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Hankel Transform of the Second Form (q, r)-Dowling Numbers

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Abstract. In this paper, using the rational generating for the second form of the q-analogue of r-Whitney numbers of the second kind, certain divisibility property for this form is established. Moreover, the Hankel transform for the second form of the q-analogue of r-Dowling numbers is derived.

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1. Introduction

The matrix of the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & a_3 & \dots & a_{n+1} \\ a_2 & a_3 & a_4 & \dots & a_{n+2} \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n+1} & a_{n+2} & \dots & a_{2n} \end{bmatrix} \tag{1}$$

whose entries are the elements of the sequence $A = (a_n)_{n=0}^{\infty}$ was defined in [16] as the *Hankel matrix* of order n of a sequence A, denoted by H_n. This can also be written as $H_n = (a_{i+j})_{0\le i,j\le n}$. In the same paper [16], the *Hankel determinant* h_n of order of n of A was defined as the determinant of the corresponding Hankel matrix of order n, (i.e. $h_n = det(H_n)$) and the *Hankel transform* of the sequence A, denoted by $H(A)$, was defined as the sequence $\{h_n\}$ of Hankel determinants of A.

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For example, the sequence of (r, β)-Bell numbers in [12, 15], denoted by $\{G_{n,r,\beta}\}$, has possessed the following Hankel transform (see [14])

$$H(G_{n,r,\beta}) = \prod_{j=0}^{n} \beta^j j!.$$

As mentioned in [16], one can easily verify that the (r, β)-Bell numbers are simply the r-Dowling numbers $D_{m,r}(n)$, which are defined in [5] as

$$D_{m,r}(n) = \sum_{k=0}^{n} W_{m,r}(n, k)$$

where $W_{m,r}(n, k)$ denotes the r-Whitney numbers of the second kind introduced by Mezo in [29]. In [14], the authors have also tried to derive the Hankel transform of the sequence of q-analogue of (r, β)-Bell numbers. In this attempt, they used the q-analogue defined in [17]. But they failed to derive it.

Just recently, another definition of q-analogue of r-Whitney numbers of the second $W_{m,r}[n, k]_q$ was introduced in [13, 16] by means of the following triangular recurrence relation

$$W_{m,r}[n, k]_q = q^{m(k-1)+r} W_{m,r}[n-1, k-1]_q + [mk + r]_q W_{m,r}[n-1, k]_q. \tag{2}$$

From this definition, two more forms of the q-analogue were defined in [13, 16] as

$$W^*_{m,r}[n, k]_q := q^{-kr-m\binom{k}{2}} W_{m,r}[n, k]_q \tag{3}$$

$$\widetilde{W}_{m,r}[n, k]_q := q^{kr} W^*_{m,r}[n, k]_q = q^{-m\binom{k}{2}} W_{m,r}[n, k]_q, \tag{4}$$

where $W^*_{m,r}[n, k]_q$ and $\widetilde{W}_{m,r}[n, k]_q$ denote the second and third forms of the q-analogue, respectively. Corresponding to these, three forms of q-analogues for r-Dowling numbers may be defined as follows:

$$D_{m,r}[n]_q := \sum_{k=0}^{n} W_{m,r}[n, k]_q \tag{5}$$

$$D^*_{m,r}[n]_q := \sum_{k=0}^{n} W^*_{m,r}[n, k]_q \tag{6}$$

$$\widetilde{D}_{m,r}[n]_q := \sum_{k=0}^{n} \widetilde{W}_{m,r}[n, k]_q. \tag{7}$$

However, among these three forms, only the third form was considered in [16] and was given the Hankel transform as follows

$$H(\widetilde{D}_{m,r}[n]_q) = q^{m\binom{n+1}{3}-rn(n+1)} [0]_{q^m}![1]_{q^m}! \dots [n]_{q^m}![m]_q^{\binom{n+1}{2}}. \tag{8}$$

This Hankel transform was derived using the Hankel transform of q-exponential polynomials in [20], the Layman's Theorem in [26] and the Spivey-Steil Theorem in [34]. This method cannot be used to derive the Hankel transform of the first and second forms of q-analogues for r-Dowling numbers. But the method used by Cigler in [8] is found to be useful to derive the Hankel transforms for the second form of the q-analogue of r-Dowling numbers.

In this paper, the Hankel transform for the sequence $\left(D^*_{m,r}[n]_q\right)_{n=0}^{\infty}$ will be established using Cigler's method [8]. However, a more general form of $D^*_{m,r}[n]_q$, denoted by $\varphi_n[x,r,m]_q$, is considered, which is defined in polynomial form as follows:

$$\varphi_n[x,r,m]_q = \sum_{k=0}^{n} W^*_{m,r}[n,k][x]_q^n, \tag{9}$$

such that, when $x=1$, $\varphi_n[1,r,m]_q = D^*_{m,r}[n]_q$.

2. A q-Analogue of $W_{m,r}(n,k)$: Second Form

The second form of q-analogue of $W_{m,r}(n,k)$ is a kind of generalization of the q-analogue considered by Cigler [8]. This q-analogue possessed several properties (see [13]) including certain combinatorial interpretation in terms of A-tableau, which is defined in [27] to be a list ϕ of column c of a Ferrer's diagram of a partition λ(by decreasing order of length) such that the lengths $|c|$ are part of the sequence $A=(r_i)_{i\geq 0}$, a strictly increasing sequence of nonnegative integers. By making use of the following explicit formula in symmetric function form [13]

$$\begin{aligned} W_{m,r}[n,k]_q &= q^{m\binom{k}{2}+kr} \sum_{S_1+S_2+\cdots S_k=n-k} \prod_{j=1}^{k} [mj+r]_q^{S_j} \\ &= \sum_{0\leq j_1\leq j_2\leq\cdots j_{n-k}\leq k} q^{m\binom{k}{2}+kr} \prod_{i=1}^{n-k} [mj_i+r]_q, \end{aligned} \tag{10}$$

we have

$$W^*_{m,r}[n,k]_q = \sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{n-k}\leq k} \prod_{i=1}^{n-k} [mj_i+r]_q. \tag{11}$$

In [16], $W^*_{m,r}[n,k]$ was expressed as

$$W^*_{m,r}[n,k] = \sum_{\phi\in T_r^A(k,n-k)} \prod_{c\in\phi} \omega(|c|)$$

where $T_r^A(h,l)$ denotes the set of A-tableau with l columns of lengths $|c|\leq h$ and $\omega(|c|) = [m|c|+r]_q$. Using the combinatorics of A-tableau, the following identities were established in [16]:

$$W^*_{m,r}[n,k]_q = \sum_{j=k}^{n}(-1)^{n-j}\binom{n}{j}q^{-nr_2}[r_2]_q^{n-j}W^*_{m,r_1}[j,k]_q \quad (12)$$

$$W^*_{m,r}[n+1,m+j+1]_q = \sum_{k=0}^{n} W^*_{m,r}[k,m]_q W^*_{m,r-m-1}[n-k,j]_q \quad (13)$$

$$W^*_{m,r}[s+p,t]_q = \sum_{k=max\{0,t-p\}}^{min\{t,s\}} W^*_{m,r}[s,k]_q W^*_{m,r+mk}[p,t-k]_q. \quad (14)$$

Moreover, the convolution-type identity (14) has been used in [13] to derive the following Hankel determinant

$$\det\left(W^*_{m,r}[s+i+j,s+j]_q\right)_{0\le i,j\le n} = \prod_{k=0}^{n}[m(s+k)+r]_q^k.$$

Another interesting property of $W^*_{m,r}[n,k]_q$ is the divisibility property. One can easily observe that, using the triangular recurrence relation of $W_{m,r}[n,k]_q$ in (2), we can generate the following table of values

n/k	0	1	2	3
0	1			
1	$[r]_q$	q^r		
2	$[r]_q^2$	$q^r\left([r]_q+[m+r]_q\right)$	q^{m+2r}	
2	$[r]_q^2$	$q^r\left([r]_q+[m+r]_q\right)$	q^{m+2r}	
3	$[r]_q^3$	$q^r[r]_q^2+q^r[r]_q[m+r]_q$ $+q^r[m+r]_q^2$	$q^{m+2r}\left([r]_q+[m+r]_q\right)$ $q^{m+2r}\left(+[2m+r]_q\right)$	q^{3m+3r}

Then, we can generate the first values of $W^*_{m,r}[n,k]_q$ as follows

n/k	0	1	2	3
0	1			
1	$[r]_q$	1		
2	$[r]_q^2$	$[r]_q+[m+r]_q$	1	
3	$[r]_q^3$	$[r]_q^2+[r]_q[m+r]_q+[m+r]_q^2$	$[r]_q+[m+r]_q+[2m+r]_q$	1

Note that $[n]_q = 1+q+q^2+\ldots+q^{n-1}$. Based on the preceding table, the constant values of $W^*_{m,r}[n,k]_q$ from row 0 to row 3 form the following triangle of numbers

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1. \end{array}$$

This can be written as

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}, \end{array}$$

which is a portion of Pascal's triangle. The following theorem generalizes the above observation.

Theorem 2.1. *The q-analogue $W^*_{m,r}[n,k]_q$ satisfies the following congruence relations*

$$W^*_{m,r}[n,k]_q \equiv \binom{n}{k} \pmod q. \tag{15}$$

Proof. We recall the rational generating function [13] for $W^*_{m,r}[n,k]_q$ is given by

$$\Psi^*_k(t) = \sum_{n\geq 0} W^*_{m,r}[n,k]_q [t]_q^n = \frac{[t]_q^k}{\prod_{j=0}^k (1-[mj+r]_q[t]_q)}.$$

Since

$$\begin{aligned} \frac{1}{1-[mj+r]_q[t]_q} &= \sum_{n\geq 0}[mj+r]_q^n[t]_q^n \\ &= \sum_{n\geq 0}(1+q+q^2+...+q^{mj+r-1})^n[t]_q^n \\ &= \sum_{n\geq 0}(1+qy)^n[t]_q^n, \end{aligned}$$

where y in q. Then

$$\frac{1}{1-[mj+r]_q[t]_q} = \sum_{n\geq 0}(1+qz_n)[t]_q^n$$

for some polynomial z_n in q. Hence,

$$\begin{aligned} \frac{1}{1-[mj+r]_q[t]_q} &= \sum_{n\geq 0}[t]_q^n + q\sum_{n\geq 0} z^n[t]_q^n \\ &\equiv \sum_{n\geq 0}[t]_q^n \pmod q \equiv \left(\frac{1}{1-[t]_q}\right) \pmod q \end{aligned}$$

Then

$$\Psi^*_k(t) = \sum_{n\geq 0} W^*_{m,r}[n,k]_q [t]_q^n = \frac{[t]_q^k}{\prod_{j=0}^k (1-[mj+r]_q[t]_q)}$$

$$\equiv [t]_q^k \left(\frac{1}{(1-[t]_q)^{k+1}}\right) \pmod q.$$

Using the Newton's Binomial Theorem, we have

$$\begin{aligned}\sum_{n\geq 0} W^*_{m,r}[n,k]_q[t]_q^n &\equiv [t]_q^k \sum_{n\geq 0}\binom{n+(k+1)-1}{n}[t]_q^n \pmod q\\ &\equiv \sum_{n\geq 0}\binom{n+k}{n}[t]_q^{n+k} \pmod q\\ &\equiv \sum_{n\geq k}\binom{n-k+k}{n-k}[t]_q^{n+k-k} \pmod q\\ &\equiv \sum_{n\geq k}\binom{n}{k}[t]_q^{n} \pmod q.\end{aligned}$$

Comparing the coefficients of $[t]_q^n$ completes the proof of the theorem.

3. Hankel Transform of $D^*_{m,r}[n]_q$

We recall that the horizontal generating function for $W_{m,r}[n,k]_q$ is given by

$$\sum_{k=0}^{n} W_{m,r}[n,k]_q[x-r|m]_{k,q} = [x]_q^n. \tag{16}$$

Using the fact that

$$[x-r|m]_{k,q} = q^{-kr-m\binom{k}{2}}\langle x\rangle_{r,m,k},$$

where $\langle x\rangle_{r,m,k} = \prod_{j=0}^{n-1}([x]_q-[r+jm]_q)$, we can write (16) as follows

$$\sum_{k=0}^{n} q^{-kr-m\binom{k}{2}} W_{m,r}[n,k]_q\langle x\rangle_{r,m,k} = [x]_q^n$$

$$\sum_{k=0}^{n} W^*_{m,r}[n,k]_q\langle x\rangle_{r,m,k} = [x]_q^n.$$

Using the method of Cigler [8], let $d[n,k] = det\ (a_{i+j+k})_{i,j=0}^{n-1}$ denote the kth Hankel determinant. That is, the $0th$ Hankel determinant is given by

$$d[n,0] = \det\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1}\\ a_1 & a_2 & a_3 & \dots & a_n\\ \dots & \dots & \dots & \dots & \dots\\ a_{n-1} & a_n & a_{n+1} & \dots & a_{2n-2}\end{bmatrix}$$

and the $1st$ Hankel determinant is given by

$$d[n,1] = \det \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2 & a_3 & a_4 & \dots & a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n+1} & a_{n+2} & \dots & a_{2n-1} \end{bmatrix}.$$

Now, define a linear functional F on the polynomial by

$$F(x^n) = a_n$$

By Gram-Schmidt orthogonalization process, there exists a sequence of orthogonal polynomials

$$p_n(x) = c_{0,n} + c_{1,n}x + \ldots + c_{n-1,n}x^{n-1} + x^n \quad (c_{n,n} = 1)$$

with respect to F such that

$$p_n(x) = \frac{1}{d[n,0]} \det \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_1 & a_2 & a_3 & \dots & a_n & x \\ a_2 & a_3 & a_4 & \dots & a_{n+1} & x^2 \\ \dots & \dots & \dots & \dots & \dots & \\ a_n & a_{n+1} & a_{n+2} & \dots & a_{2n\cdot} & x^n \end{bmatrix} \tag{17}$$

where $p_n(x) := 1$. This means that

$$F(p_n p_k) = d_n[n = k] \text{ with } d_n \neq 0.$$

Then

$$d[n,0] = \prod_{i=0}^{n-1} d_i.$$

Clearly, from (17), we have

$$p_n(0) = c_{0,n} = \frac{1}{d[n,0]}(-1)^n d[n,1].$$

Hence, we have

$$d[n,1] = d[n,0](-1)^n p_n(0). \tag{18}$$

First, let us consider the Hankel transform of $\varphi_n[x,r,m]_q$ corresponding to the $0th$ Hankel determinant.

Theorem 3.1. *The Hankel transform of $\varphi_n[x,r,m]_q$ corresponding to the* 0*th Hankel determinant is given by*

$$H(\varphi_n[x,r,m]_q) = ([m]_q[x]_q)^{\binom{n}{2}} q^{r\binom{n}{2}+\binom{n}{3}} \prod_{k=0}^{n-1} [k]_{q^m}!$$

Proof. We prove this theorem using the method of Cigler [8]. First, consider a linear operator $U_{r,q}$ on the polynomials defined by

$$U_{r,q}\langle x\rangle_{r,m,n} = [x]_q^n \quad \text{where} \quad U_{r,q}[x]_q U_{r,q}^{-1} = [x]_q(1 + [x]_q^{-r} D[x]_q^r)$$

Then, we have

$$\begin{aligned} U_{r,q}[x]_q U_{r,q}^{-1}[x]_q^n &= U_r[x]_q\langle x\rangle_{r,m,n} \\ &= U_{r,q}(\langle x\rangle_{r,m,n+1} + [r+n]_q\langle x\rangle_{r,m,n}) \\ &= [x]_q^{n+1} + [r+n]_q[x]_q^n \\ &= [x]_q(1 + [x]_q^{-r}D[x]_q^r)[x]_q^n. \end{aligned}$$

Let $F_{r,q}$ be the linear function defined by

$$F_{r,q}(\langle x\rangle_{r,m,n}) = [a]_q^n.$$

The orthogonal polynomial with respect to $F_{r,q}$ is given by

$$h_{n,q}(x,a,r,m) = \sum_{k=0}^{n}(-[a]_q)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \langle x\rangle_{r,m,n-k},$$

which is a kind of q-Poisson-Charlier polynomials satisfying the following recurrence relation

$$\begin{aligned} h_{n+1,q}(x,a,r,m) &= ([x]_q - [mn+r]_q - q^n[a]_q)h_{n,q}(x,a,r,m) \\ &\quad - q^{r+mn-1}[a]_q[n]_q h_{n-1,q}(x,a,r,m). \end{aligned}$$

Now, consider the following polynomial in $[x]_q$

$$p_{n,q}(x,a) = \prod_{k=0}^{n-1}\left([x]_q - q^k[a]_q\right) = \sum_{k=0}^{n}(-[a]_q)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q [x]_q^{n-k}.$$

By applying the linear operator $U_{r,q} : \langle x\rangle_{r,m,k} \mapsto [x]_q^k$ to $h_{n,q}(x,a,r,m)$,

$$U_r h_{n,q}(x,a,r,m) = \sum_{k=0}^{n}(-[a]_q)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q [x]_q^{n-k} = p_{n,q}(x,a).$$

This implies that

$$U_{r,q}^{-1}(p_{n,q}(x,a)) = h_{n,q}(x,a,r,m).$$

Then

$$\begin{aligned} U_r[x]_q h_{n,q}(x,a,r,m) &= U_r[x]_q U_{r,q}^{-1}(p_{n,q}(x,a)) \\ &= [x]_q(1 + [x]_q^{-r}D[x]_q^r)p_{n,q}(x,a) \end{aligned}$$

$$= [x]_q p_{n,q}(x,a) + [r+mn]_q p_{n,q}(x,a)$$

Note that

$$p_{n+1,q}(x,q) = \prod_{k=0}^{n} \Big([x]_q - q^k[a]_q\Big) = ([x]_q - q^n[a]_q)\, p_{n,q}(x,q).$$

Hence, $[x]_q p_{n,q}(x,a) = p_{n+1,q}(x,a) + [a]_q q^n p_{n,q}(x,a)$. Using the fact that

$$[r+mn]_q = [r]_q + q^r[mn]_q,$$

we have

$$\begin{aligned} U_r[x]h_{n,q}(x,a,r,m) &= p_{n+1,q}(x,a) + [a]_q q^n p_{n,q}(x,a) \\ &\quad + ([r]_q + q^r[mn]_q)p_{n,q}(x,a) \\ &= p_{n+1,q}(x,a) + [a]_q q^n p_{n,q}(x,a) + [r]_q p_{n,q}(x,a) + q^r[mn]_q p_{n,q}(x,a) \\ &= p_{n+1,q}(x,a) + [a]_q q^n p_{n,q}(x,a) + [r]_q p_{n,q}(x,a) + q^r[mn]_q[x]_q p_{n-1,q}(x,a). \end{aligned}$$

Also, $[x]_q p_{n-1,q}(x,a) = p_{n,q}(x,a) + [a]_q q^{n-1} p_{n-1,q}(x,a)$. Then

$$\begin{aligned} U_r[x]h_{n,q}(x,a,r,m) &= p_{n+1,q}(x,a) + [a]_q q^n p_{n,q}(x,a) \\ &\quad + [r]_q p_{n,q}(x,a) + q^r[mn]_q(p_{n,q}(x,a) + [a]_q q^{n-1} p_{n-1,q}(x,a)) \\ &= p_{n+1,q}(x,a) + [a]_q q^n p_{n,q}(x,a) + [r]_q p_{n,q}(x,a) + q^r[mn]_q p_n(x,a) \\ &\quad + [a]_q[mn]_q q^{r+n-1} p_{n-1,q}(x,a) \end{aligned}$$

Applying $U_{r,q}^{-1}$ yields

$$\begin{aligned} [x]_q h_{n,q}(x,a,r,m) &= h_{n+1,q}(x,a,r,m) + ([a]_q q^n + [r]_q + q^r[mn]_q)h_{n,q}(x,a,r,m) \\ &\quad + [a]_q[mn]_q q^{r+n-1} h_{n-1,q}(x,a,r,m). \end{aligned}$$

Clearly,

$$F_{r,q}(h_{n,q}(x,a,r,m)) = \sum_{k=0}^{n} (-[a]_q)^k \; q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q [a]_q^n = p_{n,q}(a,a) = 0,$$

which implies

$$\begin{aligned} d_{n,q} &= F_{r,q}([x]_q^n \; h_{n,q}(x,a,r,m)) \\ &= q^{r+n-1}[mn]_q[a]_q \; F_{r,q}([x]_q^{n-1} \; h_{n-1,q}(x,a,r,m)) \\ &= \prod_{k=1}^{n} q^{r+k-1}[mk]_q[a]_q = \prod_{k=1}^{n} q^{r+k-1}[k]_{q^m}[m]_q[a]_q \\ &= (q^r[a]_q[m]_q)^n \, q^{\binom{n}{2}}[n]_{q^m}! \end{aligned}$$

Hence, we have

$$d[n,0]_q = \prod_{k=0}^{n-1} d_{k,q}$$

$$= \prod_{k=0}^{n-1} (q^r[m]_q[x]_q)^k \;\; q^{\binom{k}{2}}[k]_{q^m}!$$

$$= (q^r[m]_q[x]_q)^{0+1+2+\ldots+n-1} \;\; q^{\binom{0}{2}+\binom{1}{2}+\binom{2}{2}+\ldots+\binom{n-1}{2}} \prod_{k=0}^{n-1}[k]_{q^m}!$$

$$= (q^r[m]_q[x]_q)^{\binom{n}{2}} \, q^{\binom{n}{3}} \prod_{k=0}^{n-1}[k]_{q^m}!.$$

This is exactly the desired Hankel transform.

As an immediate consequence of Theorem 3.1, we have the following corollary.

Corollary 3.2. *The Hankel transform of* $D^*_{m,r}[n]_q$ *is given by*

$$H(D^*_{m,r}[n]_q) = [m]_q^{\binom{n}{2}} q^{\binom{n}{3}+r\binom{n}{2}} \prod_{k=0}^{n-1}[k]_{q^m}!$$

Proof. This can easily be derived from Theorem 3.1 by letting $x = 1$.

Remark 3.3. When $m = 1$, the Hankel tranform in Corollary 3.2 yields

$$H(D^*_{1,r}[n]_q) = q^{\binom{n}{3}+r\binom{n}{2}} \prod_{k=0}^{n-1}[k]_q!,$$

which is exactly the Hankel transform of the second form of q-noncentral Bell numbers $\widehat{B}^q_{n,a}$ when $r = -a$ in [11] defined by

$$\widehat{B}^q_{n,a} = \sum_{k=0}^{n} S^*_a[n,k].$$

Remark 3.4. When $q \to 1$, Corollary 3.2 gives

$$H(D^*_{m,r}(n)) = m^{\binom{n}{2}} \prod_{k=0}^{n-1} k!,$$

which is exactly the Hankel transform of (r,β)-Bell numbers $G_{n,\beta,r}$ with $\beta = m$ in [14].

Theorem 3.5. *The Hankel transform of* $\varphi_n[x,r,m]_q$ *corresponding to the* 1*st Hankel determinant* $d[n,1]_q$ *is given by*

$$H(\varphi_n[x,r,m]_q) = d[n,1]_q$$
$$= ([m]_q[x]_q)^{\binom{n}{2}} \, q^{r\binom{n}{2}+\binom{n}{3}} \prod_{k=0}^{n-1}[k]_{q^m}! \sum_{k=0}^{n}(-1)^n[x]_q^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1}[r+jm]_q.$$

Proof. Taking $[p_n(x)]_q = h_{n,q}(x,a,r,m)$, we can compute the desired Hankel transform using (18) with

$$\begin{aligned}[p_n(0)]_q = h_{n,q}(0,a,r,m) &= \sum_{k=0}^{n} \left(-[a]_q\right)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q [0-r|m]_{k,q} \\ &= \sum_{k=0}^{n} (-1)^k [a]_q^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \prod_{j=0}^{k-1} [r+jm]_q \\ &= \sum_{k=0}^{n} [a]_q^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1} [r+jm]_q \,.\end{aligned}$$

Hence, we have

$$\begin{aligned} H\left(\varphi_n[x,r,m]_q\right) &= d[n,1]_q = d[n,0]_q(-1)^n[p_n(0)]_q \\ &= \left([m]_q[x]_q\right)^{\binom{n}{2}} q^{r\binom{n}{2}+\binom{n}{3}} \prod_{k=0}^{n-1}[k]_{q^m}! \sum_{k=0}^{n}(-1)^n[x]_q^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1}[r+jm]_q. \end{aligned}$$

4. Recommendation

We observe that the Hankel transform of the second and third forms of the q-analogue of r-Dowling numbers are obtained using different methods. It would be interesting to find a method that can be used to establish the Hankel transform of the first form of the q-analogue of r-Dowling numbers. It may be possible that this method is closely related to the one being applied in this paper.

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