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Simple Properties and Existence Theorem for the Henstock–Kurzweil–Stieltjes Integral of Functions Taking Values on $C[a, b]$ Space-valued Functions

Andrew Felix Cunanan[1,*], Julius Benitez[2]

[1] *Department of Natural Sciences and Mathematics, College of Arts and Sciences, Surigao Del Sur State University, Tandag City, Surigao del Sur, Philippines*
[2] *Department of Mathematics and Statistics, College of Sciences and Mathematics, Mindanao State University-Iligan Institute of Technology, Tibanga, Iligan City, Philippines*

Abstract. Henstock–Kurzweil integral, a nonabsolute integral, is a natural extension of the Riemann integral that was studied independently by Ralph Henstock and Jaroslav Kurzweil. This paper will introduce the Henstock–Kurzweil–Stieltjes integral of $\mathcal{C}[a, b]$-valued functions defined on a closed interval $[f, g] \subseteq \mathcal{C}[a, b]$, where $\mathcal{C}[a, b]$ is the space of all continuous real-valued functions defined on $[a, b] \subseteq \mathbb{R}$. Some simple properties of this integral will be formulated including the Cauchy criterion and an existence theorem will be provided.

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1. Introduction

The Henstock–Kurzweil–Stieltjes integral is a generalized Riemann–Stieltjes integral which has properties similar to it. In the paper [9], Ubaidillah introduce the Henstock–Kurzweil integral of functions taking values in $\mathcal{C}[a, b]$ through Riemann sums

$$S(F, D) = \sum_{D} F(t_i)[h_{i-1}, h_i]$$

where $D = \{([h_{i-1}, h_i], t_i)\}_{i=1}^{n}$ is a tagged division of $[f, g]$ Notion of integrals for Banach space-valued functions like Henstock integral for Banach space-valued functions, Henstock–Stieltjes integral of real-valued functions with respect to an increasing function and Henstock–Stieltjes integral for Banach spaces were already defined by Cao [3],

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Email addresses: deofscunananiv@yahoo.com (A. Cunanan),
julius.benitez@g.msuiit,edu.ph (J. Benitez)

Lim [7] and Tikare [8], respectively. In this paper we change the way to define the domain of the function and the integrator. We shall choose first a closed interval $[f, g]$ as our domain and a continuous real-valued function H instead of the identity map as our integrator.

2. Preliminaries

Throughout, we consider the space $\mathcal{C}[a, b]$ of all continuous real-valued functions defined on $[a, b]$. For more details of the space $\mathcal{C}[a, b]$, see [2], [5] or [9].

Let $[f, g]$ be a closed interval of $\mathcal{C}[a, b]$. A **division** of $[f, g]$ is any finite set $\{h_0, h_1, \dots, h_n\} \subset [f, g]$ such that

$$h_0 = f,\ h_n = g \text{ and } h_{i-1} < h_i$$

for all $i = 1, 2, \dots, n$. A **tagged division** of $[f, g]$ is a finite collection $\{([h_{i-1}, h_i], t_i) : i = 1, 2, \dots, n\}$ of interval−point pairs such that $\{h_0, h_1, \dots, h_n\}$ is a division of $[f, g]$ and $t_i \in [h_{i-1}, h_i]$ for every $i = 1, 2, \dots, n$. Each point t_i is referred to as the tag of the corresponding subinterval $[h_{i-1}, h_i]$. Let θ be the null element in $\mathcal{C}[a, b]$, that is, $\theta(x) = 0$, for all $x \in [a, b]$. A function $\delta : [f, g] \to \mathcal{C}[a, b]$ is said to be a **gauge** on $[f, g]$ if $\theta < \delta(h)$ for every $h \in [f, g]$.

Definition 1. [9] Let δ be a gauge on $[f, g]$. A tagged division $D = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \dots, n\}$ is said to be **δ-fine** if

$$t_i \in [h_{i-1}, h_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$$

for every $i = 1, 2, \dots, n$.

Theorem 1. [9] (Cousin's Lemma) *If δ is a gauge on $[f, g] \subset \mathcal{C}[a, b]$, then there is a δ-fine tagged division of $[f, g]$.*

3. Henstock-Kurzweil-Stieltjes Integral on $\mathcal{C}[a, b]$

Let $D = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \dots, n\}$ be a tagged division of $[f, g]$ and $F, H : [f, g] \to \mathcal{C}[a, b]$ be functions. We write

$$S(F, H; D) = \sum_{i=1}^{n} F(t_i)[H(h_i) - H(h_{i-1})],$$

called as **Henstock−Kurzweil−Stieltjes sum** of F with respect to H on $[f, g]$. For brevity, we write $D = \{([u, v], t)\}$ for a tagged division of $[f, g]$ and

$$S(F, H; D) = \sum_{D} F(t)[H(v) - H(u)].$$

Definition 2. Let $F, H : [f, g] \to \mathcal{C}[a, b]$ be functions. We say that the function F is **Henstock−Kurzweil−Stieltjes integrable** with respect to H on $[f, g]$ to $S \in \mathcal{C}[a, b]$, briefly $\mathcal{HKS}$-integrable, if for any $\epsilon > 0$, there exists a gauge δ on $[f, g]$ such that for any δ-fine tagged division D of $[f, g]$, we have

$$|S(F, H; D) - S| < \epsilon \cdot e,$$

where e is the **multiplicative identity** in $\mathcal{C}[a, b]$. The element $S \in \mathcal{C}[a, b]$ is called **Henstock−Kurweil−Stieltjes** integral, briefly $\mathcal{HKS}$-integral, of F with respect to H on $[f, g]$ and is written by

$$S = (\mathcal{HKS}) \int_f^g F \, dH.$$

The collection of all functions which are $\mathcal{HKS}$-integrable with respect to H on $[f, g]$ is denoted by $\mathcal{HKS}([f, g], H)$.

Theorem 2. (Uniqueness) *If F is $\mathcal{HKS}$-integrable with respect to H on $[f, g]$, then the $\mathcal{HKS}$-integral of F with respect to H on $[f, g]$ is unique.*

Proof. Suppose that F is $\mathcal{HKS}$-integrable with respect to H on $[f, g]$ to $S_1 \in \mathcal{C}[a, b]$ and $S_2 \in \mathcal{C}[a, b]$. Let $\epsilon > 0$. Then there exists a gauge δ_1 on $[f, g]$ such that for all δ_1-fine tagged division $D = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \ldots, n\}$ of $[f, g]$, we have

$$|S(F, H; D) - S_1| < \frac{\epsilon}{2} \cdot e. \tag{1}$$

Similarly, there exists a gauge δ_2 on $[f, g]$ such that for all δ_2-fine tagged division $Q = \{([k_{i-1}, k_i], s_i) : i = 1, 2, \ldots, m\}$ of $[f, g]$, we have

$$|S(F, H; Q) - S_2| < \frac{\epsilon}{2} \cdot e. \tag{2}$$

Define a function $\delta : [f, g] \to \mathcal{C}[a, b]$ by $\delta = \delta_1 \wedge \delta_2$. Hence, by (1) and (2)

$$|S_1 - S_2| < \frac{\epsilon}{2} \cdot e + \frac{\epsilon}{2} \cdot e = \epsilon \cdot e.$$

This shows that $S_1 = S_2$. Therefore, the $\mathcal{HKS}$-integral of F with respect to H on $[f, g]$ is unique. □

4. Simple Properties

Theorem 3. *If $F, G \in \mathcal{HKS}([f, g], H)$ and $\alpha \in \mathbb{R}$, then*

(*i*) Homogenity: $\alpha \cdot F \in \mathcal{HKS}([f, g], H)$ *and*

$$(\mathcal{HKS}) \int_f^g (\alpha \cdot F) dH = \alpha \cdot (\mathcal{HKS}) \int_f^g F dH.$$

(ii) Linearity: $F+G \in \mathcal{HKS}([f,g],H)$ *and*

$$(\mathcal{HKS})\int_f^g (F+G)dH = (\mathcal{HKS})\int_f^g FdH + (\mathcal{HKS})\int_f^g GdH.$$

Proof.

(i) Let $\epsilon > 0$. Then there exists a gauge δ on $[f,g]$ such that for any δ-fine tagged division $D = \{([h_{i-1},h_i],t_i) : i = 1,2,\ldots,n\}$ of $[f,g]$, we have

$$\left|\sum_{i=1}^{n} F(t_i)[H(h_i)-H(h_{i-1})] - (\mathcal{HKS})\int_f^g FdH\right| < \frac{\epsilon}{|\alpha|+1}\cdot e.$$

Thus, for any δ-fine tagged division $D = \{([h_{i-1},h_i],t_i) : i = 1,2,\ldots,n\}$ of $[f,g]$

$$\begin{aligned}
&\left|\sum_{i=1}^{n}(\alpha\cdot F)(t_i)[H(h_i)-H(h_{i-1})] - \alpha\cdot(\mathcal{HKS})\int_f^g FdH\right| \\
&\quad = \left|\alpha\left\{\sum_{i=1}^{n} F(t_i)[H(h_i)-H(h_{i-1})] - (\mathcal{HKS})\int_f^g FdH\right\}\right| \\
&\quad = |\alpha|\cdot\left|\sum_{i=1}^{n} F(t_i)[H(h_i)-H(h_{i-1})] - (\mathcal{HKS})\int_f^g FdH\right| \\
&\quad < |\alpha|\cdot\frac{\epsilon}{|\alpha|+1}\cdot e \\
&\quad < \epsilon\cdot e.
\end{aligned}$$

This shows that $\alpha\cdot F \in \mathcal{HKS}([f,g],H)$ and

$$(\mathcal{HKS})\int_f^g (\alpha\cdot F)dH = \alpha\cdot(\mathcal{HKS})\int_f^g FdH. \qquad \square$$

(ii) Let $\epsilon > 0$. Then there exists gauge δ_F on $[f,g]$ such that for any δ_F-fine tagged division $D = \{([h_{i-1},h_i],t_i) : i = 1,2,\ldots,n\}$ of $[f,g]$, we have

$$\left|\sum_{i=1}^{n} F(t_i)[H(h_i)-H(h_{i-1})] - (\mathcal{HKS})\int_f^g FdH\right| < \frac{\epsilon}{2}\cdot e. \tag{3}$$

Similarly, there exists gauge δ_G on $[f,g]$ such that for any δ_G-fine tagged division $Q = \{([k_{i-1},k_i],s_i) : i = 1,2,\ldots,m\}$ of $[f,g]$, we have

$$\left|\sum_{i=1}^{m} G(s_i)[H(k_i)-H(k_{i-1})] - (\mathcal{HKS})\int_f^g GdH\right| < \frac{\epsilon}{2}\cdot e. \tag{4}$$

Define $\delta = \delta_F \wedge \delta_G$. Then δ is a gauge on $[f, g]$. Let $D = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \ldots, n\}$ be a δ-fine tagged division of $[f, g]$. Then D is both δ_F and δ_G-fine. By (3) and (4),

$$\begin{aligned}
&\left|\sum_{i=1}^{n}(F+G)(t_i)[H(h_i) - H(h_{i-1})] - \left\{(\mathcal{HKS})\int_f^g FdH + (\mathcal{HKS})\int_f^g GdH\right\}\right| \\
&\leq \left|\sum_{i=1}^{n} F(t_i)[H(h_i) - H(h_{i-1})] - (\mathcal{HKS})\int_f^g FdH\right| \\
&\qquad + \left|\sum_{i=1}^{n} G(t_i)[H(h_i) - H(h_{i-1})] - (\mathcal{HKS})\int_f^g GdH\right| \\
&< \frac{\epsilon}{2}\cdot e + \frac{\epsilon}{2}\cdot e = \epsilon \cdot e.
\end{aligned}$$

Therefore, $F + G \in \mathcal{HKS}([f, g], H)$ and

$$(\mathcal{HKS})\int_f^g (F+G)dH = (\mathcal{HKS})\int_f^g FdH + (\mathcal{HKS})\int_f^g GdH. \qquad \square$$

Theorem 4. (Linearity of Integrator) *If $F \in \mathcal{HKS}([f, g], H_1) \cap \mathcal{HKS}([f, g], H_2)$, then $F \in \mathcal{HKS}([f, g], H_1 + H_2)$ and*

$$(\mathcal{HKS})\int_f^g Fd(H_1 + H_2) = (\mathcal{HKS})\int_f^g FdH_1 + (\mathcal{HKS})\int_f^g FdH_2.$$

Proof. Let $\epsilon > 0$. Then there exists gauge δ_{H_1} on $[f, g]$ such that for any δ_{H_1}-fine tagged division $D = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \ldots, n\}$ of $[f, g]$, we have

$$\left|\sum_{i=1}^{n} F(t_i)[H_1(h_i) - H_1(h_{i-1})] - (\mathcal{HKS})\int_f^g FdH_1\right| < \frac{\epsilon}{2}\cdot e. \tag{5}$$

Similarly, there exists gauge δ_{H_2} on $[f, g]$ such that for any δ_{H_2}-fine tagged division $Q = \{([k_{i-1}, k_i], s_i) : i = 1, 2, \ldots, m\}$ of $[f, g]$, we have

$$\left|\sum_{i=1}^{m} F(s_i)[H_2(k_i) - H_2(k_{i-1})] - (\mathcal{HKS})\int_f^g FdH_2\right| < \frac{\epsilon}{2}\cdot e. \tag{6}$$

Define $\delta = \delta_{H_1} \wedge \delta_{H_2}$. Then δ is a gauge on $[f, g]$. Let $D = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \ldots, n\}$ be a δ-fine tagged division of $[f, g]$. Then D is both δ_{H_1} and δ_{H_2}-fine. By (5) and (6),

$$\left|\sum_{i=1}^{n} F(t_i)[(H_1 + H_2)(h_i) - (H_1 + H_2)(h_{i-1})] - \left\{(\mathcal{HKS})\int_f^g FdH_1 + (\mathcal{HKS})\int_f^g FdH_2\right\}\right|$$

$$\begin{aligned}
&\leq \left|\sum_{i=1}^{n} F(t_i)[H_1(h_i) - H_1(h_{i-1})] - (\mathcal{HKS}) \int_f^g F dH_1\right| \\
&\quad + \left|\sum_{i=1}^{n} F(t_i)[H_2(h_i) - H_2(h_{i-1})] - (\mathcal{HKS}) \int_f^g F dH_2\right| \\
&< \frac{\epsilon}{2} \cdot e + \frac{\epsilon}{2} \cdot e = \epsilon \cdot e.
\end{aligned}$$

Therefore, $F \in \mathcal{HKS}([f,g], H_1 + H_2)$ and

$$(\mathcal{HKS}) \int_f^g F d(H_1 + H_2) = (\mathcal{HKS}) \int_f^g F dH_1 + (\mathcal{HKS}) \int_f^g F dH_2. \qquad \square$$

Theorem 5. (Additivity) *Let $f \leq r \leq g$. If $F \in \mathcal{HKS}([f,r], H)$ and $F \in \mathcal{HKS}([r,g], H)$, then $F \in \mathcal{HKS}([f,g], H)$ and*

$$(\mathcal{HKS}) \int_f^g F dH = (\mathcal{HKS}) \int_f^r F dH + (\mathcal{HKS}) \int_r^g F dH.$$

Proof. Let $\epsilon > 0$. Then there exists gauge δ_1 on $[f,r]$ such that for any δ_1-fine tagged division $D = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \ldots, n\}$ of $[f,r]$, we have

$$\left|\sum_{i=1}^{n} F(t_i)[H(h_i) - H(h_{i-1})] - (\mathcal{HKS}) \int_f^r F dH\right| < \frac{\epsilon}{2} \cdot e. \tag{7}$$

Similarly, there exists gauge δ_2 on $[r,g]$ such that for any δ_2-fine tagged division $Q = \{([k_{i-1}, k_i], s_i) : i = 1, 2, \ldots, m\}$ of $[r,g]$, we have

$$\left|\sum_{i=1}^{m} F(s_i)[H(k_i) - H(k_{i-1})] - (\mathcal{HKS}) \int_r^g F dH\right| < \frac{\epsilon}{2} \cdot e. \tag{8}$$

Define a function $\delta : [f,g] \to \mathcal{C}[f,g]$ by

$$\delta(h) = \begin{cases} \delta_1(h) \wedge (r - h) & \text{, if } f \leq h \leq r \\ \delta_1(h \wedge r) \wedge \delta_2(h \vee r) & \text{, if } h = r \text{ or } h \text{ is incomparable to } r \\ \delta_2(h) \wedge (h - r) & \text{, if } r \leq h \leq g. \end{cases}$$

Then δ is a gauge on $[f,g]$. Let $D = \{([h_{i-1}, h_i], t_i) : i = 1, 2, \ldots, n\}$ be a δ-fine tagged division of $[f,g]$. By definition of δ, we have $r = h_{i_0}$ for some $i_0 \in \{1, 2, \ldots, n\}$. Hence, $D = D_1 \cup D_2$ for some δ_1-fine tagged division D_1 of $[f,r]$ and δ_2-fine tagged division D_2 of $[r,g]$. By (7) and (8),

$$\left|\sum_{i=1}^{n} F(t_i)[H(h_i) - H(h_{i-1})] - \left\{(\mathcal{HKS}) \int_f^r F dH + (\mathcal{HKS}) \int_r^g F dH\right\}\right| < \epsilon \cdot e.$$

Therefore, $F \in \mathcal{HKS}([f,g],H)$ and

$$(\mathcal{HKS})\int_f^g FdH = (\mathcal{HKS})\int_f^r FdH + (\mathcal{HKS})\int_r^g FdH. \qquad \square$$

In the next theorem, we give an analogous form of Cauchy criterion for $\mathcal{HKS}-$integral.

Theorem 6. (Cauchy Criterion) *$F \in \mathcal{HKS}([f,g],H)$ if and only if for every $\epsilon > 0$ there exists a gauge δ on $[f,g]$ such that for any δ-fine tagged divisions $D = \{([u,v],t)\}$ and $Q = \{([u',v'],s)\}$ of $[f,g]$, we have*

$$\left|\sum_D F(t)[H(v) - H(u)] - \sum_Q F(s)[H(v') - H(u')]\right| < \epsilon \cdot e.$$

Proof. ($\Rightarrow$) Let $\epsilon > 0$. Then there exists a gauge δ on $[f,g]$ such that for any δ-fine tagged division $D = \{([u,v],t)\}$ of $[f,g]$, we have

$$\left|\sum_D F(t)[H(v) - H(u)] - (\mathcal{HKS})\int_f^g FdH\right| < \frac{\epsilon}{2} \cdot e. \tag{9}$$

Let $D = \{([u,v],t)\}$ and $Q = \{([u',v'],s)\}$ be any δ-fine tagged divisions of $[f,g]$. By (9)

$$\left|\sum_D F(t)[H(v) - H(u)] - \sum_Q F(s)[H(v') - H(u')]\right| < \epsilon \cdot e.$$

($\Leftarrow$) By assumption, for each $n \in \mathbb{N}$, there exists a gauge δ_n on $[f,g]$ such that for any δ_n-fine division $D = \{([u,v],t)\}$ and $Q = \{([u',v'],s)\}$ of $[f,g]$, we have

$$\left|\sum_D F(t)[H(v) - H(u)] - \sum_Q F(s)[H(v') - H(u')]\right| < \frac{1}{n} \cdot e. \tag{10}$$

We may assume that $\{\delta_n\}$ is decreasing; that is, $\delta_n \geq \delta_{n+1}$ for all n.

Now, for each $n \in \mathbb{N}$, fix a δ_n-fine tagged division $D_n = \{([u,v],t)\}$ of $[f,g]$ and we write

$$r_n = \sum_{D_n} F(t)[H(v) - H(u)].$$

Note that if $m \geq n$ then $\delta_n \geq \delta_m$; implying that every δ_m-fine tagged division of $[f,g]$ is also a δ_n-fine tagged division of $[f,g]$. Thus, for all $m > n$

$$|r_n - r_m| = \left|\sum_{D_n} F(t)[H(v) - H(u)] - \sum_{D_m} F(s)[H(v') - H(u')]\right| < \frac{1}{n} \cdot e.$$

Hence, $\{r_n\}$ is a Cauchy sequence in $\mathcal{C}[a,b]$. Since $\mathcal{C}[a,b]$ is complete, $\{r_n\}$ converges to some $r \in \mathcal{C}[a,b]$. We claim that

$$r = (\mathcal{HKS}) \int_f^g F dH.$$

Let $\epsilon > 0$. Since $\lim_{n\to\infty} r_n = r$ in $\mathcal{C}[a,b]$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$,

$$\left| r_n - r \right| < e \cdot \frac{\epsilon}{2}. \tag{11}$$

By Archimedean Principle, there exists $N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \frac{\epsilon}{2}$. Take $N = N_1 \wedge N_2$. Define a gauge $\delta : [f,g] \to \mathcal{C}[a,b]$ by $\delta = \delta_N$. Let $D = \{([u,v],t)\}$ be any δ-fine tagged division of $[f,g]$. Note that D is also δ_N-fine tagged division of $[f,g]$, $N \geq N_1$ and $N \geq N_2$. Thus, by (10) and (11)

$$\left| \sum_D F(t)[H(v) - H(u)] - r \right| < \epsilon \cdot e.$$

This proves our claim. □

Theorem 7. *If $F \in \mathcal{HKS}([f,g],H)$ and $[r,s] \subseteq [f,g]$, then $F \in \mathcal{HKS}([r,s],H)$.*

Proof. Let $\epsilon > 0$. By Theorem 6, there exists a gauge δ on $[f,g]$ such that for any δ-fine tagged divisions D and Q of $[f,g]$, we have

$$\left| \sum_D F(t)[H(v) - H(u)] - \sum_Q F(t)[H(v) - H(u)] \right| < \epsilon \cdot e. \tag{12}$$

Consider any δ-fine tagged divisions P_1 and P_2 of $[r,s]$. If D_1 is any δ-fine tagged division of $[f,r]$ and D_2 is any δ-fine tagged division of $[s,g]$, then

$$D = D_1 \cup P_1 \cup D_2 \text{ and } Q = D_1 \cup P_2 \cup D_2$$

are δ-fine tagged divisions of $[f,g]$ and by (12)

$$\left| \sum_{P_1} F(t)[H(v) - H(u)] - \sum_{P_2} F(t)[H(v) - H(u)] \right| < \epsilon \cdot e.$$

By Cauchy criterion, $F \in \mathcal{HKS}([r,s],H)$. □

Theorem 8. *Let $H : [f,g] \to \mathcal{C}[a,b]$ be increasing, that is, $H(k) \leq H(h)$ for any $k \leq h$ in $[f,g]$. If $F \in \mathcal{HKS}([f,g],H)$ and $F(h) \geq \theta$ for every $h \in [f,g]$, then*

$$(\mathcal{HKS}) \int_f^g F dH \geq \theta.$$

Proof. Let $\epsilon > 0$. Then there exists a gauge δ on $[f,g]$ such that for any δ-fine tagged division D of $[f,g]$, we have

$$\left|\sum_D F(t)[H(v)-H(u)] - (\mathcal{HKS})\int_f^g FdH\right| < \epsilon \cdot e. \tag{13}$$

Since $F(h) \geq \theta$ for all $h \in [f,g]$ and H is increasing,

$$\sum_D F(t)[H(v)-H(u)] \geq \theta.$$

Therefore,

$$\theta \leq \sum_D F(t)[H(v)-H(u)] < (\mathcal{HKS})\int_f^g FdH + \epsilon \cdot e.$$

Since $\epsilon > 0$ is arbitrary,

$$(\mathcal{HKS})\int_f^g FdH \geq \theta.$$

□

Theorem 9. *If $F, G \in \mathcal{HKS}([f,g],H)$ and $F(h) \leq G(h)$, for all $h \in [f,g]$, then*

$$(\mathcal{HKS})\int_f^g FdH \leq (\mathcal{HKS})\int_f^g GdH.$$

Proof. Define a function E on $[f,g]$ by setting $E(h) = G(h) - F(h)$, for all $h \in [f,g]$. Then $E(h) \geq \theta$, for all $h \in [f,g]$. Since $F, G \in \mathcal{HKS}([f,g],H)$, $E \in \mathcal{HKS}([f,g],H)$ and by Theorem 8

$$(\mathcal{HKS})\int_f^g EdH \geq \theta.$$

Hence,

$$\theta \leq (\mathcal{HKS})\int_f^g EdH = (\mathcal{HKS})\int_f^g (G-F)dH = (\mathcal{HKS})\int_f^g GdH - (\mathcal{HKS})\int_f^g FdH.$$

Therefore,

$$(\mathcal{HKS})\int_f^g GdH \leq (\mathcal{HKS})\int_f^g FdH.$$

□

5. An Existence Theorem

A function $F : [f,g] \to \mathcal{C}[a,b]$ is **bounded** on $[f,g]$ if there exists $K \geq \theta$ in $\mathcal{C}[a,b]$ such that

$$|F(h)| \leq K, \text{ for all } h \in [f,g].$$

A function $F : [f, g] \to \mathcal{C}[a, b]$ is **continuous** at $h_0 \in [f, g]$, if for any $\epsilon > 0$ there exists $\delta = \delta(h_0) > \theta$ such that whenever $h \in [f, g]$ with $|h - h_0| < \delta$, we have

$$\big|F(h) - F(h_0)\big| < \epsilon \cdot e.$$

F is said to be **uniformly continuous** on $[f, g]$, if for any $\epsilon > 0$ there exists $\delta > \theta$ such that whenever $h, h' \in [f, g]$ with $|h' - h| < \delta$, we have

$$\big|F(h') - F(h)\big| < \epsilon \cdot e.$$

If $F : [f, g] \to \mathcal{C}[a, b]$ is uniformly continuous on $[f, g]$, then it is continuous on $[f, g]$.

Definition 3. Let D_1 and D_2 be tagged divisions of $[f, g]$. We say that D_2 is **finer** than D_1, denoted by $D_1 \ll D_2$, if for every $([u, v], t) \in D_2$ there exists $([u', v'], t') \in D_1$ such that $[u, v] \subseteq [u', v']$, and every tag in D_1 is a tag in D_2. For every $([u', v'], t') \in D_1$, the tagged division $P = \{([z_{i-1}, z_i], t_i) \in D_2 : [z_{i-1}, z_i] \subseteq [u', v'], i = 1, 2, \ldots, n\}$ is the **refinement** of $([u', v'], t')$ in D_2.

We can easily see that if D_1 and D_2 are tagged divisions of $[f, g]$, then there exists a tagged division D_0 of $[f, g]$ such that $D_1 \ll D_0$ and $D_2 \ll D_0$.

Let $\mathcal{D}([f, g])$ be the collection of all divisions of $[f, g]$. For $F : [f, g] \to \mathcal{C}[a, b]$ and $D = \{[u, v]\} \in \mathcal{D}([f, g])$, the **variation of** F **over** D is given by

$$\boldsymbol{var}(F, D) = \sum_D \big|F(v) - F(u)\big|.$$

Note that for any division D of $[f, g]$, $\boldsymbol{var}(F, D)$ is a continuous function on $[a, b]$; that is, $\boldsymbol{var}(F, D) \in \mathcal{C}[a, b]$, for any $D \in \mathcal{D}([f, g])$.

Definition 4. We say that the function $F : [f, g] \to \mathcal{C}[a, b]$ is of **bounded variation** on $[f, g]$ if

$$\boldsymbol{v}_F = \boldsymbol{v}(F; [f, g]) = \sup_{D \in \mathcal{D}([f,g])} \boldsymbol{var}(F, D)$$

is continuous on $[a, b]$; that is, $\boldsymbol{v}_F \in \mathcal{C}[a, b]$.

Note that for any $F : [f, g] \to \mathcal{C}[a, b]$, v_F is a mapping from $[a, b]$ to $[0, +\infty]$; that is,

$$0 \le \boldsymbol{v}_F(x) \le +\infty, \quad \text{for all } x \in [a, b].$$

Hence, if $F : [f, g] \to \mathcal{C}[a, b]$ is of bounded variation, then

$$0 \le \boldsymbol{v}_F(x) < +\infty, \quad \text{for all } x \in [a, b].$$

Theorem 10. *Let $H : [f, g] \mapsto \mathcal{C}[a, b]$ be of bounded variation. Then the variation of H is additive; that is, if $f \le r \le g$, then*

$$\boldsymbol{v}(H; [f, g]) = \boldsymbol{v}(H; [f, r]) + \boldsymbol{v}(H; [r, g]).$$

Proof. Suppose that $H : [f, g] \to \mathcal{C}[a, b]$ is of bounded variation. Let $r \in [f, g]$ and $D = \{h_0, \ldots, h_n\}$ be a division of $[f, g]$. Then $D' = \{h_0, \ldots, h_{k-1}, r, h_k, \ldots, h_n\}$ is a refinement of D obtained by adjoining r to D. Thus

$$\sum_{D} \big|H(v) - H(u)\big| \leq \sum_{D_1} \big|H(v) - H(u)\big| + \sum_{D_2} \big|H(v) - H(u)\big|$$

where $D_1 = \{f = h_0, h_1, \ldots, h_{k-1}, r\}$ and $D_2 = \{r, h_k, \ldots, h_n = g\}$. Note that $D' = D_1 \cup D_2$ and that

$$\sum_{D_1} \big|H(v) - H(u)\big| \leq \sup_{D \in \mathcal{D}([f,r])} \left(\sum_{D} \big|H(v) - H(u)\big| \right) = \boldsymbol{v}(H; [f, r]) \quad \text{and}$$

$$\sum_{D_2} \big|H(v) - H(u)\big| \leq \sup_{D \in \mathcal{D}([r,g])} \left(\sum_{D} \big|H(v) - H(u)\big| \right) = \boldsymbol{v}(H; [r, g]).$$

Hence,

$$\boldsymbol{v}(H; [f, g]) = \sup_{D \in \mathcal{D}([f,g])} \left(\sum_{D} \big|H(v) - H(u)\big| \right) \leq \boldsymbol{v}(H; [f, r]) + \boldsymbol{v}(H; [r, g]).$$

On the other hand, for any $D_1 \in \mathcal{D}([f, r])$ and $D_2 \in \mathcal{D}([r, g])$, their union $D' = D_1 \cup D_2 \in \mathcal{D}_r([f, g])$, where $\mathcal{D}_r([f, g])$ is the set of all divisions of $[f, g]$ with r as one of the division points. Note that $\mathcal{D}_r([f, g]) \subseteq \mathcal{D}([f, g])$. Hence,

$$\sup_{D' \in \mathcal{D}_r([f,g])} \left(\sum_{D'} \big|H(v) - H(u)\big| \right) \leq \sup_{D \in \mathcal{D}([f,g])} \left(\sum_{D} \big|H(v) - H(u)\big| \right) = \boldsymbol{v}(H; [f, g])$$

Thus,

$$\begin{aligned} \boldsymbol{v}(H; [f, r]) + \boldsymbol{v}(H; [r, g]) &\leq \sup_{D' \in \mathcal{D}_r([f,g])} \left(\sum_{D'} \big|H(v) - H(u)\big| \right) \\ &\leq \boldsymbol{v}(H; [f, g]). \end{aligned}$$

Therefore, combining the two inequalities

$$\boldsymbol{v}(H; [f, r]) + \boldsymbol{v}(H; [r, g]) = \boldsymbol{v}(H; [f, g]).$$

□

Theorem 11. (Existence Theorem) *If $F : [f, g] \to \mathcal{C}[a, b]$ is continuous and $H : [f, g] \to \mathcal{C}[a, b]$ is of bounded variation on $[f, g]$, then $F \in \mathcal{HKS}([f, g], H)$.*

Proof. Let $\epsilon > 0$. Since H is of bounded variation, $\boldsymbol{v}_H \in \mathcal{C}[a, b]$. This means that there exists $K > 0$ such that $\boldsymbol{v}_H(x) \leq K$ for all $x \in [a, b]$. Since F is continuous on $[f, g]$, for all $h_0 \in [f, g]$ there exists $\delta_0(h_0) > \theta$ in $\mathcal{C}[a, b]$ such that whenever $h \in [f, g]$ with $|h - h_0| < \delta_0(h_0)$, we have

$$|F(h) - F(h_0)| < \epsilon \cdot e.$$

Define a gauge δ on $[f,g]$ by $\delta(h)=\frac{\delta_0(h)}{2}$, for all $h\in[f,g]$. Let

$$D=\{([f,h_1],t_1),([h_1,h_2],t_2),\dots,([h_{m-1},g],t_m)\}$$

and

$$Q=\{([f,k_1],r_1),([k_1,k_2],r_2),\dots,([k_{q-1},g],r_q)\}$$

be δ-fine tagged divisions of $[f,g]$. Then there exists a tagged division D_0 such that $D\ll D_0$ and $Q\ll D_0$. Now, for every $([h_{i-1},h_i],t_i)\in D$, $f=h_0, h_m=g$, $1\le i\le m$, consider the difference

$$\Delta(h_{i-1},h_i)=F(t_i)\big[H(h_i)-H(h_{i-1})\big]-S(F,H;P_i)$$

where

$$P_i=\left\{\left(\left[z_{j-1}^{(i)},z_j^{(i)}\right],s_j^{(i)}\right)\right\}_{j-1}^{n_i},\quad z_0^{(i)}=h_{i-1},\ z_{n_i}^{(i)}=h_i$$

is the refinement of $([h_{i-1},h_i],t_i)$ in D_0. Then

$$\Delta(h_{i-1},h_i)=\sum_{j=1}^{n_i}\big[F(t_i)-F(s_j^{(i)})\big]\big[H(z_j^{(i)})-H(z_{j-1}^{(i)})\big].$$

Now, $s_j^{(i)},t_i\in[h_{i-1},h_i]\subseteq(t_i-\delta(t_i),t_i+\delta(t_i))$ which implies that

$$\left|t_i-s_j^{(i)}\right|\le\big|h_i-h_{i-1}\big|<\delta(t_i).$$

By continuity of F at t_i,

$$\left|s_j^{(i)}-t_i\right|<\delta(t_i)=\frac{\delta_0(t_i)}{2}<\delta_0(t_i)\Rightarrow\big|F(s_j^{(i)})-F(t_i)\big|<\epsilon\cdot e.$$

So,

$$|\Delta(h_{i-1},h_i)|=\left|\sum_{j=1}^{n_i}\big[F(t_i)-F(s_j^{(i)})\big]\big[H(z_j^{(i)})-H(z_{j-1}^{(i)})\big]\right|.$$

Hence, by Theorem 10, we have

$$\begin{aligned}\Big|S(F,H;D)-S(F,H;D_0)\Big|&=\left|\sum_{i=1}^{m}F(t_i)[H(h_i)-H(h_{i-1})]-\sum_{i=1}^{m}S(F,H,P_i)\right|\\&=\left|\sum_{i=1}^{m}\Big\{F(t_i)[H(h_i)-H(h_{i-1})]-S(F,H,P_i)\Big\}\right|=\left|\sum_{i=1}^{m}\Delta(h_{i-1},h_i)\right|\\&\le\sum_{i=1}^{m}\Big|\Delta(h_{i-1},h_i)\Big|\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \left| \sum_{j=1}^{n_i} \left[F(t_i) - F(s_j^{(i)})\right] \left[H(z_j^{(i)}) - H(z_{j-1}^{(i)})\right] \right| \\
&\le \sum_{i=1}^{m} \left(\sum_{j=1}^{n_i} \left|F(t_i) - F(s_j^{(i)})\right| \left|H(z_j^{(i)}) - H(z_{j-1}^{(i)})\right| \right) \\
&\le \sum_{i=1}^{m} \left(\sum_{j=1}^{n_i} \frac{\epsilon}{K} \cdot e \cdot \left|H(z_j^{(i)}) - H(z_{j-1}^{(i)})\right| \right) \\
&\le \frac{\epsilon}{K} \cdot e \cdot \sum_{i=1}^{m} \left(\sum_{j=1}^{n_i} \left|H(z_j^{(i)}) - H(z_{j-1}^{(i)})\right| \right) \\
&\le \frac{\epsilon}{K} \cdot e \cdot \sum_{i=1}^{m} \upsilon(H;[h_{i-1},h_i]) \\
&= \frac{\epsilon}{K} \cdot e \cdot \upsilon_H < \frac{\epsilon}{K} \cdot e \cdot K < \epsilon \cdot e.
\end{aligned}
$$

By similar argument,

$$\left|S(F,H;Q) - S(F,H;D_0)\right| < \epsilon \cdot e.$$

Thus,

$$
\begin{aligned}
\left|S(F,H;D) - S(F,H;Q)\right| &= \left|S(F,H;D) - S(F,H;D_0) + S(F,H;D_0) - S(F,H;Q)\right| \\
&\le \left|S(F,H;D) - S(F,H;D_0)\right| + \left|S(F,H;Q) - S(F,H;D_0)\right| \\
&< \epsilon \cdot e + \epsilon \cdot e \\
&= 2\epsilon \cdot e.
\end{aligned}
$$

By Cauchy criterion, $F \in \mathcal{HKS}([f,g],H)$. □

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