S is not a total dominating set. Then S is a dominating set of G and T_x is a total dominating set of H for every $x \in S \setminus N_G(S)$, by Theorem 2.2. Since $\gamma_t(H) \neq 2$, it follows that $|T_x| > 2$ for every $x \in S \setminus N_G(S)$. By Lemma 2.1 and since $|C| = \sum_{x\in S\cap N_G(S)} |T_x| + \sum_{x\in S\setminus N_G(S)} |T_x|$, it follows that $\gamma_t(G) < |C|$. Let S_1 be a γ_t-set of G and set $Q_x = \{a\}$ for every $x \in S_1$, where $a \in V(H)$. Put $Q = \cup_{x\in S_1}(\{x\} \times Q_x)$. Then Q is a total dominating set of $G[H]$ by Theorem 2.2(i). Moreover, $|Q| = |S_1| = \gamma_t(G)$. Thus, $|Q| < |C|$, contrary to our assumption of C. Therefore, S is a total dominating set of G. Using a similar argument, it can be shown that S is a γ_t-set of G and $|T_x| = 1$ for all $x \in S$.

For the converse, suppose that $C = \cup_{x\in S}(\{x\} \times T_x)$ and S is a γ_t-set of G with $|T_x| = 1$ for all $x \in S$. By Theorem 2.2, C is a total dominating set of $G[H]$. If $C_1 = \cup_{x\in S_1}(\{x\} \times L_x)$ is another total dominating set of $G[H]$, then, by Theorem 2.2, S_1 is dominating set of G and L_x is a total dominating set of H^x for each $x \in S_1 \setminus N_G(S_1)$. Let $D_1 = S_1 \cap N_G(S_1)$ and $D_2 = S_1 \setminus N_G(S_1)$. By Theorem 2.2,

$$|D_1| + 2|D_2| \leq \sum_{x\in D_1} |L_x| + \sum_{x\in D_2} |L_x| = |C_1|.$$

Thus, by Lemma 2.1, $\gamma_t(G) = |C| \leq |C_1|$. This implies that C is a γ_t-set of $G[H]$. □

Corollary 2.5. *Let G and H be nontrivial connected graphs. Then*

$$\gamma_t(G[H]) = \gamma_t(G).$$

Proof. Let S be a γ_t-set of G. Pick $a \in V(H)$ and set $T_x = \{a\}$ and $C = \cup_{x\in S}(\{x\} \times T_x)$. By Corollary 2.3 and Corollary 2.4, C is γ_t-set of $G[H]$. Thus, $\gamma_t(G[H]) = |C| = |S| = \gamma_t(G)$. □

Theorem 2.6. *Let G and H be nontrivial connected graphs. Then*

$$f\gamma_t(G[H]) = \gamma_t(G).$$

Proof. Let $C = \cup_{x\in S}[\{x\} \times T_x]$ be a γ_t-set of $G[H]$ and let $R_C = \cup_{x\in D}[\{x\} \times R_x]$ be a forcing subset for C. First, suppose that S is a γ_t-set of G. Then $|T_x| = 1$ for all $x \in S$ by Corollaries 2.3 (i) and 2.4. Hence, $R_x = T_x$ for all $x \in D$. If $D \neq S$, say $y \in S\backslash D$, then $R_C \subseteq C^* = \cup_{x\in S}[\{x\} \times T^*_x]$, where $T^*_x = T_x$ for $x \in S\backslash\{y\}$ and T^*_y is a singleton subset of H different from T_y. Since C^* is a γ_t-set of $G[H]$ and $C^* \neq C$, R_C is not a forcing subset for C, contrary to the assumption. Thus, $D = S$, that is, $R_C = C$. Hence, $f\gamma_t(C) = |C| = |S| = \gamma_t(G) = f\gamma_t(G[H])$.

Next, suppose that S is a dominating (not a total dominating) set of G such that $|S\cap N_G(S)|+2|S\backslash N_G(S)| = \gamma_t(G)$. Then $|T_x| = 1$ for all $x \in S\cap N_G(S)$ and T_x is a γ_t-set of H for each $x \in S\backslash N_G(S)$ by Corollary 2.3(ii). (Note that in this case, $\gamma_t(H) = 2$). Let $C = C_1 \cup C_2$ where $C_1 = \cup_{x\in S\cap N_G(S)}[\{x\} \times T_x]$ and $C_2 = \cup_{x\in S\backslash N_G(S)}[\{x\} \times T_x]$. Clearly, $S \cap N_G(S) \subseteq D$, that is, $C_1 \subseteq R_C$. Now, choose $v_y \in N_G(y)$ for each $y \in S \setminus N_G(S)$ and let $F_S = \{v_y : y \in S \setminus N_G(S)\}$. Clearly, $S \cap F_S = \varnothing$. Suppose that $|F_S| < |S \setminus N_G(S)|$. Then there exist distinct $y_1, y_2 \in S \setminus N_G(S)$ such that $v_{y_1} = v_{y_2}$. Let $S_0 = S \cup F_S$. Then

$$|S_0| = |S| + |F_S| < |S \cap N_G(S)| + 2|S\backslash N_G(S)| = \gamma_t(G).$$

This is a contradiction because S_0 is a total dominating set of G. Thus, $|F_S| = |S \setminus N_G(S)|$ (hence, the v_y's are distinct). Next, suppose that there exists $q \in S \setminus N_G(S)$ such that $\{q\} \times T_q$ is not contained in R_C. Let $T_q = \{a, b\}$ and suppose, without loss of generality, that $(q, a) \notin R_C$. Let $S_q = S \cup \{v_q\}$ and set $R_q = \{b\}$, $R_{v_q} = \{a\}$, $R_x = T_x$ for each $x \in S\ \{q\}$, and $C_q = \cup_{x\in S_q}[\{x\} \times R_x]$. Then $S_q \cap N_G(S_q) = [S \cap N_G(S)] \cup \{q, v_q\}$ and $S_q \setminus N_G(S_q) = (S \setminus N_G(S)) \setminus \{q\}$. Hence,

$$|S_q \cap N_G(S_q)| + 2|S_q\backslash N_G(S_q)| = |S \cap N_G(S)| + 2 + 2|S\backslash N_G(S)| - 2 = \gamma_t(G).$$

Thus, C_q is a γ_t-set of $G[H]$ by Corollaries 2.3 and 2.4, $C_q \neq C$, and $R_C \subseteq C_q$. This implies that R_C is not a forcing subset for C, contrary to the assumption that it is. Therefore $C_2 \subseteq R_C$, showing that $R_C = C$. Accordingly, $f\gamma_t(G[H]) = |C| = \gamma_t(G)$. □

3. Connected Domination in the Lexicographic Product of Graphs

Theorem 3.1. *Let G and H be nontrivial connected graphs. Then $C = \cup_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is a connected dominating set of $G[H]$ if and only if S is a connected dominating set of G, where T_x is a connected dominating set of H whenever $|S| = 1$.*

Proof. Suppose that $C = \cup_{x\in S}(\{x\}\times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is a connected dominating set of $G[H]$. Then, clearly, S is a dominating set in G. Let $x, y \in S$, where $x \neq y$ and $xy \notin E(G)$. Let $a \in T_x$ and $b \in T_y$. Then $(x, a),(y, b) \in C$, $(x, a) \neq (y, b)$ and $(x, a)(y, b) \notin E(G[H])$. Since $\langle C\rangle$ is connected, there exists an (x, a)-(y, b) geodesic $[(x_1, a_1),(x_2, a_2), \ldots,(x_k, a_k)]$, where $(x_1, a_1) = (x, a)$, $(x_k, a_k) = (y, b)$, and $(x_i, a_i) \in C$ for all $i \in \{1, 2, \ldots, k\}$ $(k \geq 3)$.

It follows that $[x_1, x_2, \ldots, x_k]$, where $x_1 = x$ and $x_k = y$, is an x-y geodesic and $x_i \in S$ for all $i \in \{1, 2, \ldots, k\}$. This implies that $\langle S \rangle$ is connected. Now, suppose that $|S| = 1$, say $S = \{x\}$. Let $a, b \in T_x$, where $a \neq b$ and $ab \notin E(G)$. Since $(x, a), (x, b) \in C$, $(x, a) \neq (x, b)$ and $(x, a)(x, b) \notin E(G[H])$, there exists an $(x, a) - (x, b)$ geodesic $[(x, a_1), (x, a_2), \ldots, (x, a_k)]$, where $a_1 = a$, $a_k = b$, and $(x, a_i) \in C$ for all $i \in \{1, 2, \ldots, k\}$. It follows that $[a_1, a_2, \ldots, a_k]$ is an a-b geodesic and $a_i \in T_x$ for all $i \in \{1, 2, \ldots, k\}$. Hence, $\langle T_x \rangle$ is connected. Moreover, T_x is a dominating set in H.

For the converse, let $C = \cup_{x \in S}(\{x\} \times T_x)$. Assume that S is a connected dominating set of G, and that T_x is a connected dominating set of H whenever $|S| = 1$. Assume first that $|S| \geq 2$ and let $(z, c) \notin C$. Since $\langle S \rangle$ is connected, there exists $w \in S$ such that $wz \in E(G)$. Let $d \in T_w$. Then $(w, d) \in C$ and $(z, c)(w, d) \in E(G[H])$. Thus, C is a dominating set in $G[H]$. Next, let $(u, s), (v, t) \in C$, where $(u, s) \neq (v, t)$ and $(u, s)(v, t) \notin E(G[H])$. If $u = v$, then we choose $w \in S$ such that $uw \in E(G)$. Let $q \in T_w$. Then $(w, q) \in C$ and $[(u, s), (w, q), (v, t)]$ is a $(u, s) - (v, t)$ geodesic. If $u \neq v$, then there exists a $u - v$ geodesic $[u_1, u_2, \ldots, u_k]$ where $u_1 = u$, $u_k = v$ and $u_i \in S$ for each $i \in \{1, 2, \ldots, k\}$, since $\langle S \rangle$ is connected. Choose $s_i \in T_{u_i}$ for each $i \in \{1, 2, \ldots, k\}$, where $s_1 = s$ and $s_k = t$. Then $[(u_1, s_1), (u_2, s_2), \ldots, (u_k, s_k)]$ is a $(u, s) - (v, t)$ geodesic and $(u_i, s_i) \in C$ for each $i \in \{1, 2, \ldots, k\}$. Thus, $\langle C \rangle$ is connected. It is easy to show that C is a connected dominating set if $S = \{x\}$ is a dominating set and T_x is a connected dominating set in H. □

Corollary 3.2. *Let G and H be nontrivial connected graphs with $\gamma(G) = 1$. Then*

$$\gamma_c(G[H]) = \begin{cases} 1, & \gamma(H) = 1 \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. Let $\{x\}$ be a dominating set in G. If $\gamma(H) = 1$, then choose a dominating set $\{d\}$ in H. Clearly, $C_0 = \{(x, d)\}$ is a connected dominating set of $G[H]$. Hence, $\gamma_c(G[H]) = 1$. Suppose that $\gamma(H) \geq 2$ and let $S = \{x, y\}$ with $xy \in E(G)$. Choose any $a \in V(H)$. Then $C = \{(x, a), (y, a)\}$ is a connected dominating set of $G[H]$ by Theorem 3.1. Since $G[H]$ cannot be dominated by a single vertex, it follows that $\gamma_c(G[H]) = |C| = 2$. □

Corollary 3.3. *Let G and H be nontrivial connected graphs with $\gamma(G) \neq 1$. Then*

$$\gamma_c(G[H]) = \gamma_c(G).$$

Proof. Let S be a minimum connected dominating set in G. Choose any $a \in V(H)$ and set $T_x = \{a\}$ for each $x \in S$. Then $C = \cup_{x \in S}(\{x\} \times T_x)$ is a minimum connected dominating set of $G[H]$ by Theorem 3.1. Therefore, $\gamma_c(G[H]) = |C| = |S| = \gamma_c(G)$. □

Theorem 3.4. *Let G and H be nontrivial connected graphs with $\gamma(G) = 1$ and $\gamma(H) = 1$. Then*

$$f\gamma_c(G[H]) = \begin{cases} 0, & \textit{both } G \textit{ and } H \textit{ have unique } \gamma\textit{-sets,} \\ 1, & \textit{otherwise.} \end{cases}$$

Proof. Suppose that both G and H have unique γ-sets, say S and T, respectively. Then S and T are also γ_c-sets. By Theorem 3.1, $C = \cup_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is the only γ_c-set of $G[H]$, that is, $\varnothing$ is a forcing subset for C. Thus, $f\gamma_c(G[H]) = f\gamma_c(C) = 0$.

Suppose that either G or H has no unique γ-set (γ_c-set). Then by Theorem 3.1, $C = \{(x, y) : x \in S \text{ and } y \in T_x\}$, where S is a γ_c-set of G and T_x is a γ_c-set of H, is not a unique γ_c-set of $G[H]$. By Corollary 3.2, $|C| = 1$, that is, C is a forcing subset for itself. Thus, $f\gamma_c(G[H]) = f\gamma_c(C) = 1$. □

Theorem 3.5. *Let G and H be nontrivial connected graphs with $\gamma(G) = 1$ and $\gamma(H) > 1$. Then*

$$f\gamma_c(G[H]) = 2.$$

Proof. Note that by Corollary 3.2, $\gamma_c(G[H]) = 2$. Let $S = \{x, y\}$ be a γ_c-set of G. Choose any vertex $a \in V(H)$. Then $C = \{(x, a),(y, a)\}$ is a γ_c-set of $G[H]$ by Theorem 3.1. Pick $b \in V(H)\backslash\{a\}$. Then $\{(x, a)\} \subseteq C' = \{(x, a),(y, b)\}$ and $\{(y, a)\} \subseteq C^* = \{(x, b),(y, a)\}$, where C' and C^* are also γ_c-sets of $G[H]$ different from C. Thus, $f\gamma_c(C) = 2 = f\gamma_c(G[H])$. □

Theorem 3.6. *Let G and H be nontrivial connected graphs with $\gamma(G) \neq 1$. Then*

$$f\gamma_c(G[H]) = \gamma_c(G).$$

Proof. Let $C = \cup_{x\in S}[\{x\} \times T_x]$ be a γ_c-set of $G[H]$ and let $P_C = \cup_{x\in D}[\{x\} \times P_x]$ be a forcing subset for C. First, suppose that S is a γ_c-set of G. Then $|T_x| = 1$ for all $x \in S$ by Theorem 3.1 and Corollary 3.3. Hence, $P_x = T_x$ for all $x \in D$. If $D \neq S$, say $y \in S\backslash D$, then $P_C \subseteq C^* = \cup_{x\in S}[\{x\} \times T^*_x]$, where $T^*_x = T_x$ for $x \in S\backslash\{y\}$ and T^*_y is a singleton subset of H different from T_y. Since C^* is a γ_c-set of $G[H]$ and $C^* \neq C$, P_C is not a forcing subset for C, contrary to the assumption. Thus, $D = S$, that is, $P_C = C$. Hence, $f\gamma_c(C) = |C| = |S| = \gamma_c(G) = f\gamma_c(G[H])$. □

Acknowledgements

The authors would like to thank the referees for their invaluable suggestions and comments which greatly contributed in the improvement of the paper. The authors would like to thank also the following funding agencies: Department of Science and Technology-Science Education Institute-Accelerated Science and Technology Human Resource Development Program (DOST-SEI-ASTHRDP), Mindanao State University-Iligan Institute of Technology, Cebu Normal University and Commission on Higher Education, Philippines.

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