On Strong Resolving Domination in the Join and Corona of Graphs

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Abstract. Let $G$ be a connected graph. A subset $S \subseteq V(G)$ is a strong resolving dominating set of $G$ if $S$ is a dominating set and for every pair of vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that $u \in I_G[v, w]$ or $v \in I_G[u, w]$. The smallest cardinality of a strong resolving dominating set of $G$ is called the strong resolving domination number of $G$. In this paper, we characterize the strong resolving dominating sets in the join and corona of graphs and determine the bounds or exact values of the strong resolving domination number of these graphs.

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1. Introduction

All graphs considered in this study are finite, simple, and undirected connected graphs, that is, without loops and multiple edges. For some basic concepts in Graph Theory, we refer readers to [4].

Let $G = (V(G), E(G))$ be a connected graph. The open neighborhood $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Any element $u$ of $N_G(v)$ is called a neighbor of $v$. The closed neighborhood $N_G[v] = N_G(v) \cup \{v\}$. Thus, the degree of $v$ is given by $deg_G(v) = |N_G(v)|$. Customarily, for $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$.

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A nonempty set $S \subseteq V(G)$ is a dominating set in graph $G$ if $N_G[S] = V(G)$. Otherwise, we say $S$ is a non-dominating set of $G$. The domination number of a graph $G$, denoted by $\gamma(G)$, is given by $\gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\}$. If $|S| = \gamma(G)$, then $S$ is said to be a minimum dominating set or $\gamma$-set of $G$.

A vertex $w \in S$ strongly resolves two different vertices $u, v \in V(G)$ if $v \in I_G[u, w]$ or if $u \in I_G[v, w]$. A set $W$ of vertices in $G$ is a strong resolving set of $G$ if every two vertices of $G$ are strongly resolved by some vertex of $W$. The smallest cardinality of a strong resolving set of $G$ is called the strong resolving domination number $\gamma_{sr}(G)$ of $G$ and is denoted by $sdim(G)$.

A subset $S \subseteq V(G)$ is a strong resolving dominating set of $G$ if it is both strong resolving and dominating. The smallest cardinality of a strong resolving dominating set of $G$ is called the strong resolving domination number of $G$ and is denoted by $\gamma_{sr}(G)$. A strong resolving dominating set of cardinality $\gamma_{sr}(G)$ is called a $\gamma_{sr}$-set of $G$.

A clique in a graph $G$ is a complete induced subgraph of $G$. A clique $C$ in $G$ is called a superclique if for every pair of distinct vertices $u, v \in C$, there exists $w \in (V(G) \setminus C)$ such that $w \in N_G(u) \setminus N_G(v)$ or $w \in N_G(v) \setminus N_G(u)$. A superclique $C$ in $G$ is called a dominated superclique if for every $u \in C$, there exists $v \in V(G) \setminus C$ such that $uv \in E(G)$ [3]. A superclique (resp. dominated superclique) $C$ is maximum in $G$ if $|C| \geq |C^*|$ for all supercliques (resp. dominated supercliques) $C^*$ in $G$. The superclique (resp. dominated superclique) number, $\omega_S(G)$ (resp. $\omega_{DS}(G)$) of $G$ is the cardinality of a maximum superclique (resp. maximum dominated superclique) in $G$.

In recent years, the concept of domination in graphs has been studied extensively and several research papers have been published on this topic. The said concept was not formally defined mathematically until the publications of the books by Claude Berge [1] in 1958 and Oystein Ore in 1962. In 1977, a survey paper by Cockayne and Hedetniemi [2] began to study the concept of domination.

On the other hand, the problem of uniquely recognizing the possible position of an intruder such as fault in a computer network and spoiled device was the principal motivation in introducing the concept of metric dimension in graphs.

Slater [6] brought in the notion of locating sets and its minimal cardinality as locating number. The same concept was also introduced by Harary and Melter [4] but using the terms resolving sets and metric dimension to refer to locating sets and locating number, respectively.

In 2007, Oellerman and Peter-Fransen [5] introduced the strong resolving graph $G_{SR}$ of a connected graph $G$ as a tool to study the strong metric dimension of $G$.

This study aims to define and characterize the strong resolving dominating sets and determine the exact values or bounds in the join and corona of two graphs.

2. Preliminary Results

Remark 1. Every strong resolving dominating set of a connected graph $G$ is a dominating set. Hence, $\gamma(G) \leq \gamma_{sr}(G)$.

Remark 2. Every strong resolving dominating set of a connected graph $G$ is a strong resolving set. Thus, $sdim(G) \leq \gamma_{sr}(G)$. 
Remark 3. For any connected graph $G$ of order $n$, $1 \leq \gamma_{sr}(G) \leq n - 1$.

Remark 4. Any superset of a strong resolving dominating set is a strong resolving dominating set.

**Proposition 1.** Let $G$ be a connected graph of order $n \geq 2$. Then,

(i) $\gamma_{sr}(P_n) = \lceil \frac{n+1}{3} \rceil$

(ii) $\gamma_{sr}(K_n) = n - 1$

(iii)

$$
\gamma_{sr}(C_n) = \begin{cases}
2 & \text{if } n = 3 \\
2n - 2 & \text{if } n > 3 \text{ and } n \text{ is odd} \\
\left\lfloor \frac{n+2}{2} \right\rfloor & \text{if } n > 3 \text{ and } n \text{ is even}
\end{cases}
$$

**Proposition 2.** Let $G$ be a connected graph of order $n$ and let

$$A = \{x \in V(G) : \deg_G(x) = n - 1\}.$$ 

If $A \neq \emptyset$ and $C$ is a superclique in $G$, then $|C \cap A| \leq 1$. Moreover, if $C$ is a maximum superclique of $G$, then $|C \cap A| = 1$.

**Remark 5.** Let $G$ be a nontrivial connected graph with $\text{diam}(G) \leq 2$. For distinct vertices $u, v, w \in G$, $u \in I_G[v, w]$ if and only if $d_G(v, w) = 2$ and $u \in N_G(v) \cap N_G(w)$.

**Proposition 3.** Let $G$ be a nontrivial connected graph with $\text{diam}(G) \leq 2$. Then $S = V(G) \setminus C$ is a strong resolving set of $G$ if and only if $C = \emptyset$ or $C$ is a superclique in $G$. In particular, $\text{sdim}(G) = |V(G)| - \omega_S(G)$.

**Proof:** Assume that $S$ is a strong resolving set of $G$. If $S \cap V(G) = V(G)$, then $C = \emptyset$. Suppose $S \subset V(G)$. Let $C = V(G) \setminus S$. Then $S = V(G) \setminus C$. Let $x, y \in C$, where $x \neq y$. Since $S$ is a strong resolving set of $G$, $x$ and $y$ are strongly resolved by some $z \in S$. We may assume that $x \in I_G[y, z]$. Then $d_G(y, z) = 2$ and $x \in N_G(y) \cap N_G(z)$ by Remark 5. Thus, $z \in N_G(x) \cap N_G(y)$, showing that $C$ is a superclique in $G$.

Conversely, assume that $S = V(G) \setminus C$, where $C$ is a superclique in $G$. Let $x, y \notin S$, where $x \neq y$. Then $x, y \in C$. Since $C$ is a superclique in $G$, there exists $z \in S$ such that $z \in N_G(x) \cap N_G(y)$ or $z \in N_G(y) \cap N_G(x)$. Since $\text{diam}(G) = 2$, $d_G(y, z) = 2$ or $d_G(x, z) = 2$. By Remark 5, $x \in I_G[y, z]$ or $y \in I_G[x, z]$. Hence, $S$ is a strong resolving set of $G$.

Suppose $S$ is a strong resolving set of $G$. Then $S = V(G) \setminus C$, where $C$ is a superclique in $G$ and $|C| = \omega_S(G)$. Thus, $\text{sdim}(G) = |S| = |V(G)| - |C| = |V(G)| - \omega_S(G)$. \hfill \qed

3. On Strong Resolving Domination in the Join of Graphs

The **join** of two graphs $G$ and $H$ is the graph $G + H$ with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.
Remark 6. For the joins $\langle v \rangle + P_n$ and $\langle w \rangle + C_n$, it can be verified that $\gamma_{sr}(\langle v \rangle + P_n) = n - 1$ for $n \geq 3$ and $\gamma_{sr}(\langle w \rangle + C_n) = n - 2$ for $n \geq 4$.

Proposition 4. Let $G$ be a connected graph with $\gamma(G) \neq 1$ and let $K_1 = \langle v \rangle$. Then $C \subseteq V(K_1 + G)$ is a superclique of $K_1 + G$ if and only if $|C| = 1$ or $|C| \geq 2$ and $C \setminus \{v\}$ is a superclique of $G$.

Proof: The conditions follow immediately if $C \subseteq V(K_1 + G)$ is a superclique of $K_1 + G$. For the converse, the case when $|C| = 1$ is obvious. Suppose $|C| \geq 2$. Since $C \setminus \{v\}$ is a superclique of $G$, we only need to consider the pair of distinct vertices $z, v \in C$. Since $\gamma(G) \neq 1$, there exists $w \in V(G)$ such that $zw \notin E(G)$. Since $diam(K_1 + G) = 2$, $d_G(z, w) = 2$. Hence, $z \in N_G(v) \setminus N_G(w)$, showing that $C$ is a superclique of $K_1 + G$.

Theorem 1. Let $G$ be a nontrivial connected graph of order $n$ with $\gamma(G) \neq 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a strong resolving dominating set of $K_1 + G$ if and only if $S = V(G)$, or $S = V(K_1 + G) \setminus C$ or $S = V(G) \setminus C^*$ where $C$ and $C^*$ are superclique and dominated superclique, respectively, in $G$.

Proof: Let $S$ be a strong resolving dominating set of $K_1 + G$. Suppose $\gamma(G) \neq 1$. If $v \notin S$, then $S \subseteq V(G)$. By Proposition 3, $S = V(K_1 + G) \setminus (C \cup \{v\}) = V(G) \setminus C$, where $S$ is a dominating set in $K_1 + G$ and $C \cup \{v\}$ is a superclique in $K_1 + G$. By Proposition 4, $C$ is a superclique in $G$. Since $\{v\}$ is a superclique in $K_1 + G, S = V(K_1 + G) \setminus \{v\} = V(G)$. On the other hand, if $v \in S$ and $C = V(K_1 + G) \setminus S$, then $S = V(K_1 + G) \setminus C$ where $C$ is a superclique in $K_1 + G$ by Proposition 3. By Proposition 4, $C \setminus \{v\} = C$ is a superclique in $G$. Conversely, the case when $S = V(G)$ and $S = V(K_1 + G) \setminus C$ follows immediately from Proposition 3. Suppose $S = V(G) \setminus C^*$, where $C^*$ is a dominated superclique in $G$. By Proposition 4, $C \cup \{v\}$ is a superclique of $K_1 + G$. Since $v \notin S$, $S = V(G) \setminus C^* = V(K_1 + G) \setminus (C \cup \{v\})$. By Proposition 3, $S$ is a strong resolving dominating set of $K_1 + G$.

Theorem 2. Let $G$ be a nontrivial connected graph of order $n$ with $\gamma(G) = 1$ and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a strong resolving dominating set of $K_1 + G$ if and only if $S = V(G)$ or $S = V(K_1 + G) \setminus C$ or $S = (V(G) \setminus C^*) \cup \{x \in C^* : \deg(x) = n - 1\}$ where $C$ and $C^*$ are superclique and dominated superclique, respectively, in $G$.

Proof: Let $S$ be a strong resolving dominating set of $K_1 + G$. Suppose $\gamma(G) = 1$. If $v \in S$ and $C = V(K_1 + G) \setminus S$, then $S = V(K_1 + G) \setminus C = \{v\} \cup (V(G) \setminus C)$. By Proposition 3, $C$ is a superclique in $K_1 + G$. Hence for $x, y \in C, x \neq y$, there exists $w \in V(G) \setminus C$ such that $w \in N_G(x) \setminus N_G(y)$ or $w \in N_G(y) \setminus N_G(x)$, showing that $C$ is a superclique in $G$. On the other hand, if $v \notin S$, then $S \subseteq V(G)$. Let $C = V(K_1 + G) \setminus S$. Hence, $S = V(K_1 + G) \setminus C = V(G) \setminus C$. By Proposition 3, $C$ is a superclique in $K_1 + G$. Hence, $C$ is also a superclique in $G$. Since $\gamma(G) = 1$, $A_G = \{z \in V(G) : \deg_G(z) = n - 1\} \neq \emptyset$. By Proposition 2, $|C \cap A_G| = 1$. Let $z \in C \cap A_G$. Since $d_{K_1 + G}(z, v) = 1$ and $d_{K_1 + G}(v) = n$, none of the elements in $S$ strongly resolves $z$ and $v$, a contradiction. Hence, $z \in S$. Thus, $S = (V(G) \setminus C) \cup \{z\}$. In addition, since $\{v\}$ is a superclique in $K_1 + G$, $S = V(G) \setminus C \cup \{z\}$.
Let \( x,y \in V(K+G) \setminus \{v\} = V(G) \). Similarly, if \( C^* = V(G) \setminus S \), then \( C^* \) is a dominated superclique in \( G \).

For the converse, the case when \( S = V(G) \) is trivial. Suppose \( S = V(K+G) \setminus C \), where \( C \) is a superclique in \( G \). Let \( x,y \notin S, x \neq y \). Then \( x,y \in C \) and there exists \( w \in V(G) \setminus C \) such that \( xy \in E(G) \) and \( w \in N_G(x) \setminus N_G(y) \) or \( w \in N_G(y) \setminus N_G(x) \). By Remark 5, \( x \in I_{K+G}[y,w] \) or \( y \in I_{K+G}[x,w] \), showing that \( S \) is a strong resolving dominating set of \( K+G \).

Suppose \( S = (V(G) \setminus C^*) \cup \{ z \in C^* : d_{G^*}(z) = n-1 \} \), where \( C^* \) is a dominated superclique in \( G \). Let \( x,y \notin S, x \neq y \). Then \( x,y \in C^* \). By the same argument above, there exists \( w \in S \) that strongly resolves \( x \) and \( y \). Now, consider the vertices \( x \) and \( y \). Since \( x \notin S \), then \( d_{G^*}(x) < n-1 \). Hence, there exists \( z \in V(G) \) such that \( xz \notin E(G) \). It follows that \( v \in I_{K+G}[x,z] \). Thus, \( S \) is a strong resolving dominating set of \( K+G \). \( \square \)

**Corollary 1.** Let \( P_n = [v_1,v_2,\ldots,v_n] \) and \( C_m = [c_1,c_2,\ldots,c_m,c_1] \) where \( n,m \geq 3 \).

(i) The sets \( V(P_n) \setminus \{v_i,v_{i+1}\} \), for \( i = 2,\ldots,n-2 \) are the strong resolving dominating sets of \( \langle v \rangle + P_n \).

(ii) The sets \( V(C_m) \setminus \{c_i,c_{i+1}\} \) and \( V(C_m) \setminus \{c_1,c_m\} \), for \( i = 1,2,\ldots,m-1 \) are the strong resolving dominating sets of \( \langle v \rangle + C_m \).

**Corollary 2.** Let \( G \) be a nontrivial connected graph of order \( n \). Then

(i) for \( \gamma(G) = 1 \), we have \( \gamma_{sr}(K+G) = n - \omega_S(G) + 1 \);

(ii) for \( \gamma(G) \neq 1 \), we have \( \gamma_{sr}(K+G) = \min \{ \gamma_{sr}(G), n - \omega_S(G) \} \).

The next result follows from Proposition 3, Theorem 1 and Theorem 2.

**Corollary 3.** Let \( G \) be a nontrivial connected graph with \( \text{diam}(G) \leq 2 \). Then

(i) for \( \gamma(G) = 1 \), we have \( \gamma_{sr}(K+G) = s\text{dim}(G) + 1 \);

(ii) for \( \gamma(G) \neq 1 \), we have \( \gamma_{sr}(K+G) = \min \{ \gamma_{sr}(G), s\text{dim}(G) + 1 \} \).

The following theorem gives a characterization of the strong resolving dominating sets in the join of \( K_1 \) and a disconnected graph \( G \).

**Theorem 3.** Let \( K_1 = \langle v \rangle \) and \( G \) be a disconnected graph whose components are \( G_i \) for \( i = 1,2,\ldots,m \). A proper subset \( S \) of \( V(K+G) \) is a strong resolving dominating set of \( K+G \) if and only if \( S = V(G) \) or \( S = V(G) \setminus C_i^* \) or \( S = V(K+G) \setminus C_i \) where \( C_i \) is a superclique in \( G_i \), for \( i = 1,2,\ldots,m \) and \( C_i^* \) is a dominated superclique of \( G_i \).

**Proof:** Let \( S \) be a strong resolving dominating set of \( K+G \). Suppose \( v \notin S \). Then \( S \subseteq V(G) \). Let \( C_i = V(K+G) \setminus S \), for \( i = 1,2,\ldots,m \). Then \( S = V(K+G) \setminus C_i = V(G) \setminus C_i \).

Let \( x,y \in C_i, x \neq y \). Since \( d_{K+G}(w,x) = d_{K+G}(w,y) \), for all \( w \in V(G) \setminus V(G_i) \), there exists \( z \in V(G_i) \setminus C_i \) such that \( x \in I_{G_i}[y,z] \) or \( y \in I_{G_i}[x,z] \). By Remark 5,
\(x \in N_{G_i}(y) \setminus N_{G_i}(z)\) or \(y \in N_{G_i}(x) \setminus N_{G_i}(z)\). Thus, \(C_i\) is a superclique in \(G_i\). Since \(\{v\}\) is a superclique in \(K_1 + G\), by Proposition 3, \(S = (V(K_1 + G) \setminus \{v\}) = V(G)\). On the other hand, if \(v \in S\) and \(C_i = V(K_1 + G) \setminus S\), for \(i = 1, 2, \ldots, m\), then \(S = V(K_1 + G) \setminus C_i\), where \(C_i\) is a superclique in \(K_1 + G\), by Proposition 3. Hence, \(C_i\) is a superclique in \(G_i\). Similarly, if \(C_i^* = V(G) \setminus S\) for \(i = 1, 2, \ldots, m\) where \(C_i^*\) is a dominated superclique of \(G_i\) and since \(S\) is dominating, then \(V(G_i) \setminus C_i^*\) is a dominating set of \(G_i\).

For the converse, if \(S = V(G)\), then we are done. Suppose \(S = V(G) \setminus C_i^*\), or \(S = V(K_1 + G) \setminus C_i\), where \(C_i\) and \(C_i^*\) are superclique and dominated superclique, respectively, in \(G_i\) for \(i = 1, 2, \ldots, m\). Then by Theorem 1 and Theorem 2, \(V(G_i) \setminus C_i^*\) is a strong resolving dominating set of \(K_1 + G_i\). By Remark 4, \(S = V(K_1 + G) \setminus C_i\) is a strong resolving dominating set of \(K_1 + G\).

**Corollary 4.** Let \(G_i\) be connected graphs of orders \(n_i\) and \(G\) be a disconnected graph whose components are \(G_i\) for \(i = 1, 2, \ldots, m\) and \(S_i = V(G_i) \setminus C_i\) where \(C_i\) is a maximum dominated superclique of \(G_i\). Then

\[
\gamma_{sr}(K_1 + G) = \sum_{i=1}^{m} n_i - \max\{\gamma_{sr}(G_i), \omega_{DS}(G_i) + 1\} | i = 1, 2, \ldots, m \}.
\]

In the join of two graphs \(G\) and \(H\), the previous results have already considered the case when \(G\) or \(H\) is trivial. Hence, in the following theorem, a characterization of the strong resolving dominating sets in the join of nontrivial connected graphs \(G\) and \(H\) is considered.

**Theorem 4.** Let \(G\) and \(H\) be nontrivial connected graphs of orders \(m\) and \(n\), respectively. A proper subset \(S\) of \(V(G + H)\) is a strong resolving dominating set of \(G + H\) if and only if at least one of the following is satisfied:

(i) \(S = V(G + H) \setminus C_G\) where \(C_G\) is a superclique in \(G\).

(ii) \(S = V(G + H) \setminus C_H\) where \(C_H\) is a superclique in \(H\).

(iii) If \(\gamma(G) = 1\) and \(\gamma(H) = 1\),

\[
S = [V(G + H) \setminus (C_G \cup C_H)] \cup \{z \in C_G : \deg_G(z) = m - 1\}, \text{ or }
S = [V(G + H) \setminus (C_G \cup C_H)] \cup \{w \in C_H : \deg_H(w) = n - 1\}
\]

where \(C_G\) and \(C_H\) are supercliques in \(G\) and \(H\), respectively.

(iv) If \(\gamma(G) \neq 1\) and \(\gamma(H) \neq 1\),

\[
S = [V(G + H) \setminus (C_G \cup C_H)] = (V(G) \setminus C_G) \cup (V(H) \setminus C_H),
\]

where \(C_G\) and \(C_H\) are supercliques in \(G\) and \(H\), respectively.
Corollary 5. Let \( G \) and \( H \) be nontrivial connected graphs of orders \( m \) and \( n \), respectively. Then

\[
\gamma_{sr}(G + H) = \begin{cases} 
(m - \omega_S(G)) + (n - \omega_S(H)) + 1, & \text{if } \gamma(G) = 1 \text{ and } \gamma(H) = 1 \\
(m - \omega_S(G)) + (n - \omega_S(H)), & \text{if } \gamma(G) \neq 1 \text{ or } \gamma(H) \neq 1.
\end{cases}
\]

Remark 7. If \( G \) is a nontrivial connected graph with \( \gamma(G) = 1 \), then \( \text{diam}(G) \leq 2 \).

Corollary 6. Let \( G \) and \( H \) be nontrivial connected graphs with \( \gamma(G) = 1 \) and \( \gamma(H) = 1 \). Then \( \gamma_{sr}(G + H) = \text{sdim}(G) + \text{sdim}(H) + 1 \). In particular,
(i) $\gamma_{sr}(G + H) = 3$ for $G = P_m$ and $H = P_n$ \((m \geq 2, n \geq 2)\);

(ii) $\gamma_{sr}(G + H) = \left\lceil \frac{n}{2} \right\rceil + 2$ for $G = P_m$ and $H = C_n$ \((m \geq 2, n \geq 3)\);

(iii) $\gamma_{sr}(G + H) = 4$ for $G = C_m$ and $H = C_n$ \((m = n = 3)\)

(iv) $\gamma_{sr}(G + H) = \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + 1$ for $G = C_m$ and $H = C_n$ \((m, n \geq 4)\)

**Theorem 5.** Let $G$ be a disconnected graph with components $G_1, \ldots, G_m$ and $H$ a disconnected graph with components $H_1, \ldots, H_n$. A proper subset $S$ of $V(G + H)$ is a strong resolving dominating set of $G + H$ if and only if $S$ satisfies any of the following:

(i) $S = S_G \cup V(H)$ where $V(G) \setminus S_G$ is a superclique of $G_i$ for some $i \in \{1, 2, \ldots, n\}$;

(ii) $S = S_H \cup V(G)$ where $V(H) \setminus S_H$ is a superclique of $H_j$ for some $j \in \{1, 2, \ldots, m\}$;

(iii) $S = S_G \cup S_H$, where $V(G) \setminus S_G$ and $V(H) \setminus S_H$ are supercliques of $G_i$ and $H_j$, for some $i \in \{1, 2, \ldots, n\}$ and some $j \in \{1, 2, \ldots, m\}$.

**Proof:** Let $S$ be a strong resolving dominating set of $G + H$ and $x \in G_i$ and $y \in G_k$, $i \neq k$. Since $d_{G+H}(x,y) = 2$ and $d_{G+H}(x,h) = d_{G+H}(y,h) = 1$, for all $h \in V(H)$, then $x \in S$ or $y \in S$. Hence, $S \cap V(G) = \emptyset$. Similarly, $S \cap V(H) \neq \emptyset$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Suppose $S \cap V(G) \neq V(G_i)$, then $S_G \subseteq V(G_i)$ for some $i \in \{1, 2, \ldots, n\}$. Then there exists $z \in S \cap V(G_i)$ such that $u \in I_{G+H}[v, z]$ or $v \in I_{G+H}[u, z]$. It follows from Remark 5 that $C_G$ is a superclique of $G_i$. Similarly, $S \cap V(G) = V(G)$. On the other hand, if $S \cap V(G) = V(G_i)$, then $S = S_G \cup S_H$. Hence, $C_G$ and $C_H$ are supercliques of $G_i$ and $H_j$ for some $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$.

Conversely, suppose $S$ satisfies condition (i). Let $u, v \notin S, u \neq v$. Then $u, v \in C_G = V(G) \setminus S_G$. Hence, there exists $w \in V(G_i) \setminus C_G$ such that $w \in N_{G+H}(u) \setminus N_{G+H}(v)$ or $w \in N_{G+H}(v) \setminus N_{G+H}(u)$. By Remark 5, $u \in I_{G+H}[v, w]$ or $v \in I_{G+H}[u, w]$. Thus, $S$ is a strong resolving dominating set of $G + H$. Similarly, the same conclusion holds if $S$ satisfies condition (ii).

Suppose $S$ satisfies condition (iii). Let $u, v \notin S, u \neq v$. If $u, v \in C_G$ or $u, v \in C_H$, then we are done. Assume $u \in C_G$ and $v \in C_H$. Since $d_{G+H}(u, u') = 2$ for $u' \in G_k, k \neq i$ and $d_{G+H}(v, v') = 2$ for $v' \in H_p, p \neq j, v \in I_{G+H}[u, u']$ or $u \in I_{G+H}[v, v']$. Thus, $S$ is a strong resolving dominating set of $G + H$.

4. On Strong Resolving Domination in the Corona of Graphs

The **corona** of two graphs $G$ and $H$, denoted by $G \circ H$, is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining every vertex of the $i$th copy of $H$ to the $i$th vertex of $G$. For $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v, v \in V(G)$. 

Remark 8. For the coronas $P_n \circ K_1$ and $C_n \circ K_1$, it can be verified easily that

$$\gamma_{sr}(P_n \circ K_1) = \gamma_{sr}(C_n \circ K_1) = n, \forall n \geq 3.$$ 

Theorem 6. Let $G$ be a nontrivial connected graph and $H$ a connected graph. A proper subset $S$ of $V(G \circ H)$ is a strong resolving dominating set of $G \circ H$ if and only if one of the following holds:

1. $S = A \cup \left( \bigcup_{u \in V(G)} V(H^u) \right)$ where $A \subseteq V(G)$;
2. $S = A \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup B_v$ for a unique $v \in V(G)$, where $A \subseteq V(G) \setminus \{v\}$ and $B_v$ is a strong resolving dominating set of $H^v$ if $\gamma(H) = 1$ or $B_v$ is a strong resolving dominating set of $\langle v \rangle + H^v$ if $\gamma(H) \neq 1$;
3. $S = A \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup B_v$ for a unique $v \in V(G)$ where $v \in A \subseteq V(G)$ and $B_v$ is a strong resolving set of $H^v$ if $\gamma(H) = 1$ and $B_v$ is a strong resolving set of $\langle v \rangle + H^v$ if $\gamma(H) \neq 1$.

Proof: Suppose $S$ is a strong resolving dominating set of $G \circ H$. Let $A = S \cap V(G)$ and $B_v = S \cap V(H^v)$, where $v \in V(G)$. Consider the following cases:

Case 1. $S \cap V(H^v) = V(H^v)$

Then $B_v = H^v$. Thus, $S = A \cup \left( \bigcup_{u \in V(G)} V(H^u) \right)$.

Case 2. $S \cap V(H^v) \neq V(H^v)$

Let $u, v \in V(G)$, $u \neq v$ such that $S \cap V(H^u) \neq V(H^u)$ and $S \cap V(H^v) \neq V(H^v)$. Pick $p_u \in V(H^u) \setminus S$ and $p_v \in V(H^v) \setminus S$. Then, none of the vertices in $S$ strongly resolves $p_u$ and $p_v$, a contradiction. Thus, the vertex $v \in V(G)$ such that $S \cap V(H^v) \neq V(H^v)$ must be unique. Hence, $S = A \cup \left( \bigcup_{u \in V(G) \setminus \{v\}} V(H^u) \right) \cup B_v$.

Subcase 2.1 $v \in S$

Let $C_v = V(H^v) \setminus B_v$. Hence, $B_v = V(H^v) \setminus C_v$. Then it can be verified that $C_v$ is a superclique in $H^v$. If $\gamma(H) \neq 1$, by Theorem 1, $B_v$ is a strong resolving set of $\langle v \rangle + H^v$. If $\gamma(H) = 1$, then by Remark 7 and Theorem 2, $B_v$ is a strong resolving set of $H^v$.

Subcase 2.2 $v \notin S$

Since $S$ is a dominating set of $G \circ H$, $B_v$ is a dominating set of $H^v$. By similar argument in the proof of subcase 2.1, $B_v$ is a strong resolving dominating set of $H^v$ if $\gamma(H) = 1$ or $B_v$ is a strong resolving dominating set of $\langle v \rangle + H^v$ if $\gamma(H) \neq 1$.

Conversely, suppose (i), (ii) and (iii) hold. Consider the following cases:

Case 1. $p, q \in V(G) \setminus A$

Let $p, q \in V(G \circ H) \setminus S$ where $p \neq q$ and $p = v$ or $q = v$, but not both, then $p \in I_{G \circ H}[q, z]$ or $q \in I_{G \circ H}[p, z]$ for some $z \in B_v$. On the other hand, if $p \neq v$ and $q \neq v$, then $q \in I_{G \circ H}[p, w]$ for some $w \in V(H^q) \subset S$ or $p \in I_{G \circ H}[q, r]$ for some $r \in V(H^p) \setminus B_v$.

Case 2. $p, q \in V(H^v) \setminus B_v$

Since $B_v$ is a strong resolving set of $H^v$, there exists $t \in B_v \subset S$ that strongly resolves $p$ and $q$.

Case 3. $p \in V(G) \setminus (A \cup \{v\})$ and $q \in V(H^p) \setminus B_v$

Since $p \neq v$, then $V(H^p) \subset S$ and $p \in I_{G \circ H}[q, z]$ for all $z \in V(H^p)$.
Case 4. \( p = v, \ q \in V(H^v) \setminus B_v \)

Let \( t \in N_G(v) \). Then \( V(H^t) \subset S \) and \( p \in I_{G \circ H}[q, u] \), for some \( u \in V(H^t) \).

Cases 1 to 4 imply that \( S \) is a strong resolving dominating set of \( G \circ H \) and \((\text{i}), (\text{ii}), (\text{iii})\) imply that \( S \) is a dominating set of \( G \circ H \). Accordingly, \( S \) is a strong resolving dominating set of \( G \circ H \).

Corollary 7. Let \( G \) and \( H \) be connected graphs of orders \( m \) and \( n \), respectively

\[
\gamma_{sr}(G \circ H) = \begin{cases} 
(m - 1)n + \gamma_{sr}(H), & \text{if } \gamma(H) = 1 \\
(m - 1)n + \gamma_{sr}(K_1 + H), & \text{if } \gamma(H) \neq 1 
\end{cases}
\]

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References


