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Functions on n-generalized Topological Spaces

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Abstract. An n-generalized topological (n-GT) space is a pair $(X, \mathscr{G})$ of a nonempty set X and a collection $\mathscr{G}$ of n ($n \in \mathbb{N}$) distinct generalized topologies (in the sense of A. Császár [1]) on the set X. In this paper, we look into $\mathscr{G}$-continuous maps, $\mathscr{G}$-open and $\mathscr{G}$-closed maps, as well as $\mathscr{G}$-homeomorphisms in terms of n-GT spaces and establish some of their basic properties and relationships. Moreover, these notions are also examined with respect to the component generalized topologies of the underlying spaces by defining and characterizing pairwise versions of the said types of mappings.

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1. Introduction

Open sets play as a fundamental notion that underlies almost any other topological concept. When A. Császár [1] in 2002 weakened the conditions that are to be satisfied by an open set, the landscape of topological spaces widened significantly. In [1], he defined a *generalized topological space* (briefly, GT space) (X, μ) as a pair of a nonempty set X and an associated family μ of subsets of X satisfying only the conditions that $\varnothing \in \mu$ and an arbitrary union of sets in μ belongs to μ. Naturally, the elements of μ are termed *μ-open sets*. In the same space, the *μ-closure* $c_\mu(A)$ of a subset A of X is also defined as the intersection of all μ-closed sets containing A while the *μ-interior* $i_\mu(A)$ of A is the union of all μ-open sets contained in A [1]. Other basic properties of a GT space were cited in [6] and [5].

Following this generalization, the idea of utilizing two or more GTs to form a new type of topological space were deeply explored in many succeeding studies. Some of these results are seen in [5], [8], [9], [10], [12], [13], [14], [15], [16], [17], and [19]. In particular,

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an *n-generalized topological space* (briefly, n-GT space) is defined in [2] as a pair $(X, \mathscr{G})$, where X is a nonempty set and $\mathscr{G} = \{\mu_1, \ldots, \mu_n\}$ is a finite collection of n GTs on X, for some $n \in \mathbb{N}$. In this space, the *$\mathscr{G}$-closure of* $A \subseteq X$, denoted by $c_{\mathscr{G}}(A)$, is the intersection of all sets F containing A where $F = X \setminus G$ for some $G \in \cup_{j=1}^{n} \mu_j$ while its *$\mathscr{G}$-interior*, denoted by $i_{\mathscr{G}}(A)$, is the union of all sets G contained in A for some $G \in \cup_{j=1}^{n} \mu_j$. Consequently, $A \subseteq X$ is said to be *$\mathscr{G}$-closed* if $c_{\mathscr{G}}(A) = A$ and is *$\mathscr{G}$-open* if $i_{\mathscr{G}}(A) = A$. These mentioned sets in an n-GT space naturally satisfy basic properties analogous to that of a GT. Most relevant in this paper are that the arbitrary union $\mathscr{G}$-open sets is also $\mathscr{G}$-open and the arbitrary intersection of $\mathscr{G}$-closed sets is also $\mathscr{G}$-closed. Also, each μ_j-open set is $\mathscr{G}$-open for all $j = 1, \ldots, n$ so that for any $\mathscr{G}$-open set A, we can always find a j and a μ_j-open set U_j such that $U_j \subseteq A$. Additionally, by definition of $\mathscr{G}$-interior and $\mathscr{G}$-closure, for each $A \subseteq X$, we have $i_{\mathscr{G}}(A) \subseteq A$ and $A \subseteq c_{\mathscr{G}}(A)$. On a finer note, it is observed that $i_{\mathscr{G}}(A) = \bigcup_{k=1}^{n} i_{\mu_k}(A)$ and $c_{\mathscr{G}}(A) = \bigcap_{k=1}^{n} c_{\mu_k}(A)$.

Functions that arise from these new spaces were also studied in detail. They were modified to fit the definition of open sets in each of these domains. Among the popular types of functions tackled are the continuous maps, open and closed maps, as well as homeomorphisms (see [3], [4], [6] and [11], [18]). This paper intends to examine new variations of continuity, open maps and homeomorphisms in terms of n-GT spaces and to inquire into the idea of localizing these mappings to the component GTs of the underlying spaces.

2. $\mathscr{G}$-continuous Functions

We first define and establish the existence of $\mathscr{G}$-continuous maps:

Definition 2.1. *Let $(X, \mathscr{G}_X)$ and $(Y, \mathscr{G}_Y)$ be m-GT and n-GT spaces, respectively, where $\mathscr{G}_X = \{\mu_1, \ldots, \mu_m\}$ and $\mathscr{G}_Y = \{\nu_1, \ldots, \nu_n\}$ for some $m, n \in \mathbb{N}$. A function $f : X \to Y$ is said to be* $\mathscr{G}$-continuous at a point $x \in X$ *if for each $\mathscr{G}_Y$-open set V containing $f(x)$, there exists a $\mathscr{G}_X$-open set U containing x such that $f(U) \subseteq V$. The function $f : X \to Y$ is* $\mathscr{G}$-continuous *if it is continuous at all points $x \in X$.*

Example 2.2. Let $m, n \in \mathbb{N}$ where $m \leq n$ and consider the m-GT space $(X, \mathscr{G}_X)$ where $X = [0, m] \times \mathbb{R}_0^+$ and $\mathscr{G}_X = \{\mu_1, \ldots, \mu_m\}$ such that $\mu_j = \{\varnothing\} \cup \{R_s^j : s \geq 0\}$ with $R_s^j = \{(x, y) : j - 1 \leq x \leq j, y \geq s\}$, and the n-GT space $(Y, \mathscr{G}_Y)$ where $Y = [0, n] \subseteq \mathbb{R}$ and $\mathscr{G}_Y = \{\nu_1, \ldots, \nu_n\}$ such that $\nu_k = \{\varnothing\} \cup \{[0, t] : t = 1, \ldots, k\}$. Define $f : X \to Y$ such that $(a, b) \mapsto a$. For each $(a, b) \in X$, there is a j_0 such that $j_0 - 1 < a \leq j_0$ and, therefore, $(a, b) \in R_0^{j_0}$. Now, the $\mathscr{G}_Y$-open sets containing $f(a, b) = a$ are precisely the intervals $[0, t]$, where $j_0 \leq t \leq n$. Now, for each t, $f(R_0^{j_0}) = [j_0 - 1, j_0] \subseteq [0, t]$. This means that f is $\mathscr{G}$-continuous at (a, b) and because $(a, b) \in X$ is arbitrary, f is therefore $\mathscr{G}$-continuous.

This type of mapping also manifests for vertex and arc sets of a directed graph as in the following:

Example 2.3. Let D be a directed graph with no loops and multiple edges and with finite vertex set $V(D)$ and arc set $E(D)$ and let $P_1, \ldots, P_n$ be the distinct maximal nontrivial paths in D. For each $j = 1, \ldots, n$, let μ_j be a family of subsets of $V(D)$ containing precisely the empty set and all subsets of $V(D)$ whose elements induce a union of subpaths of the maximal path P_j. Then each μ_j is a GT on $V(D)$ and consequently, if $\mathscr{G} = \{\mu_1, \ldots, \mu_n\}$, then $(V(D), \mathscr{G})$ is an n-GT space. Moreover, for each $k = 1, \ldots, n$, let E_k be the collection of arcs in the maximal path P_k and $\nu_k = \mathscr{P}(E_k)$ be the power set of E_k. Clearly, each ν_k is a GT on $E(D)$ and so if we put $\mathscr{G}_E = \{\nu_1, \ldots, \nu_n\}$, then $(E(D), \mathscr{G}_E)$ is also an n-GT space. Define $\varphi : E(D) \to V(D)$ such that for $e = uv \in E(G)$, $\varphi(e) = u$. If U is a $\mathscr{G}_V$-open set containing $\varphi(e) = u$, then we can find a k and a $G_k \subseteq U$ such that $u \in G_k \in \mu_k$. Now, the vertices in G_k form a union of subpaths of the maximal path P_k and with arcs from E_k. Let E_k^* be the collection of arcs formed by the vertices in G_k. By the definition of $\mathscr{G}_E$, $E_k^* \in \nu_k$ and so it is $\mathscr{G}_E$-open, with $\varphi(E_k^*) \subseteq G_k \subseteq U$. This implies that φ is $\mathscr{G}$-continuous on $e \in E(D)$. By the arbitrary nature of e, φ is $\mathscr{G}$-continuous.

In the succeeding discussions, we assume that $(X, \mathscr{G}_X)$ and $(Y, \mathscr{G}_Y)$ are m-GT and n-GT spaces, respectively, where $\mathscr{G}_X = \{\mu_1, \ldots, \mu_m\}$ and $\mathscr{G}_Y = \{\nu_1, \ldots, \nu_n\}$ for some $m, n \in \mathbb{N}$ and adapt the notations $\mathscr{U} = \bigcup_{j=1}^{m} \mu_j$ and $\mathscr{V} = \bigcup_{k=1}^{n} \nu_k$. For convenience, we also adapt the notations $c_X(A)$ and $i_X(A)$ to signify the $\mathscr{G}_X$-closure and $\mathscr{G}_X$-interior whenever $A \subseteq X$.

Theorem 2.4. *A function $f : X \to Y$ is $\mathscr{G}$-continuous if and only if $f^{-1}(V)$ is $\mathscr{G}_X$-open for every $\mathscr{G}_Y$-open set V.*

Proof. If V is a $\mathscr{G}_Y$-open set and $x \in f^{-1}(V)$, then $f(x) \in V$ and since f is $\mathscr{G}$-continuous, we can find a $\mathscr{G}_X$-open set U_x containing x such that $f(U_x) \subseteq V$. Consequently, $U_x \subseteq f^{-1}(V)$ and so $f^{-1}(V) \subseteq \bigcup_{x \in f^{-1}(V)} U_x \subseteq f^{-1}(V)$. Hence, $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is $\mathscr{G}_X$-open. Conversely, if $x \in X$ and V is a $\mathscr{G}_Y$-open set containing $f(x)$, then $U = f^{-1}(V)$ is a $\mathscr{G}_X$-open set containing x such that $f(U) = V$. Since x is arbitrary, we see that f is continuous. □

A $\mathscr{G}$-continuous map also satisfy the following properties which are analogous to that of a continuous map as in [7]:

Theorem 2.5. *Let $f : X \to Y$ be a map. Then the following statements are equivalent:*

1. *f is $\mathscr{G}$-continuous;*
2. *$f^{-1}(B)$ is $\mathscr{G}_X$-closed for each $\mathscr{G}_Y$-closed set B;*
3. *$f(c_X(A)) \subseteq c_Y(f(A))$ for any $A \subseteq X$; and*
4. *$c_X\left(f^{-1}(B)\right) \subseteq f^{-1}(c_Y(B))$ for any $B \subseteq Y$.*

Remark 2.6. If $f : X \to Y$ is a $\mathscr{G}$-continuous map such that $f(x) \in \bigcup_{G\in\mathscr{V}} G$, then $(X, \mathscr{G}_X)$ is strong. This is so since for each $x \in X$ there is a $\mathscr{G}_Y$-open set V containing $f(x)$ and consequently a corresponding U_x containing x such that $f(U_x) \subseteq V$. This indicates that $X \subseteq \bigcup_{x\in X} U_x \subseteq X$ and $(X, \mathscr{G}_X)$ is therefore strong.

The notion of a $(\mu, \nu)^{(j,k)}$-continuous map, where $j, k = 1, 2$ are distinct, analogously defined for a bigeneralized topological space seen in [6], may be extended to maps between an m-GT space and n-GT space:

Definition 2.7. *For a pair of distinct j, k where $1 \le j \le m$ and $1 \le k \le n$, a function $f : X \to Y$ is $(\mu, \nu)^{(j,k)}$*-continuous *at a point x if for each ν_k-open set V containing $f(x)$ there is a μ_j-open set U containing x such that $f(U) \subseteq V$. If f is $(\mu, \nu)^{(j,k)}$-continuous at all points $x \in X$, then we say f is $(\mu, \nu)^{(j,k)}$*-continuous.

Example 2.8. Let $m, n \in \mathbb{N}$ where $m \ge n$ and consider the m-GT space $(X, \mathscr{G}_X)$ where X and $\mathscr{G}_X$ be the same as defined in Example 2.2. In this case, let $Y = [0, n]$ and for each $k = 1, \dots, n$, define $\nu_k = \{\varnothing\} \cup \{[0, n - t + 1]; t = 1, \dots, k\}$. Observe that $(Y, \mathscr{G}_Y)$ is an n-GT space. Define the map $f : Y \to X$ as $f(y) = \left(\frac{n-y}{n}, n\right)$. Let $y \in Y$ and V be a μ_1-open set containing $f(y) = \left(\frac{n-y}{n}, n\right)$. Then V is of the form R_b^1, where $b \le n$. Now, $Y = [0, n]$ is a ν_m-open set containing y with $f([0, n]) = \{(x, n) : 0 \le x \le 1\} \subseteq R_n^1 \subseteq R_b^1$ for all $b \le n$. Hence, f is $(\nu, \mu)^{(m,1)}$-continuous at y and since y is arbitrary, f is in fact $(\nu, \mu)^{(m,1)}$-continuous.

In the next results, some relationships between $\mathscr{G}$-continuity and $(\mu, \nu)^{(j,k)}$-continuity are established:

Theorem 2.9. *A function $f : X \to Y$ is $\mathscr{G}$-continuous at a point $x \in X$ if and only if for each $k = 1, \dots, n$, there exists $1 \le j \le m$ such that f is $(\mu, \nu)^{(j,k)}$-continuous at x.*

Proof. Suppose that f is $\mathscr{G}$-continuous at $x \in X$, $k \in \{1, \dots, n\}$ and V_k is a ν_k-open set containing $f(x)$. Then V_k is $\mathscr{G}_Y$-open and since f is $\mathscr{G}$-continuous, we can find a $\mathscr{G}_X$-open set U containing x such that $f(U) \subseteq V_k$. Since U is $\mathscr{G}_X$-open, there is a $1 \le j \le m$ such that for some $H_j \in \mu_j$, we have $x \in H_j \subseteq U$ and $f(H_j) \subseteq f(U) \subseteq V_k$. Thus, f is $(\mu, \nu)^{(j,k)}$-continuous at $x \in X$. Conversely, suppose that V is a $\mathscr{G}_Y$-open set containing $f(x)$. Then for some k, there is a $G_k \in \nu_k$ such that $f(x) \in G_k \subseteq V$. By assumption, for some $1 \le j \le m$, we can find a μ_j-open (and hence, $\mathscr{G}$-open) set U containing x such that $f(U) \subseteq G_k \subseteq V$. Thus, f is $\mathscr{G}$-continuous at x. □

Theorem 2.9 implies that to inspect for $\mathscr{G}$-continuity at a point of the domain, we may simply find for each component GT of $(Y, \mathscr{G}_Y)$ a corresponding component GT of $(X, \mathscr{G}_X)$ for which the continuity in the sense of Definition 2.7 holds.

Some properties of $(\mu, \nu)^{(j,k)}$-continuity can be drawn by a simple extension of those that are enumerated in [6]:

Theorem 2.10. Let $f : X \to Y$ be a map. The following statements are equivalent:

1. f is $(\mu, \nu)^{(j,k)}$-continuous at a point $x \in X$;
2. $x \in i_{\mu_j}(f^{-1}(V))$ for each ν_k-open set V containing $f(x)$;
3. $x \in i_{\mu_j}(f^{-1}(B))$ for each $B \subseteq Y$ such that $x \in f^{-1}(i_{\nu_k}(B))$; and
4. $x \in f^{-1}(F)$ for all ν_k-closed set F such that $x \in c_{\mu_j}(f^{-1}(F))$.

Theorem 2.11. Let $f : X \to Y$ be a map. The following statements are equivalent:

1. f is $(\mu, \nu)^{(j,k)}$-continuous;
2. $f^{-1}(V) = i_{\mu_j}(f^{-1}(V))$ for each ν_k-open set V;
3. $f^{-1}(i_{\nu_k}(B)) \subseteq i_{\mu_j}(f^{-1}(B))$ for each $B \subseteq Y$; and
4. $c_{\mu_j}(f^{-1}(F)) = f^{-1}(F)$ for all ν_k-closed set F.

In view of Definition 2.7, $(\mu, \nu)^{(j,k)}$-continuity may occur for each pair of indices j and k; and thence we have another type of continuity formally stated as follows:

Definition 2.12. *Let $(X, \mathscr{G}_X)$ and $(Y, \mathscr{G}_Y)$ be m-GT and n-GT spaces, respectively, where $\mathscr{G}_X = \{\mu_1, \dots, \mu_m\}$ and $\mathscr{G}_Y = \{\nu_1, \dots, \nu_n\}$ for some $m, n \in \mathbb{N}$. A function $f : X \to Y$ is called* pairwise (μ, ν)-continuous *if for each pair j, k where $1 \le j \le m$ and $1 \le k \le n$, f is $(\mu, \nu)^{(j,k)}$-continuous.*

Example 2.13.

1. Let $(X, \mathscr{G}_1)$ and $(X, \mathscr{G}_2)$ be strong m-GT and n-GT spaces, respectively, over the same set X and $c \in X$. If $f : X \to X$ is the constant function defined by $f(x) = c$, then for each $k = 1, \dots, n$ and for each ν_k-open set V_k containing c, $f(G_j) \subseteq \{c\} \subseteq V_k$ for all sets $G_j \in \mu_j$. Thus, f is $(\mu, \nu)^{(j,k)}$-continuous at any point $x \in X$ and any pair j, k. As a result, f is pairwise (μ, ν)-continuous.

2. In Example 2.8, we observe that for a pair $s, t \in \mathbb{N}$ where $1 \le s \le n$ and $1 \le t \le m$ and for $y \in Y$, $0 \le \frac{n-y}{n} \le 1$ which means that $f(y) \in R^1_0$ and so if $t \ne 1$, f vacuously satisfies $(\nu, \mu)^{(s,t)}$-continuity on y. Now, if $t = 1$, we note that $Y = [0, n]$ is ν_s-open for all $s = 1, \dots, n$ and recall that the μ_1-open set containing $f(y)$ are precisely the sets R^1_b where $b \le n$ with $f([0, n]) \subseteq R^1_b$. This indicates that f is $(\nu, \mu)^{(s,t)}$-continuous at y. From these cases, we can see that with the arbitrary nature of y, f is pairwise (ν, μ)-continuous.

3. Consider the graph D^* and the n-GT spaces $(V(D^*), \mathscr{G}_V)$ and $(E(D^*), \mathscr{G}_E)$ as described in Example 2.3.

   Define the mapping $g : E(D^*) \to V(D^*)$ by $g(e) = u$ for each $e = uv \in E(D^*)$. Let $j, k \in \{1, \dots, n\}$ and $e \in E(D^*)$. If $e = e_i$ for some $i = 1, \dots, s-1$, then $g(e) = g(e_i) = v_i$. Notice that μ_k-open sets V containing v_i must contain either both v_{i-1} and v_i or both v_i and v_{i+1}. For the former case, we take the ν_j-open set

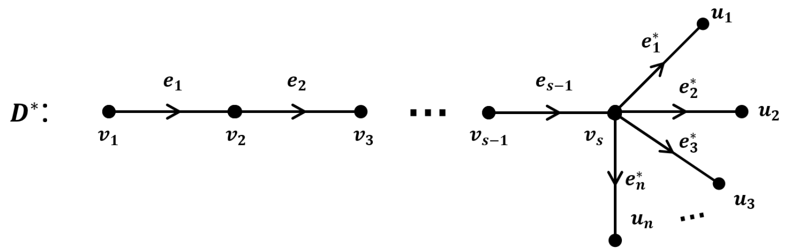

Figure 1: Directed graph D^* for Example 2.13 (3).

$U_1 = \{e_{i-1}, e_i\}$, and for the latter, take $U_2 = \{e_i\}$. These sets both contain $e = e_i$ and since $g(U_1) = \{v_{i-1}, v_i\} \subseteq V$ and $g(U_2) = \{v_i\} \subseteq V$. On the other hand, if $e = e_i^*$ for some $i = 1, \ldots, n$, then $g(e) = v_s$ for each μ_k-open set V containing v_s, $U = \{e_j\}$ is ν_j-open and $g(e_j) = \{v_s\} \subseteq V$. Hence, g is $(\nu, \mu)^{(j,k)}$-continuous at $e \in E(D^*)$. With j, k and e arbitrary, we see that g is pairwise (ν, μ)-continuous.

If every μ_j is a strong GT on X, and $f : X \to Y$ is defined by $f(x) = c$ for some $c \in Y$, then for a pair of fixed indices j, k, and for a ν_k-open set V_k containing c, $f(U_j) = \{c\} \subseteq V_k$ for all μ_j-open set U_j. Since each μ_j is strong, there is such μ_j-open set U_j^* containing x whose image is contained in V_k. With j, k held arbitrary, we see that f is (μ, ν)-continuous. If, on the other hand, $f : X \to Y$ is a map such that $f(X) \subseteq Y \backslash \left(\bigcup_{G \in \mathscr{V}} G \right) \neq \varnothing$, then f is immediately pairwise (μ, ν)-continuous.

Remark 2.14. *If $f : X \to Y$ is pairwise (μ, ν)-continuous, $x \in X$ and V is a $\mathscr{G}$-open set containing $f(x)$, then for some k there exists a ν_k-open subset G_k of V such that $f(x) \in G_k$. For any j, there is a μ_j-open (hence, $\mathscr{G}_X$-open) set U such that $x \in U$ and $f(U) \subseteq G_k \subseteq V$. Thus, f is $\mathscr{G}$-continuous whenever it is pairwise (μ, ν)-continuous.*

The next theorem provides some characterizations for a pairwise (μ, ν)-continuous map:

Theorem 2.15. *Let $f : X \to Y$ be a map. The following statements are equivalent:*

1. *f is pairwise (μ, ν)-continuous;*
2. *$f^{-1}(V)$ is μ_j-open for each $\mathscr{G}_Y$-open set V and for each $j = 1, \ldots, m$;*
3. *$f^{-1}(i_{\mathscr{G}_Y}(B)) \subseteq i_{\mu_j}(f^{-1}(B))$ for all $B \subseteq Y$ and for all $j = 1, \ldots, m$; and*
4. *$f^{-1}(F)$ is μ_j-closed for all $\mathscr{G}_Y$-closed sets F.*

Proof. (1) $\Rightarrow$ (2) If f is pairwise (μ, ν)-continuous and V is a $\mathscr{G}_Y$-open set, then $V = i_{\mathscr{G}_Y}(V) = \bigcup_{k=1}^{n} i_{\nu_k}(V)$. Now, each $i_{\nu_k(V)} \in \nu_k$. By Theorem 2.11 (2), for each $j = 1, \ldots, m$, $f^{-1}(i_{\nu_k(V)}) = i_{\mu_j}(f^{-1}(i_{\nu_k(V)}))$ and is therefore μ_j-open. As a result, $f^{-1}(V) =$

$\bigcup_{k=1}^{n} f^{-1}(i_{\nu_k}(V))$ is μ_j-open for every $j = 1, \ldots, m$.

(2) ⇒ (3) Suppose that $B \subseteq Y$. Then $i_{\mathscr{G}_Y}(B)$ is $\mathscr{G}_Y$-open. By assumption, $f^{-1}(i_{\mathscr{G}_Y}(B))$ is μ_j-open for each $j = 1, \ldots, m$, implying that $f^{-1}(i_{\mathscr{G}_Y}(B)) = i_{\mu_j}(f^{-1}(i_{\mathscr{G}_Y}(B))) \subseteq i_{\mu_j}(f^{-1}(B))$ for each $j = 1, \ldots, m$.

(3) ⇒ (1) For a pair j, k, and for $B \subseteq Y$, $i_{\nu_k}(B)$ is ν_k-open and is therefore $\mathscr{G}_Y$-open. Thus, by statement (3),

$$f^{-1}(i_{\nu_k}(B)) = f^{-1}(i_{\mathscr{G}_Y}(i_{\nu_k}(B))) \subseteq i_{\mu_j}(f^{-1}(i_{\nu_k}(B))) \subseteq i_{\mu_j}(f^{-1}(B)).$$

By the equivalence in Theorem 2.11, and since j, k are arbitrarily chosen, we see that f is pairwise (μ, ν)-continuous.

(4) ⇔ (2) This equivalence follows from the fact that $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ for any $F \subseteq Y$. □

3. $\mathscr{G}$-open and $\mathscr{G}$-closed maps

Now, we define and examine $\mathscr{G}$-open and $\mathscr{G}$-closed maps on n-GT spaces:

Definition 3.1. *Let $(X, \mathscr{G}_X)$ and $(Y, \mathscr{G}_Y)$ be m-GT and n-GT spaces, respectively, where $\mathscr{G}_X = \{\mu_1, \ldots, \mu_m\}$ and $\mathscr{G}_Y = \{\nu_1, \ldots, \nu_n\}$ for some $m, n \in \mathbb{N}$ and let $f : X \to Y$ be a map. f is called* $\mathscr{G}$-open map *[resp.* $\mathscr{G}$-closed map*] if $f(A)$ is $\mathscr{G}_Y$-open [resp. $\mathscr{G}_Y$-closed] for each $\mathscr{G}_X$-open [resp. $\mathscr{G}_X$-closed] set A.*

Example 3.2. Consider the following examples:

1. Let $(X, \mathscr{G}_X)$ and $(Y, \mathscr{G}_Y)$ and $f : X \to Y$ be as defined in Example 2.2. If $(a, b) \in X$, then there is a j_0 such that $j_0 - 1 < a \leq j_0$. Define $f^* : X \to Y$ such that $f^*(a, b) = \frac{j_0 - a}{j_0}$. For each $j = 1, \ldots, m$ and for each $s \geq 0$, $f^*(R_s^j) = [0, 1]$, which is a $\mathscr{G}_Y$-open set. But each $\mathscr{G}_X$-open set $U \subseteq X$ is the union of sets R_s^j; thus, f^* is a $\mathscr{G}$-open map. However, for the $\mathscr{G}_X$-closed set $F = \{(x, y) : 0 \leq x \leq 1, 0 \leq y < n\}$, we see that $f^*(F) = [0, 1]$ is not a $\mathscr{G}_Y$-closed set. Hence, f^* is *not* a $\mathscr{G}$-closed map.

2. Recall the spaces $(V(D), \mathscr{G}_V)$ and $(E(D), \mathscr{G}_E)$ described in Example 2.3. Observe that each $u \in V(D)$ is in some maximal path P_i in D. Let $s = \max\{i : u \in V(P_i)\}$ and denote $P_s = [u_{(s,1)}, \ldots, u_{(s,k)}]$. Then for every $u \in U$, there corresponds a unique pair (s, d) such that $u = u_{(s,d)}$ for some $1 \leq d \leq k$. Define the map $\varphi^* : V(D) \to E(D)$ by
$$\varphi^*(u) = \begin{cases} e_{(s,k-1)} = u_{(s,k-1)}u_{(s,k)}, & \text{if } d = k \\ e_{(s,d)} = u_{(s,d)}u_{(s,d+1)}, & \text{if } d \neq k \end{cases}.$$
By the uniqueness of s, we see that φ^* is a well-defined map. Also, by definition of the n-GT space $(E(D), \mathscr{G}_E)$, $\{e\}$ is $\mathscr{G}_E$-open for every arc $e \in E(D)$. Hence, if $U \subseteq V(D)$ is $\mathscr{G}_V$-open, then $\varphi^*(U) = \bigcup_{u \in U} \varphi^*(u)$ is $\mathscr{G}_E$-open. In turn, $\varphi^* : V(D) \to E(D)$ is a

$\mathscr{G}$-open map. Meanwhile, for any $\mathscr{G}_V$-closed set F, we see that $\varphi^*(F)$ is definitely a $\mathscr{G}_E$-closed set. That is, φ^* is a $\mathscr{G}$-closed map.

$\mathscr{G}$-open and $\mathscr{G}$-closed maps naturally satisfy the following inherent properties of open and closed maps in the sense of the ordinary topological spaces:

Theorem 3.3. *Let $f : X \to Y$ be a map and $S \subseteq Y$.*

1. *If f is a $\mathscr{G}$-open map and A is a $\mathscr{G}_X$-closed set containing $f^{-1}(S)$, then there is a $\mathscr{G}_Y$-closed set B containing S such that $f^{-1}(B) \subseteq A$.*
2. *If f is a $\mathscr{G}$-closed map and U is a $\mathscr{G}_X$-open set containing $f^{-1}(S)$, then there is a $\mathscr{G}_Y$-open set set V containing S such that $f^{-1}(V) \subseteq U$.*

Corollary 3.4. *If $f : X \to Y$ is a $\mathscr{G}$-closed map, $y \in Y$ and U is a $\mathscr{G}_X$-open set such that $f^{-1}(\{y\}) \subseteq U$, then $y \in i_Y(f(c_X(U)))$.*

Theorem 3.5. *Let $f : X \to Y$ be a map. The following statements are equivalent:*

1. *f is a $\mathscr{G}$-open map;*
2. *$f(i_X(A)) \subseteq i_Y(f(A))$ for every $A \subseteq X$; and*
3. *For each $x \in X$ and for every $\mathscr{G}_X$-open set U containing x, there exists a $\mathscr{G}_Y$-open set W containing $f(x)$ such that $W \subseteq f(U)$.*

Theorem 3.6. *A map $f : X \to Y$ is a $\mathscr{G}$-closed map if and only if $c_Y(f(A)) \subseteq f(c_X(A))$ for each $A \subseteq X$.*

Theorem 3.7. *Let $f : X \to Y$ be a map. The following statements are equivalent:*

1. *f is a $\mathscr{G}$-closed map;*
2. *If $A \subseteq X$ is a $\mathscr{G}_X$-open set, then $S = \{y : f^{-1}(\{y\}) \subseteq A\}$ is a $\mathscr{G}_Y$-open set; and*
3. *If $B \subseteq X$ is a $\mathscr{G}_X$-closed set, then $T = \{y : f^{-1}(\{y\}) \cap B \neq \varnothing\}$ is a $\mathscr{G}_Y$-closed set.*

We now establish the equivalence of $\mathscr{G}$-open and $\mathscr{G}$-closed maps involving bijective mappings:

Theorem 3.8. *If $f : X \to Y$ is bijective, then f is a $\mathscr{G}$-open map if and only if f is a $\mathscr{G}$-closed map.*

Proof. Observe that for a subset U of X, $f(X \setminus U) = Y \setminus f(U)$ if and only if f is bijective. Thus, if f is a $\mathscr{G}$-open map and U is $\mathscr{G}$-closed, then $f(X \setminus U) = Y \setminus f(U)$ is $\mathscr{G}$-open so that $f(U)$ is $\mathscr{G}$-closed. That is, f is also a $\mathscr{G}$-closed map. Similarly, if f is a $\mathscr{G}$-closed map, then f is also a $\mathscr{G}$-open map. □

Definition 3.9. *Let $(X, \mathscr{G}_X)$ and $(Y, \mathscr{G}_Y)$ be m-GT and n-GT spaces, respectively, where $\mathscr{G}_X = \{\mu_1, \ldots, \mu_m\}$ and $\mathscr{G}_Y = \{\nu_1, \ldots, \nu_n\}$ for some $m, n \in \mathbb{N}$. Furthermore, let $f : X \to Y$ be a map.*

1. *f is called a* $(\mu,\nu)^{(j,k)}$-open map [resp. $(\mu,\nu)^{(j,k)}$-closed map] *if $f(U)$ is a ν_k-open [resp. ν_k-closed] set for every μ_j-open [resp. μ_j-closed] set $U \subseteq X$.*

2. *f is called a* pairwise (μ,ν)-open map [resp. pairwise (μ,ν)-closed map] *if for each pair j,k where $1 \le j \le m$ and $1 \le k \le n$, f is a $(\mu,\nu)^{(j,k)}$-open map [resp. $(\mu,\nu)^{(j,k)}$-closed map].*

Notice that if both $(X,\mathscr{G}_X)$ and $(Y,\mathscr{G}_Y)$ are 1-GT spaces, then it immediately follows that $f: X \to Y$ is a $\mathscr{G}$-open map if and only if f is a $(\mu,\nu)^{(1,1)}$-open map. Furthermore, a $(\mu,\nu)^{(j,k)}$-open map [resp. $(\mu,\nu)^{(j,k)}$-closed map] may not be a $\mathscr{G}$-open map [resp. $\mathscr{G}$-closed map]. On the other hand, a $\mathscr{G}$-open map [resp. $\mathscr{G}$-closed map] may also not be a $(\mu,\nu)^{(j,k)}$-open map [resp. $(\mu,\nu)^{(j,k)}$-closed map]. These are illustrated in the following example:

Example 3.10. Let X and Y be infinite sets and $\{P_1,\dots,P_m\}$ and $\{Q_1,\dots,Q_n\}$ be some partitions of X and Y, respectively. Putting $\mathscr{G}_X = \{\mu_j : j = 1,\dots,m\}$ and $\mathscr{G}_Y = \{\nu_k : k = 1,\dots,n\}$ where $\mu_j = \mathscr{P}(P_j)$ and $\nu_k = \mathscr{P}(Q_k)$, we come up with the m-GT space $(X,\mathscr{G}_X)$ and the n-GT space $(Y,\mathscr{G}_Y)$.

1. If, in general, $f : X \to Y$ is the constant map $f(x) = c$, then for the fixed k^* (where $c \in Q_{k^*}$), f is a $(\mu,\nu)^{(j,k^*)}$-open map and, in fact, also a $\mathscr{G}$-open map.

2. In particular, consider the case where $X = Y$ and $m = n$, while $P_j \neq Q_k$ for any j,k. Then if $f : X \to Y$ is the identity map, it is easy to see that f is a $\mathscr{G}$-open map. However, f is not a $(\mu,\nu)^{(j,k)}$-open map for any pair j,k due to our choice of partitions.

3. Now, consider the case where $X \neq Y$, $\varnothing \neq A \subset X$ and $\varnothing \neq B \subset Y$, $\mu_1 = \mathscr{P}(A)$, $\mu_2 = \mathscr{P}(X \backslash A)$ and $\nu_1 = \mathscr{P}(B)$. If $\mathscr{G}_X = \{\mu_1,\mu_2\}$ and $\mathscr{G}_Y = \{\nu_1\}$, then $(X,\mathscr{G}_X)$ is a 2-GT space and $(Y,\mathscr{G}_Y)$ is a 1-GT space. Furthermore, if $c_1 \in B$ and $c_2 \in Y \backslash B$ are fixed and $f : X \to Y$ is defined as

$$f(x) = \begin{cases} c_1, & \text{if } x \in A \\ c_2, & \text{if } x \in X \backslash A \end{cases},$$

then f is a $(\mu,\nu)^{(1,1)}$-open map but not a $\mathscr{G}$-open map since $f(X) = \{c_1, c_2\}$ is not $\mathscr{G}_Y$-open.

Example 3.11. To illustrate a pairwise (μ,ν)-open map, we simply recall the mapping f in Example 3.2 (1.ii). This is so since the image of each $\mathscr{G}$-open set (and hence of any μ_j-open set) is $[0,1]$ which is ν_k-open for all k.

A $(\mu,\nu)^{(j,k)}$-open map and a $(\mu,\nu)^{(j,k)}$-closed map inherits similar and corresponding properties as established in the previous results in this section since a $(\mu,\nu)^{(j,k)}$-open map and a $(\mu,\nu)^{(j,k)}$-closed maps are analogous to a $\mathscr{G}$-open map and a $\mathscr{G}$-closed map, respectively, when the underlying spaces involved are both 1-GT spaces.

We now present some observations showing relationships of the mentioned typed of open maps:

Theorem 3.12. *If for each j there is a k such that f is a $(\mu,\nu)^{(j,k)}$-open map, then f is a $\mathscr{G}$-open map.*

Proof. Let U be a $\mathscr{G}_X$-open set. Then for each $x \in U$, there exists a j and a μ_j-open set O_x such that $x \in O_x \subseteq U$. By assumption, there exists a k such that $f(O_x)$ is ν_k-open. As a result, $f(U) = f(\bigcup_{x\in U} O_x) = \bigcup_{x\in U} f(O_x)$ is a $\mathscr{G}_Y$-open set. Since U is arbitrary, we say that f is a $\mathscr{G}$-open map. □

The converse of Theorem 3.12 is not generally true as seen in Example 3.10 (2).

Corollary 3.13. *If f is a pairwise (μ,ν)-open map, then f is a $\mathscr{G}$-open map.*

Proof. This is immediate from the definition of a pairwise (μ,ν)-open map and then applying Theorem 3.12. □

Theorem 3.14. *A mapping $f : X \to Y$ is a pairwise (μ,ν)-open map if and only if for each j and for each μ_j-open set U, $f(U) \in \bigcap_{k=1}^{n} \nu_k$.*

Proof. Let $f : X \to Y$ be a map, $j \in \{1,\dots,m\}$ and U be a μ_j-open set. If f is a pairwise (μ,ν)-open map, then for every k, $f(U)$ is μ_k-open so that $f(U) \in \bigcap_{k=1}^{n} \nu_k$. The converse is similarly outright. □

Corollary 3.15. *If for each j and for each μ_j-open set U, $f(U) \in \bigcap_{k=1}^{n} \nu_k$, then f is a $\mathscr{G}$-open map.*

Proof. This relationship is a direct consequence of Theorem 3.14 and Corollary 3.13. □

4. $\mathscr{G}$-homeomorphisms

This section basically displays the relationships of $\mathscr{G}$-continuous maps to $\mathscr{G}$-open and $\mathscr{G}$-closed maps in form of $\mathscr{G}$-homeomorphisms. In the sense of ordinary topological spaces, the existence of homeomorphisms between two topological spaces identifies an equivalence of their structures. Here, we investigate whether the existence of such likeness extends this time to n-GT spaces.

Definition 4.1. *Let $(X,\mathscr{G}_X)$ and $(Y,\mathscr{G}_Y)$ be m-GT and n-GT spaces, respectively, where $\mathscr{G}_X = \{\mu_1,\dots,\mu_m\}$ and $\mathscr{G}_Y = \{\nu_1,\dots,\nu_n\}$ for some $m,n \in \mathbb{N}$.*

1. *A bijective map $f : X \to Y$ is a $\mathscr{G}$-*homeomorphism *if both f and f^{-1} are $\mathscr{G}$-continuous maps. In this case, we use the notation $f : X \overset{\mathscr{G}}{\cong} Y$ to denote that f is a $\mathscr{G}$-homeomorphism.*

2. *If there exists a $\mathscr{G}$-homeomorphism $f : X \overset{\mathscr{G}}{\cong} Y$, then we say that the spaces X and Y are $\mathscr{G}$-*homeomorphic.

Example 4.2. By the definition of the spaces $(X, \mathscr{G}_1)$ and $(X, \mathscr{G}_2)$ in Example 3.10 (2), we see that the identity map between these particular spaces easily provides a $\mathscr{G}$-homeomorphic map. In a wider sense, if $f : X \to Y$ is any bijection and $(X, \mathscr{G}_X)$ and $(Y, \mathscr{G}_Y)$ are as defined in Example 3.10, then f is in fact a $\mathscr{G}$-homeomorphism. However, it should be noted that not every bijective mapping is a $\mathscr{G}$-homeomorphism: see this by simply dropping Q_n (or any of the component GTs) in the n-GT space $(Y, \mathscr{G}_Y)$ in the same example.

Example 4.2 suggests that even if $f : X \overset{\mathscr{G}}{\cong} Y$, it is not a guarantee that the spaces X and Y are equivalent in terms of their component GTs. In other words, a component GT μ_j in $\mathscr{G}_X$ may not coincide to any ν_k in $\mathscr{G}_Y$ even if X is $\mathscr{G}$-homeomorphic to Y.

Theorem 4.3. *Let $f : X \to Y$ be a bijective map. Then the following statements are equivalent:*

1. *f is a $\mathscr{G}$-homeomorphism;*
2. *f is a $\mathscr{G}$-continuous and $\mathscr{G}$-open map;*
3. *f is a $\mathscr{G}$-continuous and $\mathscr{G}$-closed map;*
4. *$f(c_X(A)) = c_Y(f(A))$ for every $A \subseteq X$; and*
5. *$f(i_X(A)) = i_Y(f(A))$ for every $A \subseteq X$.*

Proof. Suppose $f : X \to Y$ is a bijective map.
(1) $\Leftrightarrow$ (2) Note that f is a $\mathscr{G}$-homeomorphism if and only if f is $\mathscr{G}$-continuous and f^{-1} is also $\mathscr{G}$-continuous. As such, Theorem 2.4 provides that for every $\mathscr{G}_X$-open set U, $(f^{-1})^{-1}(U) = f(U)$ is $\mathscr{G}_Y$-open. This means that f is also $\mathscr{G}$-open map, and conversely.
(2) $\Leftrightarrow$ (3) Because f is bijective, these directions follow immediately from Theorem 3.8.
(3) $\Leftrightarrow$ (4) This follows immediately from Theorems 2.5 and 3.6.
(2) $\Leftrightarrow$ (5) From Theorem 3.5, f is a $\mathscr{G}$-open map if and only if $f(i_X(A)) \subseteq i_Y(f(A))$. Also, since f is both bijective and $\mathscr{G}$-continuous, $f^{-1}(i_Y(f(A)))$ is $\mathscr{G}_X$-open and

$$f^{-1}(i_Y(f(A))) \subseteq f^{-1}(f(A)) = A$$

implying that $f^{-1}(i_Y(f(A))) \subseteq i_X(A)$. As a result, $i_Y(f(A)) \subseteq f(i_X(A))$. Conversely, we only need to show that $f^{-1}(B)$ is $\mathscr{G}_X$-open for any $\mathscr{G}_Y$-open set B whenever (5) holds. Indeed, if B is $\mathscr{G}_Y$-open, (5) provides that

$$f(i_X(f^{-1}(B))) = i_Y(f(f^{-1}(B))) = i_Y(B) = B.$$

Thus, $i_X(f^{-1}(B)) = f^{-1}(B)$ is a $\mathscr{G}_X$-open set which, in turn, shows that f is also $\mathscr{G}$-continuous. □

Definition 4.4. *Let* $(X, \mathscr{G}_X)$ *and* $(Y, \mathscr{G}_Y)$ *be* m*-GT and* n*-GT spaces, respectively, where* $\mathscr{G}_X = \{\mu_1, \ldots, \mu_m\}$ *and* $\mathscr{G}_Y = \{\nu_1, \ldots, \nu_n\}$ *for some* $m, n \in \mathbb{N}$ *and let* $f : X \to Y$ *be a bijective map.*

1. f *is called a* $(\mu, \nu)^{(j,k)}$-homeomorphism *if* f *is a* $(\mu, \nu)^{(j,k)}$*-continuous map and* f^{-1} *is a* $(\nu, \mu)^{(k,j)}$*-continuous map and we write* $f : X \overset{(j,k)}{\cong} Y$.
2. f *is called a* pairwise (μ, ν)-homeomorphism *if for each pair* j, k, f *is a* $(\mu, \nu)^{(j,k)}$*-homeomorphism. In this case, we write* $f : X \overset{pw}{\cong} Y$.

By the definition of $(\mu, \nu)^{(j,k)}$-homeomorphism, it is easy to see that f is pairwise (μ, ν)-homeomorphism if and only if f is a pairwise (μ, ν)-continuous map and f^{-1} is a pairwise (ν, μ)-continuous map.

Theorem 4.5. *If* f *is a pairwise* (μ, ν)*-homeomorphism, then* f *is a* $\mathscr{G}$*-homeomorphism.*

Proof. If f is a pairwise (μ, ν)-homeomorphism, then f is a pairwise (μ, ν)-continuous map and f^{-1} is a pairwise (ν, μ)-continuous map. By Remark 2.14, f and f^{-1} are $\mathscr{G}$-continuous maps. Thus, f is a $\mathscr{G}$-homeomorphism. □

Theorem 4.6. If f is a pairwise (μ, ν)-homeomorphism, then $\mu_1 = \cdots = \mu_m$ and $\nu_1 = \cdots = \nu_n$.

Proof. We first show that if $f : X \overset{(j,k)}{\cong} Y$ for each pair j, k of indices, then $\{f(U) : U \in \mu_j\} = \nu_k$. Indeed, if $U \in \mu_j$, then $f(U) \in \nu_k$ since f is also a $(\mu, \nu)^{(j,k)}$-open as implied by Definition 4.4 and the equivalence of (1) and (2) in Theorems 2.11 and 2.15. Also, if $V \in \nu_k$, $f^{-1}(V) \in \mu_j$ since f is $(\mu, \nu)^{(j,k)}$-continuous, implying that $V = f(f^{-1}(V)) \in \{f(U) : U \in \mu_j\}$. Similarly, $\{f^{-1}(V) : V \in \nu_k\} = \mu_j$. Now, if $f : X \overset{pw}{\cong} Y$, then for a fixed μ_j, $\{f(U) : U \in \mu_j\} = \nu_k$ for all $k = 1, \ldots, n$. That is, $\nu_1 = \ldots = \nu_n$. In the same manner, for a fixed ν_k, $\{f^{-1}(V) : V \in \nu_k\} = \mu_j$ for all $j = 1, \ldots, m$ which indicates that $\mu_1 = \ldots = \mu_m$. □

In general, the converse of Theorem 4.6 may not hold.

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