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Upper Distance k- Cost Effective Numbers in the Join of Graphs

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Abstract. Let k be a positive integer and G be a connected graph. The open k-neighborhood set $N_G^k(v)$ of $v \in V(G)$ is the set $N_G^k(v) = \{u \in V(G) \setminus \{v\} : d_G(u,v) \leq k\}$. A set S of vertices of G is a distance k- cost effective if for every vertex u in S, $|N_G^k(u) \cap S^c| - |N_G^k(u) \cap S| \geq 0$. The maximum cardinality of a distance k- cost effective set of G is called the upper distance k- cost effective number of G. In this paper, we characterized a distance k- cost effective set in the join of two graphs. As direct consequences, the bounds or the exact values of the upper distance k- cost effective numbers are determined.

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Key Words and Phrases: Distance k-cost effective set, upper distance k-cost effective number, join, t-fringe set, t- increment

1. Introduction

We assume that all graphs $G = (V(G), E(G))$ considered throughout this paper are finite, simple, and undirected connected graphs. The basic graph theoretic concepts are adapted from [1] and [7]. The notations $V(G)$ and $E(G)$ are the vertex set and edge set, respectively, of G. The $|V(G)|$ denotes the order of G and for any set $S \subseteq V(G)$, $|S|$ is the cardinality of S.

Let G be a connected graph and $v \in V(G)$. The ***open neighborhood*** of v in G, denoted by $N_G(v)$, is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The ***degree*** of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is the cardinality of $N_G(v)$. The ***minimum degree*** of G is $\delta(G) = \min\{\deg_G(v) : v \in V(G)\}$ and the ***maximum degree*** of G is $\Delta(G) = \max\{\deg_G(v) : v \in V(G)\}$. The ***distance*** between vertices u and v in G, denoted by

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$d_G(u, v)$, is the length of the shortest path from vertex u to vertex v in G. The***diameter*** of G, denoted by $diam(G)$, is the maximum distance between any two vertices in G.

For any positive integer k and $v \in V(G)$, the ***open k- neighborhood set*** $N^k_G(v)$ of vertex v is the set of all vertices u of G such that $0 < d_G(u, v) \leq k$. That is, $N^k_G(v) = \{u \in V(G) : 0 < d_G(u, v) \leq k\}$. The degree of v in G of distance k , denoted by $\deg^k_G(v)$, is the cardinality of $N^k_G(v)$. The minimum degree $\delta^k(G)$ of G with distance k is $\delta^k(G)$ =min$\{\deg^k_G(v) : v \in V(G)\}$ and the maximum degree of G with distance k is $\Delta^k(G)$ =max$\{\deg^k_G(v); v \in V(G)\}$. Note that $\deg^1_G(v)$ =$\deg_G(v)$, $\delta^1(G) = \delta(G)$, and $\Delta^1(G) = \Delta(G)$. A simple graph G is called ***regular*** if all vertices of G have the same degree. Thus, $\Delta(G) = \delta(G)$.

Given graphs G and H with disjoint vertex sets, the ***join*** of G and H, denoted by $G + H$, is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. For a nonempty set $S \subseteq V(G)$, the ***subgraph*** $\langle S\rangle_G$ of G induced by S is the maximal subgraph of G with vertices in S. The $\delta(G : S) =$ min $\{\deg_G(v) : v \in S\}$.

A vertex v in a set $S \subseteq V(G)$ is a ***cost effective*** if $|N_G(v) \cap S^c| - |N_G(v) \cap S| \geq 0$. A set $S \subseteq V(G)$ is called ***cost effective*** if every vertex $v \in S$ is cost effective. The concept of cost effective set in graph was introduced by Haynes, et.al. in [5] which was motivated by Aharoni, et. al. in [8]. In 2018, Chellali, et. al. in [2] established a generalization of this concept. Its application in computer networks plays a vital role: in particular in maintaining edges in a network that are directly associated with the cost that should be used effectively and in a set of servers (vertices) that each server is serving a maximal number of clients (non-servers). We refer the readers to [2, 4] for some of its relevant applications and to [3, 6, 9] for some investigations of the concepts.

In this paper, we characterized the distance k- cost effective sets in the join of two graphs. As direct consequences, we determined the bounds or the exact values of the upper distance k-cost effective numbers of the join of graphs.

2. Results

Remark 1. Let G be a connected graph. If $S \subseteq V(G)$ is a distance k- cost effective then every subset of S is also a distance k-cost effective in G.

Definition 1. Let G be a connected graph and k be a positive integer. A set $S \subseteq V(G)$ is a ***distance k- cost effective*** set of G if for every $v \in S, |N^k_G(v) \cap S^c| - |N^k_G(v) \cap S| \geq 0$. The ***upper distance k- cost effective number*** of a graph G, denoted by $\alpha^k_{ce}(G)$, is the maximum cardinality of a distance k - cost effective set in G.

It is worth noting that the concept of a distance 1- cost effective set in G is just equivalent to the concept of a cost effective set in G.

Example 1. Let k and n be positive integers. For any complete graph K_n, a set S is a distance k- cost effective in K_n if and only if $|S| \leq \lfloor\frac{n+1}{2}\rfloor$. Hence, $\alpha^k_{ce}(K_n) = \lfloor\frac{n+1}{2}\rfloor$.

Remark 2. Let G and H be any connected graphs and k be any positive integer.

(i) If $S \subseteq V(G)$ is a distance k- cost effective set in G, then S is a distance k-cost effective set in $G+H$.

(ii) If $S \subseteq V(H)$ is a distance k- cost effective set in H, then S is a distance k- cost effective set in $G+H$.

Remark 3. Let G and H be any connected graphs and k be any positive integer. Then

$$\alpha_{ce}^k(G+H) \geq \max\{\alpha_{ce}^k(G), \alpha_{ce}^k(H)\}.$$

Example 2. Let $G = C_8$ and $H = K_{1,9}$. Then $\alpha_{ce}^1(G+H) = \alpha_{ce}^1(H)$ and for any connected graph G and for all integers $k \geq 2$, $\alpha_{ce}^k(G+K_1) = \alpha_{ce}^k(G)$.

Theorem 1. Let k be a positive integer and G be a connected graph such that $k \geq diam(G)$. Then S is a distance k-cost effective set in G if and only if $|S| \leq \lfloor \frac{|V(G)|+1}{2} \rfloor$.

Proof: Let k be a positive integer and G be a connected graph such that $k \geq diam(G)$. Suppose S is a distance k-cost effective set in G. Then for each $u \in S$,

$$\begin{aligned}
|N_G^k(u) \cap S^c| - |N_G^k(u) \cap S| &= |\big(V(G) \setminus \{u\}\big) \cap S^c| - |\big(V(G) \setminus \{u\}\big) \cap S| \\
&= |S^c| - \big(|S| - 1\big) \\
&= |V(G)| - |S| - |S| + 1 \\
&= |V(G)| + 1 - 2|S| \\
&\geq 0.
\end{aligned}$$

That is, $|S| \leq \frac{|V(G)|+1}{2}$. Since $|S|$ is an integer, $|S| \leq \lfloor \frac{|V(G)|+1}{2} \rfloor$.

Suppose that $|S| \leq \lfloor \frac{|V(G)|+1}{2} \rfloor$. Then $|S| \leq \frac{|V(G)|+1}{2}$. Equivalently, $2|S| \leq |V(G)| + 1$. Let $u \in S$. Then

$$\begin{aligned}
|N_G^k(u) \cap S^c| - |N_G^k(u) \cap S| &= |V(G)| - |S| - |S| + 1 \\
&= |V(G)| + 1 - 2|S| \\
&\geq |V(G)| + 1 - \big[|V(G)| + 1\big] \\
&= 0.
\end{aligned}$$

Thus, S is a distance k-cost effective set in G. $\square$

Corollary 1. If $k \geq diam(G)$, then $\alpha_{ce}^k(G) = \lfloor \frac{|V(G)|+1}{2} \rfloor$.

Corollary 2. Let G and H be connected graphs and an integer $k \geq 2$. Then S is a distance k-cost effective set in $G+H$ if and only if $|S| \leq \lfloor \frac{|V(G)|+|V(H)|+1}{2} \rfloor$. Consequently, $\alpha_{ce}^k(G+H) = \lfloor \frac{|V(G)|+|V(H)|+1}{2} \rfloor$.

Proof: Follows from Theorem 1 since $k \geq diam(G+H) = 2$. $\square$

Remark 4. For all positive integer $k \geq 2$, $\alpha_{ce}^k(G+H) = \alpha_{ce}^2(G+H)$.

Definition 2. Let G be a connected graph and t be a nonnegative integer. A set $S \subseteq V(G)$ is a t- ***fringe*** subset of G if $\Delta(\langle S\rangle_G) \leq t$. The t- fringe number of G, denoted by $\tau_t(G)$, is the maximum cardinality of a t- fringe subset of G.

Example 3. Consider the graph G as shown in Figure 1. Let $S_1 = \{v_3, v_5, v_6, v_7\}$ and $S_2 = \{v_2, v_4, v_5, v_7\}$. Then $\Delta(\langle S_1\rangle_G) = 3$ and $\Delta(\langle S_2\rangle_G) = 0$. Thus, S_1 is *t-fringe* subset of G if $t \geq 3$ while S_2 is a *t-fringe* subset of G if $t \geq 0$.

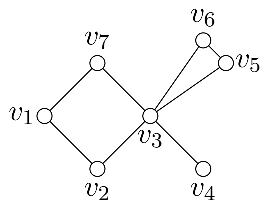

Figure 1: The Graph G with $\Delta(G) = 5$

Theorem 2. Let G be a connected graph, $S \subseteq V(G)$, and $t \geq \lfloor \frac{\Delta(G)}{2} \rfloor$. If S is a distance 1-cost effective set in G, then S is a t-fringe subset of $V(G)$.

Proof: Let $S \subseteq V(G)$ be a distance 1-cost effective set in G. Then for $u \in S$,

$$\begin{aligned}|N_G^1(u) \cap S^c| - |N_G^1(u) \cap S| &= \deg_G(u) - \deg_{\langle S\rangle_G}(u) - \deg_{\langle S\rangle_G}(u)\\ &= \deg_G(u) - 2\deg_{\langle S\rangle_G}(u)\\ &\geq 0.\end{aligned}$$

It follows that $\deg_G(u) - 2\deg_{\langle S\rangle_G}(u) \geq 0$. Hence, $2\deg_{\langle S\rangle_G}(u) \leq \deg_G(u)$. Since u is arbitrary, $2\Delta(\langle S\rangle_G) \leq \Delta(G)$. Equivalently, $\Delta(\langle S\rangle_G) \leq \frac{\Delta(G)}{2}$. Hence, $\Delta(\langle S\rangle_G) \leq \lfloor \frac{\Delta(G)}{2} \rfloor \leq t$. Accordingly, S is a t-fringe subset of $V(G)$. □

Theorem 3. Let G be a connected graph and H be any graph of order $n \geq 2$, $S \subseteq V(G)$, and $t \geq \lfloor \frac{n+\Delta(G)}{2} \rfloor$. If S is a distance 1-cost effective set in $G + H$, then S is a t-fringe subset of $V(G)$.

Proof: Let $u \in S$. Then

$$\begin{aligned}|N_{G+H}^1(u) \cap S^c| - |N_{G+H}^1(u) \cap S| &= \deg_G(u) - \deg_{\langle S\rangle_G}(u) + n - \deg_{\langle S\rangle_G}(u)\\ &= \deg_G(u) + n - 2\deg_{\langle S\rangle_G}(u)\end{aligned}$$

Since S is a distance 1-cost effective set in $G + H$, $\deg_G(u) + n - 2\deg_{\langle S\rangle_G}(u) \geq 0$. Hence, $2\deg_{\langle S\rangle_G}(u) \leq \deg_G(u) + n$. It follows that $2\deg_{\langle S\rangle_G}(u) \leq \Delta(G) + n$, $\forall\ u \in S$. Since u is arbitrary, $2\Delta(\langle S\rangle_G) \leq \Delta(G) + n$. Equivalently, $\Delta(\langle S\rangle_G) \leq \frac{\Delta(G)+n}{2}$. Hence, $\Delta(\langle S\rangle_G) \leq \lfloor \frac{\Delta(G)+n}{2} \rfloor \leq t$. Accordingly, S is a t-fringe subset of $V(G)$. □

Corollary 3. Let G be a connected graph, H be any graph of order $n \geq 2$ and $t \geq \lfloor \frac{n+\Delta(G)}{2} \rfloor$. Then $\alpha^1_{ce}(G+H) \leq \tau_t(G)$.

Theorem 4. Let G be a connected graph and H be any graph of order $n \geq 2$, $S \subseteq V(G)$, and $t \leq \lfloor \frac{n+\delta(G)}{2} \rfloor$. If S is a t-fringe subset of $V(G)$, then S is a distance 1-cost effective set in $G+H$.

Proof: Since S is a t-fringe subset of $V(G)$, then $\Delta(\langle S \rangle_G) \leq t \leq \lfloor \frac{n+\delta(G)}{2} \rfloor \leq \frac{n+\delta(G)}{2}$. It follows that $2\Delta(\langle S \rangle_G) \leq \delta(G) + n$. Let $u \in S$. Then

$$\begin{aligned}
|N^1_{G+H}(u) \cap S^c| - |N^1_{G+H}(u) \cap S| &= \deg_G(u) - \deg_{\langle S \rangle_G}(u) + n - \deg_{\langle S \rangle_G}(u) \\
&= \deg_G(u) + n - 2\deg_{\langle S \rangle_G}(u) \\
&\geq \deg_G(u) + n - 2\Delta(\langle S \rangle_G) \\
&\geq \deg_G(u) + n - (n + \delta(G) \\
&= \deg_G(u) - \delta(G) \\
&\geq 0.
\end{aligned}$$

Thus, S is a distance 1-cost effective set in $G+H$. □

Corollary 4. Let G be a connected graph, H be any graph of order $n \geq 2$, and $t \leq \lfloor \frac{n+\delta(G)}{2} \rfloor$. Then $\alpha^1_{ce}(G+H) \geq \tau_t(G)$.

The next theorem is an immediate consequence of Theorems 3 and 4.

Theorem 5. Let G be any regular graph, H be any graph of order $n \geq 2$ and $t = \lfloor \frac{n+\Delta(G)}{2} \rfloor$. Then $\alpha^1_{ce}(G+H) = \tau_t(G)$.

Definition 3. Let G be a connected graph, $S \subseteq V(G)$ and t be any integer. The $\pi(G:S)$ is defined as $\pi(G:S) = \max\{2\deg_{\langle S \rangle_G}(v) - \deg_G(v) : v \in S\}$. A set S is t- *increment* subset of G if $\pi(G:S) \leq t$.

Example 4. Consider the graph G as shown in Figure 1. Let $S_1 = \{v_2, v_4, v_6, v_7\}$, $S_2 = \{v_1, v_2, v_5, v_6\}$, and $S_3 = \{v_2, v_3, v_5, v_7\}$. Then

$$\begin{aligned}
\pi(G:S_1) &= \max\{2\deg_{\langle S_1 \rangle_G}(u) - \deg_G(u) : u \in S_1\} = -1 \\
\pi(G:S_2) &= \max\{2\deg_{\langle S_2 \rangle_G}(u) - \deg_G(u) : u \in S_2\} = 0 \\
\pi(G:S_3) &= \max\{2\deg_{\langle S_3 \rangle_G}(u) - \deg_G(u) : u \in S_3\} = 1
\end{aligned}$$

Thus, S_1 is -1-increment, S_2 is 0-increment, and S_3 is 1-increment subsets of $V(G)$.

Remark 5. Let G be a connected graph. If $S = V(G)$, then $\pi(G:S) = \Delta(G)$.

Definition 4. Let t be any integer. The t-increment number of G, denoted by $\rho_t(G)$, is the maximum cardinality of a t-increment.

Example 5. Consider the graph G as shown in Figure 1. Observe that the sets $\{v_3, v_5, v_6\}$, $\{v_1, v_3, v_5, v_6\}$, $\{v_1, v_2, v_3, v_5, v_7\}$ and $\{v_1, v_2, v_4, v_5, v_6, v_7\}$ are 2-increment subsets of G while $S_4 = V(G)$ is not a 2-increment subset of G since $\pi(G : S_4) = 5$. Thus, $\rho_2(G) = 6$.

Remark 6. Let G be a connected graph and $t_1 \leq t_2$. If $S \subseteq V(G)$ is a t_1- increment, then S is a t_2- increment. Hence, $\rho_{t_1}(G) \leq \rho_{t_2}(G)$.

Theorem 6. Let G and H be connected graphs of order m and n, respectively. Then S is a distance 1-cost effective set in $G + H$ if and only if any of the following holds:

(i) $S \subseteq V(G)$ is n-increment of G.

(ii) $S \subseteq V(H)$ is m-increment of H.

(iii) $V(G)\cap S$ is $(n-2q)$-increment and $V(H)\cap S$ is $(m-2p)$-increment, where $|V(G)\cap S| = p$ and $|V(H) \cap S| = q$.

Proof: Suppose S is a distance 1-cost effective set in $G + H$. Consider the following cases:

Case 1: $S \subseteq V(G)$
Let $u \in S$. Then

$$\begin{aligned}|N^1_{G+H}(u) \cap S^c| - |N^1_{G+H}(u) \cap S| &= \deg_G(u) - \deg_{\langle S\rangle_G}(u) + n - \deg_{\langle S\rangle_G}(u)\\ &= \deg_G(u) + n - 2\deg_{\langle S\rangle_G}(u)\\ &= n - \big(2\deg_{\langle S\rangle_G}(u) - \deg_G(u)\big)\\ &\geq 0.\end{aligned}$$

Thus, $n - \big(2\deg_{\langle S\rangle_G}(u) - \deg_G(u)\big) \geq 0$, for each $u \in S$. Equivalently, for each $u \in S$, $2\deg_{\langle S\rangle_G}(u) - \deg_G(u) \leq n$. Hence, $\pi(G : S) \leq n$. Thus, (i) holds.

Case 2: $S \subseteq V(H)$
By similar argument to case 1, (ii) holds.

Case 3: $V(G) \cap S \neq \varnothing$ and $V(H) \cap S \neq \varnothing$
Let $u \in V(G) \cap S$. Suppose $|V(G) \cap S| = p$, where $1 \leq p \leq m$. Then for each $u \in S$,

$$\begin{aligned}|N^1_{G+H}(u) \cap S^c| - |N^1_{G+H}(u) \cap S| &= \deg_G(u) - \deg_{\langle V(G)\cap S\rangle}(u) + n - q - \deg_{\langle V(G)\cap S\rangle}(u) - q\\ &= \deg_G(u) - 2\deg_{\langle V(G)\cap S\rangle}(u) + n - 2q.\end{aligned}$$

Since S is a distance 1-cost effective, $\deg_G(u) - 2\deg_{\langle V(G)\cap S\rangle}(u) + n - 2q \geq 0$, $\forall\ u \in S$. Thus, $2\deg_{\langle V(G)\cap S\rangle}(u) - \deg_G(u) \leq n-2q$. This implies that $V(G)\cap S$ is $(n-2q)$-increment of G. Similarly, let $u \in V(H) \cap S$. Then for each $u \in S$,

$$|N^1_{G+H}(u) \cap S^c| - |N^1_{G+H}(u) \cap S| = m - p + \deg_H(u) - \deg_{\langle V(H)\cap S\rangle}(u) - p - \deg_{\langle V(H)\cap S\rangle}(u) - q$$

$$\begin{aligned}
&= m - 2p + \deg_H(u) - 2\deg_{\langle V(H)\cap S\rangle}(u) \\
&\geq 0.
\end{aligned}$$

It follows that $2\deg_{\langle V(H)\cap S\rangle}(u) - \deg_H(u) \leq m - 2p$. This implies that $V(H) \cap S$ is $(m-2p)$-increment of H. Thus, (iii) holds.

Conversely, if (i) holds then $\pi(G : S) \leq n$. Let $u \in S$. Then

$$\begin{aligned}
|N^1_{G+H}(u) \cap S^c| - |N^1_{G+H}(u) \cap S| &= \deg_G(u) + n - 2\deg_{\langle S\rangle_G}(u) \\
&= n - \big(2\deg_{\langle S\rangle_G}(u) - \deg_G(u)\big) \\
&= n - \pi(G : S) \\
&\geq n - n \\
&= 0.
\end{aligned}$$

Thus, S is a distance 1-cost effective set in $G + H$.

Suppose (ii) holds. Let $u \in S$. Then by similar argument, S is a distance 1-cost effective set in $G + H$.

Suppose (iii) holds. Then $\pi(G : V(G) \cap S) \leq n - 2q$ and $\pi(H : V(H) \cap S) \leq m - 2p$. Let $u \in V(G) \cap S$. Then for each $u \in S$,

$$\begin{aligned}
|N^1_{G+H}(u) \cap S^c| - |N^1_{G+H}(u) \cap S| &= \deg_G(u) - 2\deg_{\langle V(G)\cap S\rangle}(u) + n - 2q \\
&= n - 2q - (2\deg_{\langle V(G)\cap S\rangle}(u) - \deg_G(u)) \\
&= n - 2q - \pi(G : V(G) \cap S) \\
&\geq n - 2q - (n - 2q) \\
&= 0.
\end{aligned}$$

Thus, S is a distance 1-cost effective set in $G + H$.

Similarly, let $u \in V(H) \cap S$. Then for each $u \in S$,

$$\begin{aligned}
|N^1_{G+H}(u) \cap S^c| - |N^1_{G+H}(u) \cap S| &= m - 2p + \deg_H(u) - 2\deg_{\langle V(H)\cap S\rangle}(u) \\
&= m - 2p - \big(2\deg_{\langle V(H)\cap S\rangle}(u) - \deg_H(u)\big) \\
&= m - 2p - \pi(H : V(H) \cap S) \\
&\geq m - 2p - (m - 2p) \\
&= 0.
\end{aligned}$$

Thus, S is a distance 1-cost effective set in $G + H$. □

Corollary 5. Let G be a connected graph, $S \subseteq V(G)$ and $n \geq 2$. Then S is a distance 1-cost effective set in $G + K_n$ if and only if S is n-increment.

Corollary 6. Let G be a connected graph and $n \geq 2$. A set $S \subseteq V(K_n)$ is a distance 1-cost effective in $G + K_n$ if and only if $|S| \leq \lfloor \frac{|V(G)|+n+1}{2} \rfloor$.

Corollary 7. Let G be a graph, $n \geq 2$ and $S \subseteq (G + K_n)$ such that $V(G) \cap S \neq \varnothing$ and $V(K_n) \cap S \neq \varnothing$. Then S is a distance 1-cost effective set in $G + K_n$ if and only if the following hold:

(i) $|S| \leq \lfloor \frac{|V(G)|+n+1}{2} \rfloor$; and

(ii) $V(G) \cap S$ is $(n - 2p)$-increment, where $p = |V(K_n) \cap S|$ and $1 \leq p \leq n$.

Theorem 7. Let G and H be connected graphs of order m and n, respectively. Then

$$\alpha^1_{ce}(G + H) = \max\{\rho_n(G), \rho_m(H), \rho_{n-2q}(G) + \rho_{m-2p}(H), 1 \leq p \leq m,\ 1 \leq q \leq n\}.$$

Proof: Let S be an upper distance 1-cost effective set in $G + H$. Suppose $S \subseteq V(G)$. Then by theorem 6(i), $|S| = \rho_n(G)$. Suppose $S \subseteq V(H)$. Then by Theorem 6(ii), $|S| = \rho_m(H)$. Suppose $V(G) \cap S \neq \varnothing$ and $V(H) \cap S \neq \varnothing$. Then by Theorem 6(iii), $|S| = \rho_{n-2q}(G) + \rho_{m-2p}(H)$. Therefore,

$$\alpha^1_{ce}(G + H) = \max\{\rho_n(G), \rho_m(H), \rho_{n-2q}(G) + \rho_{m-2p}(H), 1 \leq p \leq m,\ 1 \leq q \leq n\}.$$

□

Corollary 8. Let G be a graph and $n \geq 2$. Then

$$\alpha^1_{ce}(G + K_n) = \max\{\lfloor \frac{|V(G)|+n+1}{2} \rfloor, \rho_n(G)\}.$$

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