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On a System of Linear Singular Partial Differential Equations with Weight Functions

Euler Yoland B. Guerrero

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, MSU-Iligan Institute of Technology, Iligan City, Philippines*

Abstract. Let X be a Banach space, Ω an open bounded subset of X, and Y a complex Banach space. We consider a Volevič system of singular linear partial differential equations of the form

$$t\frac{\partial u_i}{\partial t} = \sum_{j=1}^{N} a_{ij}(t,x)u_j(t,x) + \sum_{(j,k)\in\mathcal{N}(i)} b_{jk}(t,x)((\mu_0(t)D)^k u_j(t,x)\cdot x_k^{(k)})_{(j,k)} + g_i(t,x), \qquad (1)$$

$1 \le i \le N$, in the unknown function $u = (u_1, u_2, ..., u_N) \in Y^N$ of $t \ge 0$ and $x \in \Omega$, where $a_{ij}, b_{jk} \in \mathbb{C}$, $x_k = (x, ..., x)$ (x is k times) D denotes the Frechet differentiation with respect to x, and

$$\mathcal{N}(i) = \{(j,k) : j \text{ and } k \text{ are integers}, 1 \le j \le N, 0 < k \le n(i,j)\}, \qquad (2)$$

$n(i,j) = n(i) - n(j) + 1$, where $n(i)$, $i = 1, 2, ..., N$, are nonnegative integers. The map μ_0 belongs to $C^0([0,T],\mathbb{C})$. We express growth estimates in terms of weight functions and we establish an existence and uniqueness theorem for our system in the class of ultradifferentiable maps with respect to the space variable x.

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1. Introduction

The study of partial differential equations have been a very fruitful endeavor both in pure and applied mathematics. Its practical use cannot be underestimated as many recent scientific and engineering works such as in [8] uses partial differential equations to model real-world problems.

Gerard and Tahara [2], and Baouendi and Goulaouic [1] were some of the authors who worked on nonlinear or linear differential equations with singularity. Lope [5], extended

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Email addresses: `euleryoland.guerrero@g.msuiit.edu.ph` (EY Guerrero)

the work of Baouendi and Galaouic using the concept of weight functions. These weight functions are used to describe growth estimates on the coefficients of the partial Taylor expansion of a function.

In [3], Koike considered a Volevič system of singular nonlinear partial differential equations with general singularity. He established the existence and uniqueness theorem of the solution in the ultradifferentiable class using the Banach fixed point theorem and Nirenberg-Nishida [6, 7] iteration method. This method was also used in [4].

In this paper, we will establish an existence and uniqueness theorem on (1) in the ultradifferentiable class with growth estimates in terms of weight functions.

2. Preliminaries

We first give the definition of a weight function as defined by Tahara [9]. We then give the definitions and basic results about ultradifferentiable maps as proved by Koike [3].

Definition 1. *Let $T > 0$. we say that $\mu(t)$ is a weight function on $[0, T]$ if it is continuous, nonnegative, increasing function on $(0, T]$ such that*

$$\int_0^T \frac{\mu(t)}{t} dt < +\infty.$$

Let V and W be Banach spaces, and U be an open subset of V. We denote by $C^0(V, W)$ the set of all continuous mappings from V to W and $L(V, W)$ the Banach space of all bounded (continuous) linear mappings from V to W. Moreover, we let $L^p(U, W)$ to be the space of all p-linear continuous mappings of U^p into W.

Definition 2. *Let M_j, $j = 0, 1, ...,$ be a sequence of positive numbers with*

$$M_0 = M_1 = 1.$$

A map $v \in C^\infty(\Omega, Y)$ is said to belong to the ultradifferentiable class $\{M_p\}(\Omega, Y)$ (or $\{M_p\}$ for short) if

$$\|D^j v(x)\| \leq C^{1+j} M_j$$

$x \in \Omega$, $j = 0, 1, 2, ...,$ and constant C.

As was done in Koike's paper, in our problem, we impose on the sequence $\{M_p\}$ the following conditions:

(C1) If $\sum_{i=1}^{n} k_i = n$, $k_i \geq 0$, $n = 1, 2, ...,$ then $\prod_{i=1}^{n} N_{k_i+1} \leq N_{n+1}$, where $N_p = \dfrac{M_p}{p!}$.

(C2) There is a constant K such that $M_{j+1} \leq K(j + 1)M_j$, $j = 0, 1, 2, ...$

For $s > 0$, we write

$$\|u\|_s = \|u\|_s(U) = \sup_{x\in U} \sum_{j=0}^{\infty} \frac{\|D^j u(x)\| s^j}{M_j},$$

$$\|u\|'_s = \|u\|'_s(U) = \sup_{x\in U} \sum_{j=1}^{\infty} \frac{\|D^j u(x)\| s^j}{M_j},$$

and

$$B_s(U,V) = \{u \in C^\infty(U,V) : \|u\|_s(U) < \infty\},$$

where V is a subset of a Banach space.

Remark 1. *It is not difficult to show that $u \in \{M_p\}(U,V)$ if and only if $u \in B_s(U,V)$ for some $s > 0$.*

Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be Banach spaces, U an open subset of $\mathcal{X}$, and V an open subset of $\mathcal{Y}$. The next theorem states the multiplication-closedness of the $\{M_p\}$ class.

Theorem 1. *Let $G \in C^\infty(U, L^m(\mathcal{Y},\mathcal{Z}))$, $u_i \in C^\infty(U,\mathcal{Y})$, $i = 1,2,...,m(m = 1,2,...)$. Then*

$$\|Gu_1,...,u_m\|_{s/H}(U) \leq C_1^m \|G\|_s(U) \prod_{i=1}^{m} \|u_i\|_s(U),$$

where $(Gu_1,...,u_m)(x) = G(x)u_1(x),....,u_m(x)$ and $C_1 = max\{\frac{1}{N_2},1\}$.

Theorem 2. *Let $f \in C^\infty(V,\mathcal{Z})$ and $u \in C^\infty(U,V)$. If $\|u\|s'(U) \leq R$ for some $s > 0$ and $R > 0$, then*

$$\|f \circ u\|_{s/H}(U) \leq \|f\|_R(V).$$

Corollary 1. *Let $f \in C^\infty(V,\mathcal{Z})$, and $u, v \in C^\infty(U,V)$. Then*

$$\|f \circ u - f \circ v\|_{s/H^2}(U) \leq C_1 \|Df\|_R(V) \|u - v\|_{s/H}(U)$$

if $\|u\|'_s(U) \leq R$ and $\|v\|'_s(U) \leq R$.

Theorem 3. *Assume (C2). Then there exists $K_n > 0$ such that*

$$\|D^n u\|_r \leq K_n (s-r)^{-n} \|u\|_s \tag{3}$$

$0 < r < s \leq s_1$, where K_n is independent of u, r and s.

Remark 2. *The preceding theorem implies that if $u \in \{M_p\}$, then $Du \in \{M_p\}$.*

We will now give our assumptions for (1). Let Y be a complex Banach space and $L^k(X,Y)$ the Banach space of all bounded multi-k-linear maps from X^k to Y, while $L^0(X,Y)$ denotes Y. Let Ω be an open subset of X and U_i a neighborhood of the origin in the Banach space $\{(\xi_{jk})_{(j,k)\in\mathcal{N}(i)} : \xi_{jk} \in L^k(X,Y)\}$, where $\mathcal{N}(i)$ is the set defined in (2). Let

$$f_i(u,w)(t,x) = \sum_{j=1}^{N} a_{ij}(t,x)u_j(t,x) + \sum_{(j,k)\in\mathcal{N}(i)} b_{jk}(t,x)((\mu_0(t)D)^k u_j(t,x)\cdot x_k^{(k)})_{(j,k)}.$$

We work on (1) under the following assumptions:

(A_1) μ_0 belong to $C^0([0,T],\mathbb{C})$ for a $T>0$ and $f_i \in C^0([0,T], B_{s_1}(\Omega\times U_i, Y))$, for some $s_1 > 0$.

(A_2) $f_i(0,0)(0,x) = 0$, for all $x\in\Omega$, $1\le i\le N$

(A_3) The spectrum of the $N\times N$ matrix $A(x) = (A_{ij}(x)) \in L(Y^N)$, where

$$A_{ij} = -D_{u_j} f_i(u,w)(0,x)|_{(u,w)=(0,0)}$$

is contained in the half plane$\{z\in\mathbb{C} : \mathrm{Re} z > b_0\}$ for a positive number b_0.

(A_4) For some $\kappa\in(0,1)$,

$$\int_0^T \frac{(\mu(t))^\kappa}{t}\,dt < \infty,$$

where $\mu(t) = \sup_{0\le\tau\le t} |\mu_0(\tau)|$.

Condition (A_1) states that f_i is continuous in t and ultradifferentiable in the other variables.

The next results are proved in [3] assuming (C1), (C2), and (A_1)-(A_4).

Let κ be the number as in (A_2) and

$$c = \max_{1\le i\le N}\{n(i)\} + 1 + \frac{\kappa}{1-\kappa} \qquad d = \max_{1\le i,j\le N}\{n(i,j)\}. \tag{4}$$

Then $c\ge d+1$ and $d\ge 1$. The function ω in the following lemma plays an important role.

Lemma 1. *There exists a function $\omega\in C^0([0,T],\mathbb{R})\cap C^1((0,T],\mathbb{R})$ such that $\omega(0)=0$,*

$$\omega(t)^c\omega'(t) \ge \frac{\mu_(t)^\kappa}{t} \tag{5}$$

and

$$\omega(t)^c \ge \mu(t)^\kappa \tag{6}$$

for $t\in(0,T]$.

Now put

$$\rho(i,j) = max\{n(j,i), 1\}$$

$$\nu(\tau,t) = \ln\left(\frac{t}{\tau}\right)$$

and

$$E(\tau,t)(x) = (E_{ij}(\tau,t)(x)) = \exp\left[\ln\frac{\tau}{t}A(x)\right] \in L(Y^N)$$

for $(\tau, t) \in \Delta$, where $A(x) = (A_{ij}(x))$ is the matrix operaton as in (A_3) and

$$\Delta = \{(\tau,t) : 0 = \tau < t \leq T \text{ or } 0 < \tau \leq t \leq T\}.$$

Lemma 2. *There exists a positive number b such that for every $x_0 \in \Omega$ there are positive numbers $s_0 (s_0 < s_1)$, C_0 and an open neighborhood $U \subset \Omega$ of x_0 such that $E \in C^0(\Delta, B_{s_0}(U, L(Y^N)))$,*

$$\|E(\tau,t)\|_{s_0}(U) \leq C_0\left(\frac{\tau}{t}\right)^b \tag{7}$$

and

$$\|E_{ij}(\tau,t)\|_{s_0}(U) \leq C_0 e_{\rho(i,j)}(\tau,t), \tag{8}$$

where

$$e_{\rho(i,j)}(\tau,t) = \left(\frac{\tau}{t}\right)^b \frac{\nu(\tau,t)^{\rho(i,j)-1}}{(\rho(i,j)-1)!}.$$

Note that 0 does not belong to the spectrum of $A(x)$, thus the map $\overline{A} : x \to A(x)^{-1}$ is well-defined and ultradifferentiable with respect to x, that is, we can assume that $\overline{A} \in B_{s_0}(\Omega, L(Y^N))$, for some $s_0 > 0$.

Lemma 3. *Let $s \in (0, s_0], \delta \in (0,T]$ and $v \in C^0([0,\delta), B_s(U, Y^N))$. Then $u(0) = \overline{A}v(0)$ and*

$$u(t) = \int_0^t \frac{1}{\tau} E(\tau,t) v(\tau) d\tau$$

for $t \in (0,\delta)$, if and only if $u \in C^0([0,\delta), B_s(U, Y^N))$ and

$$t\frac{\partial u}{\partial t}(t) + Au(t) = v(t)$$

for $t \in (0,\delta)$.

We write, for $t > 0$,

$$\mathcal{H}[h](t) = \int_0^t \frac{\tau^{b-1}}{t^b} h(\tau) d\tau.$$

Note that $\mathcal{H}[h](0) = h(0)/b$. We may assume $b \leq 1$ without loss of generality. Note that $\mathcal{H}[1](t) = 1/b$.

Lemma 4. *Let $\delta \in (0, T]$, $a > 0$, $\beta \geq 0$and $\gamma \geq 1$, and let $m = 0$ or $m = 1$. If $\alpha \geq \kappa m$, $\omega(t) < a$ and $h(t) \leq \mu(t)^\alpha \omega(t)^\beta (1 - \omega(t)/a))^{-\gamma}$ for $t \in [0, \delta)$, then*

$$\mathcal{H}[h](t) \leq C_\gamma a^m \mu(t)^{\alpha - \kappa m} \omega(t)^{\beta + cm} \left(1 - \frac{\omega(t)}{a}\right)^{-Max\{1, \gamma - m\}}$$

for $t \in [0, \delta)$, where $C_\gamma = \max\left\{\frac{1}{\gamma - 1}, \frac{1}{b}\right\}$ if $\gamma > 1$ and $C_\gamma = \frac{1}{b}$ if $\gamma = 1$.

Lemma 5. *Let $h \in C^0([0, \delta), \mathbb{R})$, $\delta \in (0, T]$. Then it holds that*

$$\int_0^t \frac{1}{\tau} e_p(\tau, t) h(\tau) d\tau = \mathcal{H}^p[h](t)$$

for $t \in (0, \delta)$ and $p = 1, 2, \ldots$.

3. Existence and Uniqueness Theorem

We first state our main theorem and then prove the existence and uniqueness parts in two sections.

Theorem 4 (Main Theorem)**.** *Let $C1, C2, and A_1 - A_4$ hold and $\alpha \in (0, 1]$. For every $x_0 \in \Omega$, there exists a positive number R small enough and a neighborhood $U \subset \Omega$ such that if the map $g_i : t \to (x \mapsto g_i(t, x))$ belongs to $C^0([0, T], B_{s_1}(\Omega, Y)$ for some $s_1 > 0$ with*

$$\|g_i(t, x)\|_{s_1}(\Omega) \leq CR\mu(t)^\alpha, \qquad (t, x) \in [0, T] \times \Omega, \qquad 1 \leq i \leq N$$

for some constant $C > 0$, then (1) has a unique solution $u = (u_1, ..., u_N)$ in $[0, T_0) \times U$ for a positive number $T_0 \leq T$ and a neighborhood $U \subset \Omega$ of x_0, satisfying

$$u_j \in C^0([0, T_0), B_s(U, Y)) \cap C^1((0, T_0), B_s(U, Y)), \qquad 1 \leq j \leq N$$

and

$$\|u_j(t, x)\|_s(U) \leq R\mu(t)^\alpha \quad \text{and} \quad \|((\mu_0 D)^k u_j(t, x))_{(j,k) \in \mathcal{N}(i)}\|_s(U) \leq R\mu(t)^\alpha,$$

for all $t \in [0, \delta)$ and some $s > 0$.

3.1. Existence

Let $\alpha \in [0,1]$, $\mu(t)$ be a weight function, and $x_0 \in \Omega$. Let U be the set obtained by Lemma 2. For brevity, we abbreviate $\|\cdot\|_s(U)$ to $\|\cdot\|_s$, and (t,x) to (t) if t is the only variable needed in our analysis. We let $w_{jk}(t,x) = ((\mu_0 D)^k u_j(t,x))_{(j,k)}$ then

$$f_i(u,w)(t,x) = \sum_{j=1}^{N} a_{ij}(t)u_j(t,x) + \sum_{(j,k)\in\mathcal{N}(i)} b_{jk}w_{(j,k)}(t,x),$$

and write

$$F_i(u,w)(t,x) = f_i(t,x,u_j(t,x),w_{jk}(t,x))$$

for $u = (u_j)_{1\le j\le N}$ and $w = (w_{jk})_{(j,k)\in\mathcal{N}(i)}$, where the values of u_j and w_{jk} belong to Y and $L^k(X,Y)$, respectively. Further, we set $F = (F_i)_{1\le i\le N}$ and

$$\Psi(u,w)(t) = \int_0^t \frac{E(\tau,t)}{\tau}(F(u,w)(\tau) + Au(\tau) + g(\tau))d\tau.$$

We need to show that for a fixed w, the operator $\Psi(\cdot,w)$ is a contraction mapping from a function space to itself. Let $u = (u_1,...,u_N)$ and W_T be the set

$$W_T = \{u \in C^0([0,T),(B_s(U,Y))^N) : \|u(t)\|_s \le C\mu(t)^\alpha \text{ for some } C > 0\}.$$

For a $u \in W_T$ we define the norm $\|u\|_W$ as

$$\|u(t)\|_W = \max_{1\le j\le N} \|u_j(t)\|_s.$$

Then $(W_T, \|\cdot\|_W)$ is a Banach space. For $R > 0$, we set

$$W_{T,R} = \{u \in W_T : \|u\|_W \le R\mu(t)^\alpha\}.$$

This is a closed subset of W_T and so it is a complete metric space. $W_{T,R}$ will be the form of our function space. We note that if $u \in W_{T,R}$, then $\|u(t)\|_s \le R\mu(t)^\alpha$.

Similarly, we define $W'_{T,R}$ by just replacing $(B_s(U,Y))^N$ in our definition of W_T by $B_s(U,L^k(X,Y))$.

Let $C_2 = \sup_{0\le t\le T} \|D_w f_i(t)\|_s \left(\Omega \times Y^N \times \prod_{(j,k)\in\mathcal{N}(i),k>0} L^k(X,Y)\right)$. This is finite by (A_1) and Remark 2.10. Further, we let $C' = N^2C_1^2C_0C_2$, where C_1 and C_0 are the constants in Theorem 2.6 and Lemma 2.12, respectively.

Set $r_0 = \min\{b^d/C', 1\}$ and b is the positive constant obtained in Lemma 2.

Proposition 1. *There exists $T_0 \in (0,T]$ and $R < s_1$ such that if*

$$\|g_i(t)\|_s(\Omega) \le \frac{br^2(1-r)}{C_0}R\mu(t)^\alpha, \qquad t \in [0,T], \qquad \|x_k^{(k)}\| \le 1$$

and fixed $w \in W'_{T_0,R}$ *with*

$$\|w_{jk}(t)\|_s \leq r_0 r R \mu(t)^\alpha, \qquad t \in [0, T_0), \tag{9}$$

then the following are true:

(a) $\Psi[\cdot, w]$ *is a mapping from* $W_{T_0,R}$ *to itself.*

(b) $\Psi[\cdot, w]$ *is a contraction map.*

Proof. By Remark 2 and (A_1), $D_u f_i \in \{M_p\}$. Hence, f_i is continuous with respect to $t, u,$ and w. Thus, we can find $T_0 \in [0, T]$ and $R < s_1$ such that if $u, v \in W_{T_0,R}$ and $w, \overline{w} \in W'_{T_0,R}$, then

$$N^2 C_1^2 C_0 \|D_u f_i(\tau, P, Q) - D_u f_i(0,0,0)\|_s \quad \leq \quad r b^d \tag{10}$$

where $P = \theta u + (1-\theta)v$, $Q = \theta w + (1-\theta)\overline{w}$. Now, since $F_i(0,0,0) = 0$

$$\begin{aligned} F_i(u,w)(t) &= F_i(u,v)(t) - F_i(0,0)(t) \\ &= \sum_{j=1}^{N} \int_0^1 D_u f_i(t, \theta u, \theta w) u_j(t) d\theta + \sum_{(j,k)\in\mathcal{N}(i)} \int_0^1 D_w f_i(t, \theta u, \theta w) w_{jk}(t) \cdot x_k^{(k)} d\theta. \end{aligned}$$

Using the definition of A we may rewrite $A_{ij} u_j(t)$ as

$$A_{ij} u_j(t) = -\int_0^1 D_u f_i(0,0,0) \cdot u_j(t) d\theta.$$

Hence,

$$\begin{aligned} F_i(u,w)(t) + \sum_{j=1}^{N} A_{ij} u_j(t) + g_i(t) &= \sum_{j=1}^{N} \int_0^1 [D_u f_i(t, \theta u, \theta w) - D_u f_i(0,0,0)] u_j(t) d\theta \\ &\quad + \sum_{(j,k)\in\mathcal{N}(i)} \int_0^1 D_w f_i(t, \theta u, \theta w) w_{jk}(t) \cdot x_k^{(k)} d\theta + g_i(t). \end{aligned}$$

Thus,

$$\begin{aligned} \left\| F_i(u,w)(t) + \sum_{j=1}^{N} A_{ij} u_j(t) + g_i(t) \right\|_s &\leq \sum_{j=1}^{N} C_1 \|D_u f_i(t, \theta u, \theta w) - D_u f_i(0,0,0)]\|_s \|u_j(t)\|_s \\ &\quad + \sum_{(j,k)\in\mathcal{N}(i)} C_1 \|D_w f_i(t, \theta u, \theta w)\|_s \|w_{jk}(t) \cdot x_k^{(k)}\|_s \\ &\quad + \|g_i(t)\|_s. \end{aligned}$$

Using Lemma 2, we have

$$\|\Psi_i(u,w)(t)\|_s \quad \leq \quad \int_0^t \left\| E(\tau, t) \left(F_i(u,w)(\tau) + \sum_{j=1}^{N} A_{ij} u_j(\tau) + g_i(\tau) \right) \right\|_s \frac{d\tau}{\tau}.$$

$$
\begin{aligned}
&\leq \int_0^t \Bigg\{ C_0 \frac{\tau^{b-1}}{t^b} \Bigg(\sum_{j=1}^{N} C_1 \| D_u f_i(t,\theta u,\theta w) - D_u f_i(0,0,0)] \|_s \| u_j(\tau)\|_s \\
&\quad + \sum_{(j,k)\in\mathcal{N}(i)} C_1 \| D_w f_i(t,\theta u,\theta w)\|_s \| w_{jk}(\tau)\cdot x_k^{(k)}\|_s + \|g_i(\tau)\|_s \Bigg) \Bigg\} d\tau \\
&= \Bigg(\sum_{j=1}^{N} C_0 C_1 \| D_u f_i(t,\theta u,\theta w) - D_u f_i(0,0,0)] \|_s \| u_j(t)\|_s \\
&\quad + \sum_{(j,k)\in\mathcal{N}(i)} C_0 C_1 \| D_w f_i(t,\theta u,\theta w)\|_s \| w_{jk}(t)\|_s \| x_k^{(k)}\|_s + C_0\|g_i(t)\|_s \Bigg) \frac{1}{b}.
\end{aligned}
$$

Note also that $b^d < b \leq 1$ and $d > 1$. Thus, by our assumptions, (10) and our defined constant C',

$$
\begin{aligned}
\|\Psi_i(u,w)(t)\|_s &\leq \Bigg(N C_0 C_1 \| D_u f_i(t,\theta u,\theta w) - D_u f_i(0,0,0)] \|_s \max_{1\leq j\leq N} \|u_j(t)\|_s \\
&\quad + N C_0 C_1 \| D_w f_i(t,\theta u,\theta w)\|_s \max_{(j,k)\in N(i)} \|w_{jk}(t)\|_s + C_0 \|g_i(t)\|_s \Bigg) \frac{1}{b} \\
&\leq \Bigg(rb \max_{1\leq j\leq N} \|u_j(t)\|_s + C' \max_{(j,k)\in N(i)} \|w_{jk}(t)\|_s + C_0 \frac{br^2(1-r)}{C_0} R\mu(t)^\alpha \Bigg) \frac{1}{b} \\
&\leq r \max_{1\leq j\leq N} \|u_j(t)\|_s + \frac{C'}{b} \max_{(j,k)\in N(i)} \|w_{jk}(t)\|_s + r^2(1-r)R\mu(t)^\alpha \\
&\leq \sup_{0\leq \tau\leq t} \Bigg\{ r \max_{1\leq j\leq N} \|u_j(t)\|_s + \frac{C'}{b} \max_{(j,k)\in N(i)} \|w_{jk}(t)\|_s \Bigg\} + r^2(1-r)R\mu(t)^\alpha.
\end{aligned}
$$

Thus, using the definition of r_0 and with $r \leq \frac{1}{3}$, we have

$$
\begin{aligned}
\|\Psi_i(u,w)(t)\| &\leq rR\mu(t)^\alpha + \frac{C'}{b} r r_0 R\mu(t)^\alpha + r^2(1-r)R\mu(t)^\alpha \\
&\leq \frac{1}{3}R\mu(t)^\alpha + \frac{C'}{b}\cdot\frac{b^d}{C'}\frac{1}{3}R\mu(t)^\alpha + \frac{1}{3}R\mu(t)^\alpha \\
&\leq \left(\frac{1}{3} + \frac{C'}{b}\cdot\frac{b^d}{3C'} + \frac{1}{3} \right) R\mu(t)^\alpha \\
&= R\mu(t)^\alpha
\end{aligned}
$$

proving (a). Furthermore, note that

$$
\begin{aligned}
F_j(u,w)(t) &- F_j(v,\overline{w})(t) \\
&= \sum_{k=1}^{N} \int_0^1 \Big[D_u f_j(t,P,Q)\cdot(u_k - v_k)(t) + D_w f_j(t,P,Q)(w_{k\eta} - \overline{w}_k \eta) \Big] d\theta.
\end{aligned}
$$

Hence, similar to the previous approach,

$$\begin{aligned} F_j(u,w)(t) &- F_j(v,\overline{w})(t) + \sum_{k=1}^{N} A_{jk}(u_k(t)-v_k(t)) \\ &= \sum_{k=1}^{N}\int_0^1 \Big[D_u f_j(t,P,Q) - D_u f_j(0,0,0)](u_k-v_k)(\tau)]d\theta \\ &\quad + \sum_{(k,\eta)} D_w f_j(t,P,Q)(w_{k\eta}-\overline{w}_{k\eta})(\tau)d\theta \end{aligned}$$

Thus, by Lemma 2 and (10) we have

$$\begin{aligned} \Bigg\| \sum_{j=1}^{N} E_{ij} &\Big[F_j(u,w)(t) - F_j(v,\overline{w})(t) + \sum_{k=1}^{N} A_{jk}(u_k(t)-v_k(t))\Big]\Bigg\|_s \\ &\le \sum_{j=1}^{N} C_0 e_{\rho(i,j)}(\tau,t)\Bigg[\sum_{k=1}^{N} C_1\|D_u f_j(t,P,Q) - D_u f_j(0,0,0)\|_s\|(u_k-v_k)(t)\|_s \\ &\quad + \sum_{(k,\eta)} C_1\|D_w f_j(t,P,Q)\|_s\|(w_{k\eta}-\overline{w}_k\eta)(t)\|_s\Bigg] \\ &\le N\max_{1\le j\le N} e_{\rho(i,j)}(\tau,t)\Bigg[NC_1C_0\|D_u f_j(t,P,Q) - D_u f_j(0,0,0)\|_s \\ &\quad \times \max_{1\le k\le N}\|(u_k-v_k)(t)\|_s + NC_1C_0\|D_w f_j(t,P,Q)\|_s \\ &\quad \times \max_{(k,\eta)\in\mathcal{N}(i)}\|(w_{k\eta}-\overline{w}_k\eta)(t)\|_s\Bigg] \\ &\le rb^d \max_{1\le j,k\le N} e_{\rho(i,j)}(\tau,t)\|(u_k-v_k)(t)\|_s + C' \max_{\substack{1\le j\le N \\ (k,\eta)\in\mathcal{N}(i)}}\Bigg[e_{\rho(i,j)}(\tau,t) \\ &\quad \times\|(w_{k\eta}-\overline{w}_k\eta)(t)\|_s\Bigg] \end{aligned}$$

By Lemma 5,

$$\begin{aligned} \|\Psi_i(u,w)(t) - \Psi_i(v,\overline{w})(t)\| &\le rb^d \max_{1\le j,k\le N}\mathcal{H}^{\rho(i,j)}[\|u_k-v_k\|_s](t) \\ &\quad + C' \max_{\substack{1\le j\le N \\ (k,\eta)\in\mathcal{N}(i)}}\mathcal{H}^{\rho(i,j)}[\|w_{l\eta}-\overline{w}_{l\eta}\|_s](t). \end{aligned} \tag{11}$$

Hence, when $w=\overline{w}$, we have

$$\|\Psi_i(u,w)(t) - \Psi_i(v,\overline{w})\| \le rb^d \max_{1\le j\le N}\mathcal{H}^{\rho(i,j)}[\|u_k-v_k\|_s](t),$$

proving (b).

It follows from the Banach fixed point theorem that there exists a unique $u \in W_{T_0,R}$ such that $u = \Psi(u, w)$. Denote this u by $S[w]$. We have by (11),

$$\begin{aligned}\|S_i[w](t) - S_i[\overline{w}(t)]\|_s\| &= \|\Psi_i(S[w], w)(t) - \Psi_i(S[\overline{w}], \overline{w})(t)\| \\ &\leq rb^d \max_{1\leq j\leq N} \mathcal{H}^{\rho(i,j)}[\|S_k[w] - S_k[\overline{w}]\|_s](t) \\ &\quad + C' \max_{(k,\eta)\in\mathcal{N}(i)} \mathcal{H}^{\rho(i,j)}[\|w_{k\eta} - \overline{w}_{k\eta}\|_s](t).\end{aligned}$$

Using (11) n-times, we get

$$\begin{aligned}\|S_i[w](t) - S_i[\overline{w}(t)]\|_s\| &\leq r^{n+1}b^d \max_{1\leq j\leq N} \sup_{0\leq\tau\leq t} \mathcal{H}^{\rho(i,j)}[\|S_k[w] - S_k[\overline{w}]\|_s](\tau) \\ &\quad + \sum_{p=0}^{n} r^p C' \max_{(k,\eta)\in\mathcal{N}(i)} \sup_{0\leq\tau\leq t} \mathcal{H}^{\rho(i,j)}[\|w_{k\eta} - \overline{w}_{k\eta}\|_s](\tau).\end{aligned}$$

As $n \to \infty$, we have the following Proposition:

Proposition 2. *For $w, \overline{w} \in W_{T_0,R}$ satisfying (9), we have*

$$\|S_i[w](t) - S_i[\overline{w}(t)]\|_s\| \leq C \max_{(k,\eta)\in\mathcal{N}(i)} \sup_{0\leq\tau\leq t} \mathcal{H}^{\rho(i,j)}[\|w_{k\eta} - \overline{w}_{k\eta}\|_s](\tau), \tag{12}$$

where $C = C'/(1 - r)$.

From (11), when $w = 0$ and $u \in W_{T_0,R}$, we have

$$\begin{aligned}\|S_i[0](t)\|_s &= \|\Psi_i(S[0], 0)(t)\|_s \\ &\leq r\|S_i[0](t)\|_s + r^2(1 - r)R\mu(t)^\alpha.\end{aligned}$$

Hence, since $r \in (0, 1)$, we have

$$\begin{aligned}(1 - r)\|S_i[0](t)\|_s &\leq r^2(1 - r)R\mu(t)^\alpha \\ \|S_i[0]\|_s &\leq r^2R\mu(t)^\alpha.\end{aligned} \tag{13}$$

To solve the equation $u = S[((\mu_0 D)^\eta u_k)_{(k,\eta)\in\mathcal{M}}]$ we use the method of Nirenberg-Nishida. We define $u_n = (u_{n,1}, u_{n,2} ..., u_{n,N})$, $n = 0, 1, ...,$ recursively by

$$u_0 = 0, \qquad u_{n+1} = S[((\mu_0 D)^\eta u_{n,k})_{(k,\eta)\in\mathcal{N}(i)}] \qquad (n = 0, 1, ...).$$

we write $v_n = u_{n+1} - u_n$. Let $a_0 \in (0, 1)$ be a small number to be determined later and

$$a_n = a_0 \prod_{j=1}^{n} (1 + j^{-2})^{-1}.$$

Then, $\{a_n\}_{n\geq 0}$ is a decreasing sequence of positive numbers tending to a positive limit a_∞. Observe that

$$a_\infty = a_0 \prod_{j=1}^{\infty}(1+j^{-2})^{-1} = a_0\left(\prod_{j=1}^{\infty}(1+j^{-2})\right)^{-1}.$$

Since $\sum_{j=1}^{\infty} j^{-2}$ is convergent, a_∞ is convergent.

Corresponding to each a_n, we have the t-interval

$$I_n(s) = \{t \geq 0 : \omega(t) < a_n(s_0 - s)\} \qquad (0 < s < s_0),$$

and

$$\sigma_{n,s}(t) = \left(1 - \frac{\omega(t)}{a_n(s_0 - s)}\right)^{-1}.$$

Note that for all n, $\sigma_{n,s}(t) \geq 1$ and $I_{n+1}(s) \subset I_n(s)$. Let $a_0 s_0 \leq w(T_0)$. Then $I_0(s) \subset [0, T_0)$. Put $s(t) = (s_0 + s - \frac{\omega(t)}{a_n})/2$. Then, for $0 < s < s(t) < s_0$, we have the following remark.

Remark 3. *If $t \in I_n(s)$, then*

(1) $t \in I_n(s(t))$

(2) $\sigma_{n,s(t)} \leq 2\sigma_{n,s}(t)$

(3) $(s(t) - s)^{-\eta} = 2^\eta (s_0 - s)^{-\eta}\sigma_{n,s}(t)^\eta$

(4) $1 \leq \sigma_{n,s}(t) \leq (n+1)^2 + 1.$

(5) $(s_0 - s)^{-\eta} \leq \dfrac{a_0\omega(t)^{c\eta}}{\omega(t)^\eta \mu(t)^{\kappa\eta}}$

We now prove the following proposition. Proving it means proving the convergence of our solution $u(t,x) = \lim_{n\to\infty} u_n(t,x)$, for $x \in U$ and

$$t \in I_\infty(s) = \{t \geq 0 : \omega(t) < a_0(s_0 - s)\} \qquad (0 < s < s_0).$$

Proposition 3. *Let $v_{n,i} = u_{n+1,i} - u_{n,i}$. For $n \geq 0$ the following hold:*

(a) $u_{n+1,i} := S_i[(\mu_0 D)^\eta u_{n,k}]_{(k,\eta)\in\mathcal{N}(i)}$ *exists on* $I_n(s) \times U_i$.

(b) For $t \in I_n(s)$,

$$\|v_{n,i}(t)\|_s \leq R r^{n+2} \mu(t)^{(1-\kappa)n} \omega(t)^n \sigma_{n,s}(t)^{dn} \mu(t)^\alpha.$$

(c) For $t \in I_n(s)$,

$$\|(\mu_0(t)D)^\eta v_{n,i}(t)\|_s \le Rr^{n+2}2^{dn+\eta}K_\eta(s_0-s)^{-\eta}\mu(t)^{(1-\kappa)n+\eta}\omega(t)^n\sigma_{n,s}(t)^{dn+\eta}\mu(t)^\alpha.$$

implying that for $t \in I_{n+1}$,

$$\|(\mu_0(t)D)^\eta v_{n,i}(t)\|_s \le Rr^{n+2}2^{dn+\eta}K_\eta a_0\mu(t)^{(1-\kappa)n+(1-\kappa)\eta}\omega^{n+(c-1)\eta}\sigma_{n,s}(t)^{dn+\eta}\mu(t)^\alpha$$

and thus,

$$\|(\mu_0(t)D)^\eta u_{n+1,i}\|_s \le R\mu(t)^\alpha.$$

Proof. Since $u_{0,i}=0$, Proposition 1 assures us that $u_{1,i}=S_i[0]$ exists for $t \in I_0(s)$. By (13),

$$\begin{aligned}\|v_{0,i}(t)\|_s &= \|u_{1,i}(t)-u_{0,i}(t)\|_s\\ &= \|u_{1,i}(t)\|_s\\ &= \|S_i[0](t)\|_s\\ &\le Rr^2\mu(t)^\alpha.\end{aligned}$$

By (3) and Remark 3 (3) we have

$$\begin{aligned}\|(\mu_0D)^\eta v_{0,i}(t)\|_s &\le \mu(t)^\eta K_\eta(s(t)-s)^{-\eta}\|v_{0,i}(t)\|_{s(t)}\\ &\le \mu(t)^\eta K_\eta 2^\eta(s_0-s)^{-\eta}\sigma_{0,s}(t)^\eta Rr^2\mu(t)^\alpha\\ &= Rr^2 2^\eta K_\eta(s_0-s)^{-\eta}\mu(t)^\eta\sigma_{0,s}^\eta\mu(t)^\alpha.\end{aligned}$$

Hence, by Remark 3 (5), for $t \in I_1(s)$, we have

$$\begin{aligned}\|(\mu_0D)^\eta v_{0,i}(t)\|_s &= \|(\mu_0D)^\eta u_{1,i}(t)\|\\ &\le Rr^2 2^\eta K_\eta(s_0-s)^{-\eta}\mu(t)^\eta\sigma_{0,s}^\eta\mu(t)^\alpha\\ &\le Rr^2 2^\eta K_\eta\frac{a_0^\eta\omega(t)^{c\eta}}{\omega(t)^\eta\mu(t)^{\kappa\eta}}\mu(t)^\eta\sigma_{0,s}^\eta\mu(t)^\alpha\\ &\le Rr^2 2^\eta K_\eta a_0\mu(t)^{(1-\kappa)\eta}\omega(t)^{(c-1)\eta}\sigma_{0,s}^\eta\mu(t)^\alpha\\ &\le R\mu(t)^\alpha,\end{aligned}$$

provided a_0 is small enough.

Suppose (a)-(c) hold for $n=0,1,...,p$ with $n \le l$. Proposition 1 and (c) imply that $u_{p+2,i}=S[(\mu_0D)^\eta u_{p+1,k}]$ exists for $t \in I_{p+1}(s)$, showing (a) for $n=p+1$. Now, for $t \in I_{p+1}(s)$ and Proposition 2,

$$\begin{aligned}\|v_{p+1,i}(t)\|_s &= \|S_i[((\mu_0D)^\eta u_{p+1,k})](t)-S_i[((\mu_0D)^\eta u_{p,k})](t)\|_s\\ &\le C\max_{(k,\eta)\in\mathcal{N}(i)}\sup_{0\le\tau\le t}\mathcal{H}^{\rho(i.j)}[\|((\mu_0D)^\eta v_{p,k})\|_s](\tau).\end{aligned}$$

Using Lemma 4 $\rho(i,j)$-times, we have by Proposition 2 and (c) that

$$\begin{aligned} \|v_{p+1,i}(t)\|_s &\leq \max_{(k,\eta)\in\mathcal{N}(i)} \overset{(i,j)}{\min_m} hC(\gamma)(s_0-s)^{-\eta}(a_{p+1}(s_0-s))^m \mu(t)^{(1-\kappa)p+\eta-\kappa m} \\ &\quad \times\omega(t)^{p+cm}\sigma_{p+1,s}(t)^{\max\{1,dp+\eta-m\}}\mu(t)^{\alpha}, \end{aligned}$$

where

$$\overset{(i,j)}{\min_m} = \min_{0\leq m\leq \min\{\rho(i,j),\frac{\alpha+\eta}{\kappa}\}}$$

m an integer, and $C(\gamma)$ depends only on γ. Thus, since $w(t) < a_0(s_0-s)$ and

$$a_{p+1}(s_0-s) < 1,$$

we have

$$\begin{aligned} \|v_{p+1,i}(t)\|_s &\leq \max_{(k,\eta)\in\mathcal{N}(i)} \overset{(i,j)}{\min_m} hC(\gamma)a_0\mu(t)^{(1-\kappa)p+\eta-\kappa m} \\ &\quad \times\omega(t)^{p+cm-\eta}\sigma_{p+1,s}(t)^{\max\{1,dp+\eta-m\}}\mu(t)^{\alpha}. \end{aligned}$$

If $m=1$, then

$$\|v_{p+1,i}(t)\|_s \leq Rr^{p+3}\mu(t)^{(1-\kappa)p+1-\kappa}\omega(t)^{p+1}\sigma_{p+1,s}(t)^{d(p+1)}\mu(t)^{\alpha},$$

where $h = Rr^2$, $C(\gamma)a_0 \leq r^{p+1}$, since $\sigma_{p+1,s} \geq 1$ and $dp \leq d(p+1)$. Thus,

$$\|v_{p+1,i}(t)\|_s \leq Rr^{q+3}\mu(t)^{(1-\kappa)(p+1)}\omega(t)^{p+1}\sigma_{p+1,s}(t)^{d(p+1)}\mu(t)^{\alpha}.$$

Hence, by (3) we have

$$\begin{aligned} \|(\mu(t)D)^{\eta}v_{p+1,i}\|_s &\leq \mu(t)^{\eta}K_{\eta}(s(t)-s)^{-\eta}\|v_{p+1,i}\|_{s(t)} \\ &\leq Rr^{p+3}K_{\eta}2^{d(p+1)+\eta}a_0\mu(t)^{(1-\kappa)(p+1)+(1-\kappa)\eta} \\ &\quad \times\omega(t)^{(p+1)+(c-1)\eta}\sigma_{p+1,s}(t)^{d(p+1)+\eta}\mu(t)^{\alpha}. \end{aligned}$$

Then, by (3) and Remark 3.1.3 , we have for $t\in I_p(s)$ $(n<p\leq l)$

$$\begin{aligned} \|(\mu_0 D)^{\eta}u_{p,i}(t)\|_s &= \left\|\sum_{n=0}^{p-1}(\mu_0 D)^{\eta}v_{n,i}\right\|_s \\ &\leq \sum_{n=0}^{p-1} Rr^{n+2}K_n 2^{dn+\eta}a_0\mu(t)^{(1-k)n+(1-k)\eta}\omega(t)^{n+(c-1)\eta}\sigma_{n,s}(t)^{dn+\eta}\mu(t)^{\alpha}. \end{aligned}$$

Thus, by Remark 3.1.3(4),

$$\|(\mu_0 D)^{\eta}u_{p,i}(t)\|_s \leq Rr^2 a_0 2^{dl+d}((l+1)^2+1)^{dl+d}K_{\eta}\mu(t)^{\alpha}\sum_{n=0}^{p-1} r^n$$

$$\leq \quad R\mu(t)^{\alpha},$$

provided r and a_0 are small enough.

Now let l be an arbitrary integer satisfying $l \geq \frac{cd+c}{1-\kappa}$. We prove by induction on (p,q) $(p \geq 0,\ 0 \leq q \leq c)$ that the estimation

$$\|v_{l+pc+q,i}(t)\|_s \leq Rr^{l+pc+q+2} \max_{G,\phi}^{(q,i)} \min_{0\leq L\leq G} \mu(t)^{\alpha(\phi,L)}\omega(t)^{\beta(\phi,L)}\sigma_{l+pc+q,s}(t)^{\gamma(\phi,L)}\mu(t)^{\alpha} \quad (14)$$

holds for $t \in I_{l+pc+q}(s)$, where

$$\alpha(\phi,L) = cd+\phi-\kappa L, \qquad \beta(\phi,L) = cd+c-\phi+L, \qquad \gamma(\phi,L) = ld+\phi-L,$$

and

$$\max_{G,\phi}^{(q,i)} = \max_{\substack{q\leq G\leq qd,\\ q\leq\phi\leq n(i)+G}},$$

with G, L denoting integers and ϕ a real number. When (p,q)=(0,0),

$$\|v_{l,i}(t)\|_s \leq Rr^{l+2}\mu(t)^{cd}\omega(t)^{cd+c}\sigma_{l,s}(t)^{ld}\mu(t)^{\alpha},$$

where $G = 0 = L$ and $\phi = 0$. Thus,

$$\|v_{l,i}(t)\|_s \leq Rr^{1+2}\mu(t)^{(1-\kappa)l}\omega(t)^{l}\sigma_{l,s}(t)^{ld}\mu(t)^{\alpha},$$

since $l(1-\kappa) \geq cd+c \geq cd$ and $l \geq \frac{cd+c}{1-k} \geq cd+c$. Hence, (b) shows that (14) holds. Assume that (14) holds for some (p,q) with $q < c$. If

$$a_0 \max\{C(\gamma(\phi,L)) : 0 \leq \phi \leq cd+c, 0 \leq L \leq cd\} \leq r,$$

then, applying Lemma 4 $\rho(i,j)$-times, we have

$$\begin{aligned}\|v_{l+pc+q+1,i}(t)\|_s \quad \leq \quad & Rr^{l+pc+q+3} \max_{\substack{(k,\eta)\in\mathcal{M}(j),\\ 1\leq j\leq N}} \max_{G,\phi}^{(q,k)} \min_{m}^{(i,j)} \min_{0\leq L\leq G} \mu(t)^{\alpha(\phi+\eta,L+m)} \\ & \times\omega(t)^{\beta(\phi+\eta,L+m)}\sigma_{1+pc+q,s}(t)^{\gamma(\phi+\eta,L+m)}\mu(t)^{\alpha},\end{aligned}$$

and

$$\max_{G,\phi}^{(q,i)} = \max_{\substack{q+1\leq G+\rho(i,j)\leq qd+d,\\ q+1\leq\phi+\eta\leq n(i)+(G+\rho(i,j))}},$$

which implies (14) for $(p,q+1)$, since the conditions $(k,\eta) \in \mathcal{M}(j)$ and $m \leq \rho(i,j)$ yield that $q+1 \leq \phi+\eta$,

$$\begin{aligned}\phi+\eta \quad &\leq \quad (n(k)+G)+n(j,k) \\ &= \quad n(k)+G+n(j)-n(k)+1 \\ &= \quad G+n(j)+1\end{aligned}$$

$$\begin{aligned} &= n(i) + G + n(j) - n(i) + 1 \\ &= n(i) + G + n(j, i) \\ &\leq n(i) + (G + \rho(i, j)), \end{aligned}$$

$$L + m \leq G + \rho(i, j)$$

and

$$\begin{aligned} q + 1 &\leq G + \rho(i, j) \\ &\leq qd + d. \end{aligned}$$

Now, assume that (14) holds for (p, c) (i.e., $c = q$) with some p. Then

$$0 \leq c - n(i) \leq \phi - n(i) \leq G,$$

so we can put

$$L = \phi - n(i)$$

$(-[-z])$ is the smallest integer which is not less than z. We have then

$$\begin{aligned} \alpha(\phi, L) &\geq cd + \phi - \kappa(\phi + 1 - n(i)) \\ &= cd + n(i) + (1 - \kappa)\left(\phi - n(i) - \frac{\kappa}{1 - \kappa}\right) \\ &\geq \alpha(n(i), 0) + (1 - \kappa)\left(c - n(i) - \frac{\kappa}{1 - \kappa}\right) \\ &\geq \alpha(n(i), 0), \end{aligned}$$

$$\begin{aligned} \beta(\phi, L) &\geq cd + c - \phi + (\phi - n(i)) \\ &= \beta(n(i), 0) \end{aligned}$$

and

$$\gamma(\phi, L) \leq \gamma(n(i), 0).$$

Therefore, (14) holds for $(p+1, 0)$. This completes the proof of (14). Note that $\alpha(\phi, L) \geq 0$, $\beta(\phi, L) \geq 0$ and $\gamma(\phi, L)$ is bounded (indeed $\gamma(\phi, L) \leq ld + cd + c$). Hence, for $n \geq l$,

$$\begin{aligned} \|(\mu_0 D)^\eta u_{n+1,i}(t)\|_s &= \left\| \sum_{x=0}^{n} (\mu_0 D)^\eta v_{x,i}(t) \right\|_s \\ &\leq \sum_{x=0}^{n} \mu(t)^\eta \|D^\eta v_{x,i}\|_s \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{x=0}^{n} \mu(t)^{\eta} K_{\eta}(s(t)-s)^{-\eta}\|v_{x,i}(t)\|_{s(t)}\mu(t)^{\alpha} \\
&\leq \sum_{x=0}^{n} K_{\eta}2^{\eta}a_0\mu(t)^{(1-\kappa)\eta}\omega(t)^{(c-1)\eta}\sigma_{x,s}(t)^{\eta} \\
&\quad \times Rr^{x+2}\max_{G,\phi}^{(q,i)}\min_{0\leq L\leq G}\mu(t)^{\alpha(\phi,L)}\omega(t)^{\beta(\phi,L)}\sigma_{x,s(t)}(t)^{\gamma(\phi,L)}\mu(t)^{\alpha} \\
&\leq \sum_{x=0}^{n} K_{\eta}2^{\eta}a_0\mu(t)^{(1-\kappa)\eta}\omega(t)^{(c-1)\eta}\sigma_{x,s}(t)^{\eta} \\
&\quad \times Rr^{x+2}\max_{G,\phi}^{(q,i)}\min_{0\leq L\leq G}\mu(t)^{\alpha(\phi,L)}\omega(t)^{\beta(\phi,L)}2^{\gamma(\phi,L)}\sigma_{x,s}(t)^{\gamma(\phi,L)}\mu(t)^{\alpha}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(\mu_0 D)^{\eta}u_i^{n+1}(t)\|_s &\leq Rr^2a_0K_{\eta}2^{d+ld+\phi}((x+1)^2+1)^{d+ld+\phi}\sum_{x=0}^{n} r^x \\
&\leq R\mu(t)^{\alpha}.
\end{aligned}$$

Therefore, we have shown the well-definedness of $u^{n+1}(t)$ for $n \geq l$ and $u^n(t)$ converges to $u(t) \in B_s$ uniformly in $I(s) = \{t \geq 0 : \omega(t) < \lim_{n\to\infty} a_n(s_0 - s)\}$. This $u \in C^0(I(s), B_s)$ is the solution of (1).

3.2. Uniqueness of the Solution

The next proposition implies the uniqueness of our solution, but the proof is similar to that of Proposition 3 and so we omit it here.

Proposition 4. *Suppose $u_n = (u_{n,i})_{1\leq n\leq N}$ and $v_n = (v_{n,i})_{1\leq n\leq N}$ are two solutions of (1) in*

$$C^0(I_{\infty}, (B_s(U,Y))^N)$$

with estimate

$$\{\|u_{n,i}\|_s, \|v_{n,i}\|_s\} \leq R\mu(t)^{\alpha}$$

for all $t \in I_{\infty}(s)$. Then, for $t \in I_{\infty}(s)$, $n = 0, 1, 2, \dots,$ we have

$$\|(u_{n,i} - v_{n,i})(t)\|_s \leq 2Rr^{n+2}\mu(t)^{(1-\kappa)n}\omega(t)^n\sigma_{n,s}(t)^{dn}\mu(t)^{\alpha}$$

and

$$\begin{aligned}
\|(\mu_0(t)D)^{\eta}(u_{n,i} - v_{n,i})(t)\|_s &\leq Rr^{n+2}2^{dn+\eta+1}K_{\eta}a_0\mu(t)^{(1-\kappa)n+(1-\kappa)\eta} \\
&\quad \times\omega(t)^{n(c-1)\eta}\sigma_{n,s}(t)^{dn+\eta}\mu(t)^{\alpha}.
\end{aligned}$$

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