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Some Structural Properties of Fully UP-semigroups

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Abstract. This paper investigates a new class of algebra related to UP-algebras and semigroups called fully UP-semigroups (or f-UP-semigroups). It establishes some structural properties of f-UP-semigroups. It also introduces and examines f-UP-fields, f-UP-domains, f-UP-ideals, and quotient f-UP-semigroups. Moreover, it investigates the relationship between an f-UP-field and an f-UP-domain.

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1. Introduction

In 1966, Y. Imai and K. Iseki [5] introduced the idea of BCK-algebra as a generalization of the concept of set-theoretic difference and propositional calculi. In the same year, K. Iseki [6] introduced the notion of BCI-algebra as a generalization of BCK-algebra. Studies on different types of algebraic structures followed, among them B-algebras, G-algebras, BCH-algebras, BE-algebras, and SU-algebras. In 2009, C. Prabpayak and U. Leerawat [11] introduced the notion of KU-algebra and investigated some related properties. In 2017, A. Iampan [3] introduced a class of algebra called UP-algebra (UP means the University of Phayao). He established its structure and defined some concepts such as UP-subalgebras, UP-ideals, congruences, and UP-homomorphism. He determined some properties of UP-homomorphism, which led to four isomorphism theorems for UP-algebras. He also presented some connections between UP-algebras and KU-algebras and showed that the notion of UP-algebra is a generalization of KU-algebra.

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In 1993, Jun, Hong, and Roh [7] introduced a class of algebra related to BCI-algebras and semigroups with distributive laws property, called a BCI-semigroup. Jun et al. [8, 9] renamed the BCI-semigroup as the IS-algebra and studied related properties. In 2018, F. Kareem and E. Hasan [10] introduced the concept of KU-semigroup which is a combination of KU-algebra and semigroup. In the same year, A. Iampan [4] introduced a new class of algebra called a fully UP-semigroup (or f-UP-semigroup) which is a combination of UP-algebra and semigroup. In this study, the notion of f-UP-semigroup is investigated and some of its properties are established.

2. Preliminaries

An algebra of type $(2, 0)$ is an algebra with a binary operation and a constant element.

Definition 1. [11] A KU-*algebra* is an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following axioms: for all $x, y, z \in X$,

(KU1) $(x * y) * [(y * z) * (x * z)] = 0$,

(KU2) $0 * x = x$,

(KU3) $x * 0 = 0$,

(KU4) $x * y = y * x = 0$ implies $x = y$.

Example 1. [11] Let $X = \{0, a, b, c\}$ be a set with a binary operation $*$ defined by the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	0	0	0

Then, $(X; *, 0)$ is a KU-algebra.

Definition 2. [3] A UP-*algebra* is an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following axioms: for all $x, y, z \in X$,

(UP1) $(y * z) * [(x * y) * (x * z)] = 0$,

(UP2) $0 * x = x$,

(UP3) $x * 0 = 0$,

(UP4) $x * y = y * x = 0$ implies $x = y$.

Example 2. [3] Let $X = \{0, a, b, c\}$ be a set with a binary operation $*$ defined by the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	0	0	b	b
b	0	a	0	b
c	0	a	0	0

Then, $(X;*,0)$ is a UP-algebra.

Definition 3. [3] Let X be a UP-algebra. A subset S of X is called a UP-*subalgebra* of X if the constant zero of X is in S and $(S;*,0)$ itself forms a UP-algebra.

Definition 4. [1] Define $x \wedge y = (y*x)*x$. Then X is said to be a *commutative* UP-*algebra* if for any $x, y \in X$, $(y*x)*x = (x*y)*y$, that is, $x \wedge y = y \wedge x$.

Definition 5. [3] Let X be a UP-algebra. Then, a subset I of X is called a UP-*ideal* of X if it satisfies:

(i) the constant zero of X is in I, and

(ii) for any $x, y, z \in X$, $x*(y*z) \in I$ and $y \in I$ imply $x*z \in I$.

Proposition 1. [3] *In a* UP-*algebra* $(X;*,0)$*, the following properties hold: for any* $x, y, z \in X$*,*

(i) $x*x = 0$*,*

(ii) $x*y = 0$ *and* $y*z = 0$ *imply* $x*z = 0$*,*

(iii) $x*y = 0$ *implies* $(z*x)*(z*y) = 0$*,*

(iv) $x*y = 0$ *implies* $(y*z)*(x*z) = 0$*,*

(v) $x*(y*x) = 0$*,*

(vi) $(y*x)*x = 0$ *implies* $x = y*x$*, and*

(vii) $x*(y*y) = 0$*.*

The next result gives a relationship between UP-algebras and KU-algebras.

Theorem 1. [3] *Any* KU-*algebra is a* UP-*algebra.*

The converse of Theorem 1 does not hold. To see this, consider the UP-algebra $(X;*,0)$ in Example 2. Let $x = 0$, $y = a$, and $z = c$. Observe that $(x*y)*[(y*z)*(x*z)] = (0*a)*[(a*c)*(0*c)] = a*(b*c) = a*b = b \neq 0$, so (KU1) is not satisfied. Thus, $(X;*,0)$ is not a KU-algebra.

In view of Theorem 1, the notion of UP-algebras is a generalization of KU-algebras.

Proposition 2. [3] *A nonempty subset* S *of a* UP-*algebra* $(X;*,0)$ *is a* UP-*subalgebra of* X *if and only if* S *is closed under the* $*$ *operation.*

Let X be a UP-algebra and A be a nonempty subset of X. Then $X * A$ is given by $X * A = \bigcup_{x\in X, a\in A} (x * a)$.

Theorem 2. [3] *Let X be a* UP*-algebra and B a* UP*-ideal of X. Then $X * B \subseteq B$. In particular, B is a* UP*-subalgebra of X.*

Let $(X; *, 0)$ be a UP-algebra and B be a UP-ideal of X. Define the binary relation $\sim_B$ on X as follows: for all $x, y \in X$, $x \sim_B y$ if and only if $x * y \in B$ and $y * x \in B$. An equivalence relation ρ on X is called a *congruence* if for any $x, y, z \in X$, $x\rho y$ implies $(x * z)\rho(y * z)$ and $(z * x)\rho(z * y)$.

If $x \in X$, then the ρ-class of x is $[x]_\rho$ defined as $[x]_\rho = \{y \in X : y\rho x\}$. The set of all ρ-classes is called the *quotient set of X by ρ*, and is denoted by X/ρ. That is, $X/\rho = \{[x]_\rho : x \in X\}$.

Theorem 3. [3] *Let $(X; *, 0)$ be a* UP*-algebra and B a* UP*-ideal of X. Then the following hold:*

(*i*) *the $\sim_B$-class $[0]_{\sim_B}$ is a* UP*-ideal and a* UP*-subalgebra of X,*

(*ii*) *a $\sim_B$-class $[x]_{\sim_B}$ is a* UP*-ideal of X if and only if $x \in B$,*

(*iii*) *a $\sim_B$-class $[x]_{\sim_B}$ is a* UP*-subalgebra of X if and only if $x \in B$, and*

(*iv*) *$(X/\sim_B; *, [0]_{\sim_B})$ is a* UP*-algebra under the operation $*$ defined by $[x]_{\sim_B} * [y]_{\sim_B} = [x * y]_{\sim_B}$ for all $x, y \in X$, called the quotient* UP*-algebra of X induced by the congruence $\sim_B$.*

Definition 6. [10] A KU-*semigroup* is a nonempty set X together with two binary operations $*$ and $\cdot$ and a constant 0 satisfying the following:

(KUS1) $(X; *, 0)$ is a KU-algebra;

(KUS2) $(X, \cdot)$ is a semigroup; and

(KUS3) the operation $\cdot$ is left and right distributive over the operation $*$, that is, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$.

Example 3. [10] Let $X = \{0, a, b, c\}$ be a set with the binary operations $*$ and $\cdot$ defined by the following Cayley tables:

$*$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	0	0	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	b
c	0	0	b	c

Then, $(X; *, \cdot, 0)$ is a KU-semigroup.

Definition 7. [4] A *fully* UP-*semigroup* (or f-UP-*semigroup*) is a nonempty set X together with two binary operations $*$ and $\cdot$ and a constant 0 satisfying the following:

(fUP1) $(X; *, 0)$ is a UP-algebra;

(fUP2) $(X, \cdot)$ is a semigroup; and

(fUP3) the operation $\cdot$ is left and right distributive over the operation $*$.

A. Iampan [4] analogously introduced a *left* [resp., *right*] UP-*semigroup* as a nonempty set X together with two binary operations $*$ and $\cdot$ and a constant 0 satisfying (fUP1), (fUP2), and the operation $\cdot$ is left [resp. right] distributive over the operation $*$. Thus, an f-UP-semigroup is both a left and a right UP-semigroup.

Example 4. [4] Let $X = \{0, a, b, c\}$ be a set with the binary operations $*$ and $\cdot$ defined by the following Cayley tables:

∗	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	a	b	0

·	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	0

Then, $(X; *, \cdot, 0)$ is an f-UP-semigroup.

Example 5. Let $X = \{0, a, b, c\}$ be a set with the binary operations $*$ and $\cdot$ defined by the following Cayley tables:

∗	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	0	0	0

·	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	a	b	c

Then, routine calculations show that $(X; *, \cdot, 0)$ is an f-UP-semigroup.

Example 6. Let $X = \{0, a, b, c, d\}$ be a set with the binary operations $*$ and $\cdot$ defined by the following Cayley tables:

∗	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	a	b	0

·	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	c	a
c	0	c	a	b

Then, routine calculations show that $(X;*,\cdot,0)$ is an f-UP-semigroup.

Hereinafter, let X denote the f-UP-semigroup $(X;*,\cdot,0)$, unless otherwise indicated.

Definition 8. A nonempty subset S of an f-UP-semigroup X is called an f-UP-*subsemigroup* of X if the constant 0 of X is in S and $(S;*,\cdot,0)$ itself forms an f-UP-semigroup.

Obviously, $\{0\}$ and X are f-UP-subsemigroups of X. In Example 4, the set $S_1 = \{0, b\}$ is an f-UP-subsemigroup of X, while the set $S_2 = \{0, b, c\}$ is not an f-UP-subsemigroup since $b \cdot c = a \notin S_2$.

The following remark immediately follows from Definitions 8, 7, and 3.

Remark 1. *Every f-UP-subsemigroup of $(X;*,\cdot,0)$ is a UP-subalgebra of X with respect to $*$.*

The converse of Remark 1 does not hold. To see this, consider Example 4. It can be easily verified that $S = \{0, b, c\}$ is a UP-subalgebra of $(X;*,0)$ but S is not an f-UP-subsemigroup of $(X;*,\cdot,0)$ since $b \cdot c = a \notin S$.

Definition 9. An f-UP-semigroup X is said to be *commutative* if $a \cdot b = b \cdot a$ for all $a, b \in X$. If X is not commutative, then it is called a *noncommutative f-UP-semigroup.*

Routine calculations show that the f-UP-semigroups in Examples 4 and 6 are commutative while the f-UP-semigroup in Example 5 is noncommutative since $a \cdot c = 0 \neq a = c \cdot a$.

Definition 10. Let X be an f-UP-semigroup. An element $e \in X$ is called a *unity* in X if $x \cdot e = x = e \cdot x$ for all $x \in X$.

Proposition 3. *Let X be an f-UP-semigroup. If the unity of X exists, then it is unique.*

Proof. Let X be an f-UP-semigroup with unity. Suppose $1, 1' \in X$ both satisfy the properties of being a unity. Then, for all $x \in X$, $x \cdot 1 = 1 \cdot x = x$ and $x \cdot 1' = 1' \cdot x = x$. If $x = 1$, we have $1 \cdot 1' = 1$. If $x = 1'$, we have $1 \cdot 1' = 1'$. Therefore, $1 = 1'$. □

If an f-UP-semigroup X has unity, it shall be denoted by 1.

Definition 11. Let X be an f-UP-semigroup with unity 1. An element a of X is called 1-*invertible* if there exists $b \in X$ such that $a \cdot b = 1 = b \cdot a$.

We next introduce the concepts of f-UP-field and f-UP-domain analogous to the definitions of JB-field and JB-domain given by J. Endam and J. Vilela [2].

Definition 12. Let X be an f-UP-semigroup with unity 1. Then X is called an f-UP-*field* if the following hold:

(i) the semigroup $(X, \cdot)$ is commutative; and

(ii) every $0 \neq a \in X$ is 1-invertible.

Definition 13. A nonzero element a of an f-UP-semigroup X is called a 0-*divisor* if there exists $b \in X$ such that $b \neq 0$ and either $a \cdot b = 0$ or $b \cdot a = 0$.

Note that 0 is not a 0-divisor.

Remark 2. *An element cannot be* 1*-invertible and a* 0*-divisor at the same time. Thus, an* f*-*UP*-field has no* 0*-divisors.*

Definition 14. Let X be an f-UP-semigroup with unity 1. Then X is called an f-UP-*domain* if the following hold:

(i) the semigroup $(X, \cdot)$ is commutative; and

(ii) X has no 0-divisors.

The f-UP-semigroup in Example 6 is an f-UP-domain.

Remark 3. *Every* f*-*UP*-field is an* f*-*UP*-domain.*

3. Elementary Properties of f-UP-semigroups

This section presents some elementary properties of f-UP-semigroups. Throughout this section, X means an f-UP-semigroup $(X; *, \cdot, 0)$.

Theorem 4. *Let* $a, b, c \in X$*. Then the following properties hold:*

(i) $a \cdot 0 = 0 \cdot a = 0,$

(ii) $a \cdot (0 * b) = (0 * a) \cdot b = a \cdot b,$

(iii) $a \cdot (b * (0 * c)) = (a \cdot b) * (a \cdot c)$ *and* $(b * (0 * c)) \cdot a = (b \cdot a) * (c \cdot a),$

(iv) $a \cdot (b \wedge c) = (a \cdot b) \wedge (a \cdot c)$ and $(a \wedge b) \cdot c = (a \cdot c) \wedge (b \cdot c),$

(v) *If* $a \cdot b = 0$*, then* $a \cdot (b * c) = a \cdot c,$

(vi) *If* $a \cdot c = 0$*, then* $(a * b) \cdot c = b \cdot c.$

Proof. Let $a, b, c \in X$.

(i) By Proposition 1(i) and (fUP3), $a \cdot 0 = a \cdot (0 * 0) = (a \cdot 0) * (a \cdot 0) = 0$. Similarly, $0 \cdot a = 0$.

(ii) By (UP2), $a \cdot (0 * b) = a \cdot b = (0 * a) \cdot b$.

(iii) By (UP2) and (fUP3), $a \cdot (b * (0 * c)) = a \cdot (b * c) = (a \cdot b) * (a \cdot c)$. Similarly, $(b * (0 * c)) \cdot a = (b * c) \cdot a = (b \cdot a) * (c \cdot a)$.

(iv) By Definition 4 and (fUP3), $a \cdot (b \wedge c) = a \cdot [(c * b) * b] = [a \cdot (c * b)] * (a \cdot b) = [(a \cdot c) * (a \cdot b)] * (a \cdot b) = (a \cdot b) \wedge (a \cdot c)$ and $(a \wedge b) \cdot c = [(b * a) * a] \cdot c = [(b * a) \cdot c] * (a \cdot c) = [(b \cdot c) * (a \cdot c)] * (a \cdot c) = (a \cdot c) \wedge (b \cdot c)$.

(*v*) Suppose $a \cdot b = 0$. Then by (fUP3) and (UP2), $a \cdot (b * c) = (a \cdot b) * (a \cdot c) = 0 * (a \cdot c) = a \cdot c$.

(*vi*) If $a \cdot c = 0$, then by (fUP3) and (UP2), $(a * b) \cdot c = (a \cdot c) * (b \cdot c) = 0 * (b \cdot c) = b \cdot c$. □

The following theorem gives a necessary and sufficient condition for a subset of an f-UP-semigroup to be an f-UP-subsemigroup.

Theorem 5. *A nonempty subset S of an f-UP-semigroup $(X; *, \cdot, 0)$ is an f-UP-subsemigroup of X if and only if $x * y, x \cdot y \in S$ for all $x, y \in S$.*

Proof. Let $\varnothing \neq S \subseteq X$. Suppose S is an f-UP-subsemigroup of X. Then by Definition 8, $(S; *, \cdot, 0)$ is an f-UP-semigroup. Thus, the binary operations $*$ and $\cdot$ are closed in S, that is, $x * y, x \cdot y \in S$ for all $x, y \in S$. Conversely, suppose $x * y, x \cdot y \in S$ for all $x, y \in S$. Then $0 = x * x \in S$. By Proposition 2, $(S; *, 0)$ is a UP-subalgebra of X, hence (fUP1) holds. Let $x, y, z \in S \subseteq X$. Then $x \cdot y \in S$ by our assumption and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ by associativity in X. Hence, $(S, \cdot)$ is a semigroup and (fUP2) is satisfied. Moreover, (fUP3) holds for all $x, y, z \in S \subseteq X$. Thus, S is an f-UP-subsemigroup of X. □

Theorem 6. *Let X be an f-UP-semigroup and $\{A_i : i \in I\}$ a family of f-UP-subsemigroups of X. Then $\bigcap_{i \in I} A_i$ is an f-UP-subsemigroup of X.*

Proof. Since A_i is an f-UP-subsemigroup of X, $0 \in A_i$ for all $i \in I$. Thus, $0 \in \bigcap_{i \in I} A_i$ and $\bigcap_{i \in I} A_i \neq \varnothing$. Let $x, y \in \bigcap_{i \in I} A_i$. Then for all $i \in I$, $x, y \in A_i$ and by Theorem 5, $x * y, x \cdot y \in A_i$. Hence, $x * y, x \cdot y \in \bigcap_{i \in I} A_i$. Therefore, $\bigcap_{i \in I} A_i$ is an f-UP-subsemigroup of X. □

The next result shows a relationship between KU-semigroups and f-UP-semigroups.

Theorem 7. *Any* KU-*semigroup is an* f-UP-*semigroup.*

Proof. Let $X = (X; *, \cdot, 0)$ be a KU-semigroup. By Theorem 1, $(X; *, 0)$ is a UP-algebra. By Definition 6, $(X, \cdot)$ is a semigroup and left and right distributivity hold for $\cdot$ over $*$, thus X is an f-UP-semigroup. □

Remark 4. *The converse of Theorem 7 does not hold.*

To see this, let $X = \{0, a, b, c, d\}$ be a set with the binary operations $*$ and $\cdot$ defined by the following Cayley tables:

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	0	0
b	0	b	0	0	0
c	0	b	b	0	0
d	0	b	b	d	0

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	0
c	0	0	0	0	0
d	0	0	0	0	0

Then by routine calculations, $(X;*,\cdot,0)$ is an f-UP-semigroup. Let $x=0, y=c$, and $z=a$. Observe that $(x*y)*[(y*z)*(x*z)]=(0*c)*[(c*a)*(0*a)]=c*(b*a)=c*b=b$, so (KU1) is not satisfied. Thus, $(X;*,\cdot,0)$ is not a KU-semigroup.

Theorem 8. *Let X be an f-UP-semigroup with unity 1 and let T be the set of all 1-invertible elements of X. Then*

(*i*) $1\in T$,

(*ii*) $0\notin T$, *and*

(*iii*) $a\cdot b\in T$ *for all* $a,b\in T$.

Proof. Let T be the set of all 1-invertible elements of X.

(*i*) Since $1\cdot 1=1$, $1\in T$. Thus, $T\neq\varnothing$.

(*ii*) Suppose $0\in T$. Then there exists $b\in X$ such that $0\cdot b=1=b\cdot 0$. But $0\cdot b=0$ and so, $0=1$, a contradiction. Thus, $0\notin T$.

(*iii*) Let $a,b\in T$. Then there exist $c,d\in X$ such that $a\cdot c=1=c\cdot a$ and $b\cdot d=1=d\cdot b$. Moreover, $d\cdot c\in X$. By (fUP2), $(a\cdot b)\cdot(d\cdot c)=((a\cdot b)\cdot d)\cdot c=(a\cdot(b\cdot d))\cdot c=(a\cdot 1)\cdot c=a\cdot c=1$ and $(d\cdot c)\cdot(a\cdot b)=((d\cdot c)\cdot a)\cdot b=(d\cdot(c\cdot a))\cdot b=(d\cdot 1)\cdot b=d\cdot b=1$. Hence, $a\cdot b\in T$. □

The next result establishes a relation between 0-divisors and the cancellation property of an f-UP-semigroup.

Theorem 9. *If an f-UP-semigroup X has no 0-divisors, then left and right cancellation laws hold, that is, for all $a,b,c\in X$, $a\neq 0$, $a\cdot b=a\cdot c$ implies $b=c$ (left cancellation) and $b\cdot a=c\cdot a$ implies $b=c$ (right cancellation). If either left or right cancellation law holds, then X has no 0-divisors.*

Proof. Let $a,b,c\in X$ such that $a\cdot b=a\cdot c$ and $a\neq 0$. Then $a\cdot(b*c)=(a\cdot b)*(a\cdot c)=0$ by Proposition 1(*i*). Since X has no 0-divisors and $a\neq 0$, we have $b*c=0$. Since $a\cdot b=a\cdot c$, we have $0=a\cdot(b*c)=(a\cdot b)*(a\cdot c)=(a\cdot c)*(a\cdot b)=a\cdot(c*b)$ and so, $c*b=0$. By (UP4), $b=c$. Hence, the left cancellation law holds. Similarly, the right cancellation law holds.

Conversely, suppose one of the cancellation laws holds, say, the left cancellation. Let a be a nonzero element of X and $b\in X$. Suppose $a\cdot b=0$. Then by Theorem 4(*i*), $a\cdot b=a\cdot 0$ and so by left cancellation, $b=0$. Suppose $b\cdot a=0$ and $b\neq 0$. Then by Theorem 4(*i*), $b\cdot a=b\cdot 0$ and so by left cancellation, $a=0$, a contradiction. Therefore, $b=0$ and X has no 0-divisors. Similarly, the right cancellation law implies that X has no 0-divisors. □

Theorem 10. *A finite commutative f-UP-semigroup X with more than one element and without 0-divisors is an f-UP-field.*

Proof. Let $a_1, a_2, \ldots, a_n$ be the distinct elements of X. Let $a \in X$ with $a \neq 0$. Now, $a \cdot a_i \in X$ for all $i = 1, 2, \ldots, n$ and so $\{a \cdot a_1, a \cdot a_2, \ldots, a \cdot a_n\} \subseteq X$. If $a \cdot a_i = a \cdot a_j$, then by Theorem 9, $a_i = a_j$. Thus, the elements $a \cdot a_1, a \cdot a_2, \ldots, a \cdot a_n$ are distinct and so $X = \{a \cdot a_1, a \cdot a_2, \ldots, a \cdot a_n\}$. Hence, one of the elements, say $a \cdot a_i$, must be equal to a. Since X is commutative, $a_i \cdot a = a \cdot a_i = a$. Let $b \in X$. Then there exists $a_j \in X$ such that $b = a \cdot a_j$. Thus, $b \cdot a_i = a_i \cdot b = a_i \cdot (a \cdot a_j) = (a_i \cdot a) \cdot a_j = a \cdot a_j = b$. This implies that a_i is the unity of X. We denote the unity of X by 1. Now, $1 \in X = \{a \cdot a_1, a \cdot a_2, \ldots, a \cdot a_n\}$ and so one of the elements, say $a \cdot a_k$, must be equal to 1. By commutativity, $a_k \cdot a = a \cdot a_k = 1$. Hence, every nonzero element of X is 1-invertible. Therefore, X is an f-UP-field. □

As a consequence of Theorem 10, the following corollary holds.

Corollary 1. *Every finite f-UP-domain is an f-UP-field.*

4. f-UP-Ideal and the Quotient f-UP-semigroup

Definition 15. A nonempty subset I of an f-UP-semigroup X is called an f-UP-*ideal* of X if the following hold:

(fUPI1) the constant 0 of X is in I,

(fUPI2) for any $x, y, z \in X$, $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$, and

(fUPI3) for any $a \in I, x \in X$, $a \cdot x, x \cdot a \in I$.

Obviously, the subsets $\{0\}$ and X are f-UP-ideals of X. Consider the f-UP-semigroup in Example 4. Routine calculations show that the set $I_1 = \{0, a, b\}$ is an f-UP-ideal of X while the set $I_2 = \{0, b, c\}$ is not an f-UP-ideal of X since $b \cdot c = a \notin I_2$.

Theorem 11. *Let $(X; *, \cdot, 0)$ be an f-UP-semigroup and I an f-UP-ideal of X. Then I is an f-UP-subsemigroup of X.*

Proof. By (fUP1), $(X; *, 0)$ is a UP-algebra and by definition, I is a UP-ideal of the UP-algebra X. By Theorem 2, I is a UP-subalgebra of X. Let $x, y \in I \subseteq X$. Then by Proposition 2, $x * y \in I$. Since I is an f-UP-ideal of the f-UP-semigroup X, $x \cdot y \in I$ by (fUPI3). Thus, I is an f-UP-subsemigroup of X by Theorem 5. □

Theorem 12. *Let X be an f-UP-semigroup and $\{A_i : i \in \mathscr{I}\}$ be a nonempty collection of f-UP-ideals of X. Then $\bigcap_{i \in \mathscr{I}} A_i$ is an f-UP-ideal of X.*

Proof. Suppose $\{A_i : i \in \mathscr{I}\}$ is a nonempty collection of f-UP-ideals of X. Since $0 \in A_i$ for all $i \in \mathscr{I}$, $0 \in \bigcap_{i \in \mathscr{I}} A_i$ and so $\bigcap_{i \in \mathscr{I}} A_i \neq \varnothing$. Suppose $x, y, z \in X$ such that $x * (y * z) \in \bigcap_{i \in \mathscr{I}} A_i$ and $y \in \bigcap_{i \in \mathscr{I}} A_i$. Then $x * (y * z) \in A_i$ and $y \in A_i$ for all $i \in \mathscr{I}$. Since each A_i is an f-UP-ideal for all $i \in \mathscr{I}$, it follows that $x * z \in A_i$ for all $i \in \mathscr{I}$. Hence, $x * z \in \bigcap_{i \in \mathscr{I}} A_i$. Let $a \in \bigcap_{i \in \mathscr{I}} A_i$ and $x \in X$. Then $a \in A_i$ for all $i \in \mathscr{I}$. Since each A_i is an f-UP-ideal for all $i \in \mathscr{I}$, $a \cdot x, x \cdot a \in A_i$ for all $i \in \mathscr{I}$. Hence, $a \cdot x, x \cdot a \in \bigcap_{i \in \mathscr{I}} A_i$. Therefore, $\bigcap_{i \in \mathscr{I}} A_i$ is an f-UP-ideal of X. □

Let $(X; *, \cdot, 0)$ be an f-UP-semigroup and I be an f-UP-ideal of X. Define the binary relation $\sim_I$ on X as follows: for all $x, y \in X$, $x \sim_I y$ if and only if $x * y \in I$ and $y * x \in I$. Denote $[x]_I$ as the equivalence class containing $x \in X$ and X/I as the set of all equivalence classes of X with respect to "$\sim_I$", that is, $[x]_I = \{y \in X : x \sim_I y\}$ and $X/I = \{[x]_I : x \in X\}$.

Remark 5. *Let X be an f-UP-semigroup and I be an f-UP-ideal of X. Then $x \in [x]_I$ for all $x \in X$.*

Lemma 1. *Let X be an f-UP-semigroup and I be an f-UP-ideal of X. Then $[x]_I = [y]_I$ if and only if $x \sim_I y$.*

Proof. Suppose $[x]_I = [y]_I$. Since $y \in [y]_I = [x]_I$, we have $x \sim_I y$. Conversely, suppose $x \sim_I y$. Let $z \in [x]_I$. Then $x \sim_I z$. By symmetric property, $z \sim_I x$. By transitivity, $z \sim_I y$ and by symmetric property, $y \sim_I z$ and so, $z \in [y]_I$. Thus, $[x]_I \subseteq [y]_I$. Let $z \in [y]_I$. Then $y \sim_I z$. By transitivity, $x \sim_I z$, that is, $z \in [x]_I$. Thus, $[y]_I \subseteq [x]_I$. Hence, $[x]_I = [y]_I$. □

Proposition 4. *Let X be an f-UP-semigroup and I be an f-UP-ideal of X. Then*

(*i*) $[0]_I = I$,

(*ii*) $[x]_I = I$ *if and only if* $x \in I$, *for all* $x \in I$, *and*

(*iii*) $I * [x]_I = [x]_I$ *for all* $x \in X$.

Proof. Let I be an f-UP-ideal of X.

(*i*) If $x \in [0]_I$, then by definition, $0 \sim_I x$ and by (UP2), $x = 0 * x \in I$. Thus, $[0]_I \subseteq I$. Let $x \in I$. By (UP2), $0 * x = x \in I$. By (UP3) and (fUPI1), $x * 0 = 0 \in I$. Thus, $0 \sim_I x$ and so, $x \in [0]_I$. Hence, $I \subseteq [0]_I$. Therefore, $[0]_I = I$.

(*ii*) Suppose $[x]_I = I$. Then by Remark 5, $x \in I$. Conversely, let $x \in I$. By (UP2), $0 * x = x \in I$. By (UP3) and (fUPI1), $x * 0 = 0 \in I$. Thus, $0 \sim_I x$, and by Lemma 1, $[0]_I = [x]_I$. By (*i*), $I = [x]_I$.

(*iii*) For all $x \in X$, $[x]_I = [0 * x]_I = [0]_I * [x]_I$ as defined in Theorem 3(*iv*). By (*i*), $[x]_I = I * [x]_I$. □

Theorem 13. *If X is an f-UP-semigroup and I an f-UP-ideal of X, then $(X/I; *, \cdot, [0]_I)$ is an f-UP-semigroup, where $*$ and $\cdot$ are defined by $[x]_I*[y]_I = [x*y]_I$ and $[x]_I\cdot[y]_I = [x\cdot y]_I$, respectively. If X is commutative, then X/I is commutative and if X has unity, then X/I has unity.*

Proof. Let I be an f-UP-ideal of X. Then I is a UP-ideal of the UP-algebra $(X; *, 0)$. By Theorem 3, $(X/I; *, [0]_I)$ is a UP-algebra, where $*$ is defined by $[x]_I * [y]_I = [x*y]_I$. We show that the binary operation $\cdot$ on X/I is well-defined. Let $[x]_I = [x']_I$ and $[y]_I = [y']_I$. Then $x \sim_I x'$ and $y \sim_I y'$ which imply $x * x', x' * x, y * y', y' * y \in I$. By Theorem 4($iii$), (UP2), and ($f$UPI3), $(x \cdot y) * (x \cdot y') = x \cdot (y * (0 * y')) = x \cdot (y * y') \in I$ and $(x \cdot y') * (x \cdot y) = x\cdot(y'*(0*y)) = x\cdot(y'*y) \in I$. Thus, $x\cdot y \sim_I x\cdot y'$. Similarly, $(x\cdot y')*(x'\cdot y') = (x*(0*x'))\cdot y' = (x*x')\cdot y' \in I$ and $(x'\cdot y')*(x\cdot y') = (x'*(0*x))\cdot y' = (x'*x)\cdot y' \in I$. Thus, $x\cdot y' \sim_I x'\cdot y'$. By transitivity, $x \cdot y \sim_I x' \cdot y'$. By Lemma 1, $[x]_I \cdot [y]_I = [x \cdot y]_I = [x' \cdot y']_I = [x']_I \cdot [y']_I$.

Let $[x]_I, [y]_I, [z]_I \in X/I$. Since $(X, \cdot)$ is a semigroup, then

$$\begin{aligned} [x]_I \cdot ([y]_I \cdot [z]_I) &= [x]_I \cdot [y \cdot z]_I \\ &= [x \cdot (y \cdot z)]_I \\ &= [(x \cdot y) \cdot z]_I \\ &= [x \cdot y]_I \cdot [z]_I \\ &= ([x]_I \cdot [y]_I) \cdot [z]_I. \end{aligned}$$

Hence, $(X/I, \cdot)$ is semigroup. Moreover, by distributive property on X,

$$\begin{aligned} [x]_I \cdot ([y]_I * [z]_I) &= [x]_I \cdot [y * z]_I \\ &= [x \cdot (y * z)]_I \\ &= [(x \cdot y) * (x \cdot z)]_I \\ &= [x \cdot y]_I * [x \cdot z]_I \\ &= ([x]_I \cdot [y]_I) * ([x]_I \cdot [z]_I) \end{aligned}$$

and

$$\begin{aligned} ([x]_I * [y]_I) \cdot [z]_I &= [x * y]_I \cdot [z]_I \\ &= [(x * y) \cdot z]_I \\ &= [(x \cdot z) * (y \cdot z)]_I \\ &= [x \cdot z]_I * [y \cdot z]_I \\ &= ([x]_I \cdot [z]_I) * ([y]_I \cdot [z]_I). \end{aligned}$$

Thus, the distributive property holds on X/I. Therefore, $(X/I; *, \cdot, [0]_I)$ is an f-UP-semigroup. Suppose X is commutative. Then $x \cdot y = y \cdot x$ for all $x, y \in X$. Let $[x]_I, [y]_I \in X/I$. Then $[x]_I \cdot [y]_I = [x \cdot y]_I = [y \cdot x]_I = [y]_I \cdot [x]_I$. Hence, X/I is commutative. If X has unity 1, then X/I has unity $[1]_I$ since $[x]_I\cdot[1]_I = [x\cdot 1]_I = [x]_I$ and $[1]_I\cdot[x]_I = [1\cdot x]_I = [x]_I$ for any $x \in X$. □

The f-UP-semigroup $(X/I; *, \cdot, [0]_I)$ in Theorem 13 is called the *quotient* f-UP-*semigroup* of X by I.

5. Conclusion

This paper investigated fully UP-semigroups, a new class of algebra related to UP-algebras and semigroups, which was introduced by A. Iampan [4] in 2018. It established some structural properties of f-UP-semigroups. It also introduced and examined f-UP-fields, f-UP-domains, f-UP-ideals, and quotient f-UP-semigroups. Moreover, the relationship between an f-UP-field and an f-UP-domain is determined. In the subsequent study, we introduce and investigate homomorphisms on f-UP-semigroups, which lead to the isomorphism theorems on f-UP-semigroups.

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References

[1] M. Ansari A. Haidar and A. Koam. On a Graph Associated to UP-algebras. *Mathematical and Computational Applications*, 23(61), 2018.

[2] J. Endam and J. Vilela. On JB-semigroups. *Applied Mathematical Sciences*, 9(59): 2901-2911, 2015.

[3] A. Iampan. A new branch of the logical algebra: UP-algebras. *Journal of Algebra and Related Topics*, 5(1):35-54, 2017.

[4] A. Iampan. Introducing fully UP-semigroups. *Discussiones Mathematicae, General Algebra and Applications*, 38:297-306, 2018.

[5] Y. Imai and K. Iseki. On Axiom Systems of Propositional Calculi XIV. *Proc. Japan Academy*, 42:19-22, 1996.

[6] K. Iseki. Algebra Related with a Propositional Calculus. *Proc. Japan Academy*, 42:351-366, 1966.

[7] Y. Jun S. Hong and E.Roh. BCI-semigroups. *Honam Math.*, 15(1):59-64, 1993.

[8] Y. Jun E. Roh and X. Xin. A Class of Algebras Related to BCI-algebras and Semigroups. *Soochow J. Math.*, 4 (24):309-321.

[9] Y. Jun E. Roh and X. Xin. I-ideals Generated by a Set in IS-algebras. *Bull Kcrean Math. Soc.*, 35: 615-624, 1998.

[10] F. Kareem and E. Hasan. On KU-semigroups. *International Journal of Science and Nature*, 9(1):79-84, 2018.

[11] C. Prabpayak and U. Leerawat. On ideals and congruence in KU-algebras. *Scientia Magna Journal*, 5(1):54-57, 2009.