



Coefficients Characterization of Entire Harmonic Functions in Terms of Norm of Gradients at Origin in

$$\mathbb{R}^n, n \geq 3$$

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Abstract. Coefficient characterizations of generalized order, lower order and generalized type of entire harmonic function having the spherical harmonic expansion throughout a neighborhood of the origin in \mathbb{R}^n have been obtained in terms of norm of gradients at origin.

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1. Introduction

In the study of entire functions of one complex variable, the main issues are the relationship between the growth of such functions and behavior of Taylor coefficients. Several authors such as Srivastava and Kumar [20], Kumar [11,14], Harfaoui [8] and others investigated growth parameters of entire functions in terms of Taylor's series coefficients and polynomial approximation errors in different norms. Similar studies have been done for harmonic functions by Kumar [12,13], Kumar and Kasana [15] and Armitage [1] as they have series expansion in terms of spherical harmonics in \mathbb{R}^n . Some times it is useful to study the growth of harmonic functions in terms of norm of their gradient at the origin in n-dimensional space. Such results are equivalent to characterization in terms of spherical harmonic coefficients. Results of one kind can not obtained directly from the other, and thus require separate study. Also, the problem to investigate the growth characteristics of harmonic functions in terms that are not related to series expansion coefficients. The

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derivatives of a harmonic functions at the origin are equal to complicated linear combinations of the spherical harmonic coefficients. The relevance of this study is due to the fact that the harmonic functions play very important role in theoretical mathematical research, physics and mechanics to express stationary processes.

Therefore, the aim of this paper is to characterize the generalized growth parameters (generalized order, lower order and generalized type) in the sense of Sheremeta [18] of entire harmonic functions in terms of norm of gradient at origin .

It is significant to mention here that time dependent problems in \mathbb{R}^3 leads to the study of entire harmonic functions in \mathbb{R}^4 .

A function $H(x), x \in \mathbb{R}^n$ which has continuous partial derivatives of second order and satisfies Laplace,s differential equation

$$\sum_{i=1}^n \frac{\partial^2 H}{\partial x_i^2} = 0$$

is said to be harmonic in n-dimensional space \mathbb{R}^n .

The function H has the spherical harmonic expansion throughout a neighborhood of the origin in \mathbb{R}^n as

$$H(x) = \sum_{k=0}^{\infty} H_k(x), \quad (1.1)$$

where $H_k(x)$ is a harmonic homogeneous polynomial of degree k in x_1, x_2, \dots, x_n having real coefficients [3, pp. 47]. These polynomials are known as spherical harmonics.

Let $S^n = \{x \in \mathbb{R}^n : |x| = 1\}$ be a unit sphere in \mathbb{R}^n . The series (1.1) also can be expressed as

$$H(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} a_k^j Q_k^j\left(\frac{x}{r}\right) r^k, \quad |x| = r, \quad (1.2)$$

where $\{Q_k^j\}_{j=1}^{d_k}$ be an orthonormal basis for H_k with respect to the scalar product

$$\langle f, g \rangle = \frac{1}{w_n} \int_{S^n} f(x)g(x) d\sigma_1,$$

while

$$w_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

denotes the area of S^n and $d\sigma_1$ is the element of surface area on S^n .

Also (See [21, pp.145])

$$d_k = \frac{(n + 2k - 2)(n + k - 3)!}{k!(n - 2)!}$$

is the dimension of vector space H_k and

$$a_k^j = \frac{1}{w_n} \int_{S^n} H(x) Q_k^j(x) d\sigma_1.$$

If the series (1.2) converges uniformly on the sphere $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}} = \delta$, then we have

$$a_k^j = \frac{1}{\delta^{2k+n-1} w_n} \int_{|x|=\delta} H(x) Q_k^j(x) d\sigma,$$

where $d\sigma = \delta^{n-1} d\sigma_1$ is the element of surface area on the sphere $|x| = \delta$.

For each n -tuple $a = (a_1, a_2, \dots, a_n)$ of non-negative integers, we define $|a| = a_1 + a_2 + \dots + a_n$, $a! = a_1! a_2! \dots a_n!$ and $D^a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \partial x_2^{a_2} \dots \partial x_n^{a_n}}$.

In view of [2] for each $\xi \in C^\infty(\mathbb{R}^n)$ and non-negative integer k , we define the norm of the k^{th} gradients of H at origin by

$$|\nabla_k H(0)| = \left(\frac{k!}{2^k} \sum_{|a|=k} \frac{[D^a H(0)]^2}{a!} \right)^{\frac{1}{2}}. \tag{1.3}$$

It has been proved [5] that the series (1.1) converges absolutely and uniformly on compact subsets of the open ball $|x| < R$, where

$$R^{-1} = \sqrt{2} \limsup_{k \rightarrow \infty} \left(\frac{|\nabla_k H(0)|}{k!} \right)^{\frac{1}{k}}. \tag{1.4}$$

Definition (1.3) and equality (1.4) immediately show that the series (1.1) converges absolutely and uniformly on compact subsets of open ball $|x| < R$, where

$$R^{-1} = \limsup_{k \rightarrow \infty} \left(\frac{|\nabla_k H(0)|}{k!} \right)^{\frac{1}{k}}, \tag{1.5}$$

and such convergence can not obtain within any larger ball centered at origin.

Fryant and Shankar [4] proved the following lemma.

Lemma A. Let $H(x) = \sum_{k=0}^\infty H_k(x)$ is uniformly convergent in a neighborhood of the origin in \mathbb{R}^n . Then for all $r < R$,

$$M_2(r, H) \leq M(r, H) \leq N(r, H),$$

where $M(r, H) = \max_{|x|=r} |H(x)|$,

$$M_2(r, H) = [\Gamma(n/2) \sum_{k=0}^{\infty} \frac{|\nabla_k H(0)|^2}{k! \Gamma(k + n/2)} r^{2k}]^{\frac{1}{2}},$$

and

$$N(r, H) = \sqrt{\Gamma(n/2)} \sum_{k=1}^{\infty} \sqrt{d_k} \frac{|\nabla_k H(0)|}{\sqrt{k! \Gamma(k + n/2)}} r^k.$$

Here the upper bound of $M(r, H)$ holds for all $r \geq 0$, and the lower bound of $M(r, H)$ obtains for all r such that the spherical harmonic series H is uniformly convergent on the sphere $|x| = r$.

We define the order ρ of H as

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, H)}{\log r}, 0 \leq \rho \leq \infty,$$

and when $0 < \rho < \infty$, the type T is defined as

$$T = \limsup_{r \rightarrow \infty} \frac{\log M(r, H)}{r^\rho}, 0 \leq T \leq \infty.$$

Fugard [6] characterized the order and type of an entire harmonic function in terms of the m^{th} gradient defined above. Also, Kumar and Singh [16] investigated these results for non entire case. Srivastava [19] improved Fugard’s results and obtained generalized order and generalized type. In this paper, we extend the results of Srivastava [19].

2. Generalized Growth

Let $\xi : [a, \infty) \rightarrow \mathbb{R}$ for some $a \geq 0$, such that $\xi(x)$ is positive, strictly increasing and differentiable and tends to ∞ as $x \rightarrow \infty$. Then ξ is said to belong to the class L^0 if for every real valued function $\phi(x)$ such that $\phi(x) \rightarrow 0$ as $x \rightarrow \infty$, ξ satisfies

$$\lim_{x \rightarrow \infty} \frac{\xi[(1 + \phi(x))x]}{\xi(x)} = 1,$$

and belongs to the class Λ if for all $c, 0 < c < \infty$, we have the stronger condition

$$\lim_{x \rightarrow \infty} \frac{\xi(cx)}{\xi(x)} = 1.$$

Let $\alpha, \beta \in L^0, \Lambda$, following the analogy with [18], we define generalized and lower generalized order of the entire harmonic function $H \in \mathbb{R}^n$ by

$$\rho(\alpha, \beta, H) = \limsup_{r \rightarrow \infty} \frac{\alpha(\log M(r, H))}{\beta(r)}, \lambda(\alpha, \beta, H) = \liminf_{r \rightarrow \infty} \frac{\alpha(\log M(r, H))}{\beta(r)}.$$

Now we prove

Theorem 2.1. Let H be a harmonic function in a neighborhood of the origin in \mathbb{R}^n , satisfying one of the following conditions:

(i). For $\alpha, \beta \in \Lambda, F(t, c) = \beta^{-1}(c\alpha(t)), 0 < c < \infty,$

$$\lim_{t \rightarrow \infty} \frac{d(\log F(t, c))}{d(\log t)} = O(1).$$

(ii). For $\alpha, \beta \in L^0,$

$$\lim_{t \rightarrow \infty} \frac{d(\log F(t, c))}{d(\log t)} = p, 0 < p < \infty,$$

then the generalized order $\rho(\alpha, \beta, H)$ of entire harmonic function H is determined by

$$\rho(\alpha, \beta, H) = \limsup_{k \rightarrow \infty} \frac{\alpha(pk)}{\beta(e^p R [\frac{|\nabla_k H(0)|}{k!}]^{\frac{-1}{k}})}.$$

Proof. Consider the entire functions of single complex variable z :

$$f_1(z) = \sqrt{\Gamma(n/2)} \sum_{k=1}^{\infty} \frac{|\nabla_k H(0)|}{\sqrt{k! \Gamma(k + n/2)}} \left(\frac{z}{R}\right)^k,$$

and

$$f_2(z) = \sqrt{\Gamma(n/2)} \sum_{k=1}^{\infty} \sqrt{d_k} \frac{|\nabla_k H(0)|}{\sqrt{k! \Gamma(k + n/2)}} \left(\frac{z}{R}\right)^k.$$

Since

$$\frac{\Gamma(k + n/2)}{\Gamma(k + 1)} = k^{\frac{n}{2}-1},$$

we have

$$\frac{\sqrt{d_k} \sqrt{\Gamma(n/2)} |\nabla_k H(0)|}{\sqrt{k! \Gamma(k + n/2)}} \simeq \frac{\sqrt{d_k} \sqrt{\Gamma(n/2)} |\nabla_k H(0)|}{k! k^{(n-2)/4}},$$

as

$$(d_k)^{\frac{1}{2}} = \left[\frac{(n + 2k - 2)(n + k - 3)!}{k!(n - 2)!} \right]^{\frac{1}{2}} = \left[\frac{(n + k - 3)!}{(k - 1)!} \right]^{\frac{1}{2}} = k^{\frac{(n-2)}{2}}.$$

Therefore,

$$\frac{\sqrt{d_k} \sqrt{\Gamma(n/2)} |\nabla_k H(0)|}{\sqrt{k! \Gamma(k + n/2)}} \simeq \frac{\sqrt{\Gamma(n/2)} |\nabla_k H(0)| k^{(n-2)/4}}{k!},$$

or

$$\lim_{k \rightarrow \infty} \left[\frac{\sqrt{d_k} \sqrt{\Gamma(n/2)} |\nabla_k H(0)|}{\sqrt{k! \Gamma(k + n/2)}} \right]^{-\frac{1}{k}} \simeq \left[\frac{|\nabla_k H(0)|}{k!} \right]^{-\frac{1}{k}}.$$

Hence $f_1(z)$ and $f_2(z)$ defined above are entire functions in view of (1.5).

Using Lemma A, we obtain

$$\mu(r, f_1) \leq M(r, H) \leq M(r, f_2), \tag{2.1}$$

where $\mu(r, f_1)$ is the maximum term of the power series expansion of function $f_1(z)$ on the circle $|z| = r$ and $M(r, f_2) = \max_{|z|=r} |f_2(z)|$.

We see that

$$\begin{aligned} |f_1(z)|^2 &= \sum_{k=0}^{\infty} \left\{ \frac{\sqrt{\Gamma(n/2)} |\nabla_k H(0)|}{\sqrt{k! \Gamma(k + n/2)}} \right\}^2 \left(\frac{r}{R}\right)^{2k} \\ &+ \sum_{k \neq m} \left\{ \frac{\sqrt{\Gamma(n/2)} |\nabla_k H(0)| |\nabla_m H(0)|}{\sqrt{k! m! \Gamma(k + n/2) \Gamma(m + n/2)}} \right\} \left(\frac{z}{R}\right)^k \left(\frac{\bar{z}}{R}\right)^m. \end{aligned} \tag{2.2}$$

Using (2.2) with the estimate [6, p.290] of $M_2(r, H)$, we get

$$M(r, f_1) \geq M_2(r, H) \geq B \left(\frac{r}{R}\right)^k \frac{|\nabla_k H(0)|}{k!}, \tag{2.3}$$

where B is a finite constant. Since $\mu(r, f_1)$ is the maximum term of $f_1(z)$ then by a result of Valiron [22, p.34], we obtain

$$\log M(r, f_1) \simeq \log \mu(r, f_1) \text{ as } r \rightarrow \infty. \tag{2.4}$$

Now taking into account the definition of $\rho(\alpha, \beta, H)$ with (2.1) and (2.4) we get

$$\rho(\alpha, \beta, f_1) \leq \rho(\alpha, \beta, H) \leq \rho(\alpha, \beta, f_2).$$

Applying the coefficient formula of generalized order of an entire function of one complex variable [18] and bearing in mind that $\beta \in \Lambda$ or L^0 , we obtain

$$\rho(\alpha, \beta, f_1) = \rho(\alpha, \beta, f_2) = \limsup_{k \rightarrow \infty} \frac{\alpha(pk)}{\beta(e^p R \left[\frac{|\nabla_k H(0)|}{k!} \right]^{-\frac{1}{k}})}. \tag{2.5}$$

Remark 2.1. If $\alpha(x) = \beta(x) = \log x$, we get the classical order $\rho(H)$ in terms of norm of gradient at the origin studied by Fugard [6, Thm.2.1],

$$\rho(H) = \limsup_{k \rightarrow \infty} \frac{\log k}{\left[\frac{|\nabla_k H(0)|}{k!} \right]^{-\frac{1}{k}}}, \rho(H) = \rho.$$

Remark 2.2. If $\alpha(x) = x, \beta(x) = x^\rho, p = \frac{1}{\rho}$, then (2.5) gives the formula for the classical type $T(H)$ obtained by Fugard [6, Thm.2.6],

$$R(T(H)\rho e)^{\frac{1}{\rho}} = \limsup_{k \rightarrow \infty} k^{\frac{1}{\rho}} \left(\frac{|\nabla_k H(0)|}{k!} \right)^{\frac{1}{k}}.$$

Remark 2.3. If $\alpha(x) = x, \beta(x) = x^{\rho(x)}$, where $\rho(x)$ is the proximate order of the entire function H , then the formula for the generalized type $T^*(H)$ with respect to proximate order $\rho(x)$ is given by

$$R(T^*(H)\rho e)^{\frac{1}{\rho}} = \limsup_{k \rightarrow \infty} \theta(k) \left(\frac{|\nabla_k H(0)|}{k!} \right)^{\frac{1}{k}},$$

where $x = \theta(k) \Leftrightarrow k = x^{\rho(x)}$.

Theorem 2.2. Let H be a harmonic function in a neighborhood of the origin in $\mathbb{R}^n, n \geq 3$, for which

$$\lambda(\alpha, \beta, H) \geq \liminf_{k \rightarrow \infty} \frac{\alpha(pk)}{\beta(e^p R \left[\frac{|\nabla_k H(0)|}{k!} \right]^{-\frac{1}{k}})}. \tag{2.6}$$

If the function

$$\mu(k) = \left\{ \frac{|\nabla_k H(0)|}{|\nabla_{k+1} H(0)|} \right\} \sqrt{(k+1)(k+n/2)}$$

be a nondecreasing function of k for all large values of k and one of the (i),(ii) conditions of Theorem 2.1 is satisfied, then inequality in (2.6) converts in equality.

Proof. As $f_1(z)$ is defined above is an entire function and

$$\log M(r, f_1) \simeq \log M(r, H) \text{ as } r \rightarrow \infty.$$

Hence $f_1(z)$ is also of generalized lower order $\lambda(\alpha, \beta, f_1)$. Since under the assumption

$$\frac{\frac{\sqrt{\Gamma(n/2)}|\nabla_k H(0)|}{\sqrt{k!\Gamma(k+n/2)}}}{\frac{\sqrt{\Gamma(n/2)}|\nabla_{k+1} H(0)|}{\sqrt{(k+1)!\Gamma(k+1+n/2)}}} \simeq \frac{|\nabla_k H(0)|}{|\nabla_{k+1} H(0)|} \sqrt{(k+1)(k+n/2)}$$

is non-decreasing function of k . Now applying [17, Thm.2] with (2.1), for the function $f_1(z)$ and $f_2(z)$ we get the required result.

3. Growth of Entire Harmonic Functions of Zero Order

To study the growth of entire functions of zero order, Kapoor and Nautiyal [10] defined a new class of functions as follows:

The class of functions $\xi(x)$ denoted by $\overline{\Omega}$ which satisfies:

- (i). $\xi(x)$ is positive, defined on $[a, \infty)$, differentiable, strictly increasing and tends to ∞ as $x \rightarrow \infty$.
- (ii). $\xi(x)$ such that

$$\lim_{x \rightarrow \infty} \frac{d(\xi(x))}{d(\log x)} = K, 0 < K < \infty.$$

The generalized order $\rho(\alpha, \alpha, f)$, generalized lower order $\lambda(\alpha, \alpha, f)$ and generalized type of the entire function $f(z)$ were defined as:

$$\rho(\alpha, \alpha, H) = \limsup_{r \rightarrow \infty} \frac{\alpha(\log M(r, f))}{\alpha(\log r)}, \lambda(\alpha, \alpha, f) = \liminf_{r \rightarrow \infty} \frac{\alpha(\log M(r, f))}{\alpha(\log r)},$$

$$T(\alpha, \alpha, f) = \limsup_{r \rightarrow \infty} \frac{\alpha(\log M(r, f))}{[\alpha(\log r)]^\rho},$$

where $\alpha(x) \in \overline{\Omega}$ and $1 \leq \lambda(\alpha, \alpha, f) \leq \rho(\alpha, \alpha, f) \leq \infty$.

The coefficient characterizations of entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ were also obtained as follows:

$$\rho(\alpha, \alpha, f) = 1 + \limsup_{k \rightarrow \infty} \frac{\alpha(k)}{\alpha(\log |a_k|^{-\frac{1}{k}})}. \quad (3.1)$$

If $|\frac{a_k}{a_{k+1}}|$ be a non-decreasing function of k , then

$$\lambda(\alpha, \alpha, f) = 1 + \liminf_{k \rightarrow \infty} \frac{\alpha(k)}{\alpha(\log |a_k|^{-\frac{1}{k}})}. \quad (3.2)$$

Also, for $\alpha(x) \in \overline{\Omega}$, Ganti and Srivastava [7] obtained

$$T(\alpha, \alpha, f) = \limsup_{k \rightarrow \infty} \frac{\alpha(\frac{k}{\rho})}{\{\alpha(\frac{\rho}{\rho-1} \log |a_k|^{-\frac{1}{k}})\}^{\rho-1}},$$

provided $\frac{dF(k; T, \rho)}{d(\log x)} = O(1)$ as $x \rightarrow \infty$ for all $T, 0 < T < \infty$.

Ning Juhong and Chen Qing [9] improved above results by introducing a new class Ω^* (the extension of $\overline{\Omega}$).

The class of functions $\xi(x) \in \Omega^*$ satisfies (i) and (iii)
 (iii). $\lim_{x \rightarrow \infty} \frac{d(\xi(x))}{d(\log^{[q]} x)} = K, 0 < K < \infty, q \geq 1, q \in \mathbb{N}^+,$
 where $\log^{[q]} x = \log^{[q-1]} \log x, \log^{[0]} = x.$

The class $\xi(x)$ also satisfies L^0 and $\Lambda.$
 It is clear that $\alpha(x) \in \bar{\Omega}$ is a particular case of $\alpha(x) \in \Omega^*.$ for $q = 1.$

Ning Juhong and Chen Qing [9] obtained the following coefficient characterization:

Let $\alpha(x) \in \Omega^*,$ then some necessary and sufficient conditions of the entire function $f(z)$ having generalized order ρ is

$$\limsup_{r \rightarrow \infty} \frac{\alpha(\log M(r, f))}{\alpha(\log r)} - 1 = \limsup_{k \rightarrow \infty} \frac{\alpha(k)}{\alpha(\log |a_k|^{-\frac{1}{k}})} \text{ for } q = 1, \tag{3.3}$$

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{\alpha(k)}{\alpha(\log |a_k|^{-\frac{1}{k}})} &\leq \limsup_{r \rightarrow \infty} \frac{\alpha(\log M(r, f))}{\alpha(\log r)} \\ &\leq \limsup_{k \rightarrow \infty} \frac{\alpha(k)}{\alpha(\log |a_k|^{-\frac{1}{k}})} + 1, \text{ for } q = 2, 3, \dots, \end{aligned} \tag{3.4}$$

For $\alpha(x) \in \Omega^*,$ the entire function $f(z)$ of generalized order $\rho, 1 < \rho < \infty$ having the generalized type T if and only if

$$\limsup_{r \rightarrow \infty} \frac{\alpha(\log M(r, f))}{[\alpha(\log r)]^\rho} = \limsup_{k \rightarrow \infty} \frac{\alpha(\frac{k}{\rho})}{\{\alpha(\log |a_k|^{-\frac{1}{k}})\}^{\rho-1}} \text{ for } q = 1, \tag{3.5}$$

$$\limsup_{r \rightarrow \infty} \frac{\alpha(\log M(r, f))}{[\alpha(\log r)]^\rho} = \limsup_{k \rightarrow \infty} \frac{\alpha(\frac{k}{\rho})}{\{\alpha(\log |a_k|^{-\frac{1}{k}})\}^\rho} \text{ for } q = 2, 3, \dots \tag{3.6}$$

Now we prove

Theorem 3.1. Let $\alpha(x) \in \Omega^*,$ then necessary and sufficient conditions for H to be continued to the entire harmonic function in space $\mathbb{R}^n, n \geq 3$ having generalized order $\rho_1(\alpha, \alpha, H)$ is

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{\alpha(k)}{\alpha(\log[\frac{|\nabla_k H(0)|}{k!}]^{-\frac{1}{k}})} &\leq \limsup_{r \rightarrow \infty} \frac{\alpha(\log M(r, H))}{\alpha(\log r)} \\ &\leq \limsup_{k \rightarrow \infty} \frac{\alpha(k)}{\alpha(\log[\frac{|\nabla_k H(0)|}{k!}]^{-\frac{1}{k}})} + 1 \text{ for } q = 2, 3, \dots \end{aligned} \tag{3.7}$$

Proof. Applying the method of proving Theorem 2.1 and taking (3.4) into account with properties of $\alpha(x),$ we obtain the required result (3.7).

Theorem 3.2. Let $\alpha(x) \in \Omega^*$, then the function H can be continued to the entire harmonic function in space \mathbb{R}^n , $n \geq 3$, having generalized order $\rho_1(\alpha, \alpha, H)$, $1 < \rho_1(\alpha, \alpha, H) < \infty$, is of generalized type $T_1(\alpha, \alpha, H)$ if, and only if

$$\limsup_{r \rightarrow \infty} \frac{\alpha(\log M(r, H))}{[\alpha(\log r)]^{\rho_1}} = \limsup_{k \rightarrow \infty} \frac{\alpha(k)}{[\alpha(\log[\frac{|\nabla_k H(0)|}{k!}]^{-\frac{1}{k}})]^{\rho_1}} \quad \text{for } q = 2, 3, \dots$$

Proof. The result follows on using (3.6) for the entire function $f_1(z)$.

Remark 3.1. Theorems 3.1 and 3.2 have been proved by Srivastava [19] for $q = 1$.

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