EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 13, No. 2, 2020, 351-368
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Approximation of Function in generalized Hölder Class

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Abstract. In the present work, we study error estimation of a function $g \in H_r^{(\eta)}$ $(r \geq 1)$ class using Matrix-Hausdorff $(T\Delta_H)$ means of its Fourier series. Our Theorem 1 generalizes twelve previously known results. Thus, the results of [4, 5, 11–16, 18, 26, 29, 30] become the particular cases of our Theorem 1. Several useful results in the form of corollaries are also deduced from our Theorem 1.

2020 Mathematics Subject Classifications: 41A10, 41A25, 42B05, 42A10, 40G05, 40C05

Key Words and Phrases: Error estimation, Generalized Hölder class, Fourier series, Matrix (T) means, Hausdorff (Δ_H) means, Matrix-Hausdorff $(T\Delta_H)$ product means

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1. Introduction

In the past few decades, the researchers have been greatly interested in studying the error estimation of functions in different function spaces using summability operators due to their variety of applications in science and engineering. In this direction, several researchers like [2, 3, 9, 10, 19–23, 25, 28] have obtained results on error estimation of functions in different Lipschitz classes and Hölder classes with different single summability operators. Taking a view point that a product summability is more effective than the individual single summability operator, researchers like [11, 13, 18, 27–29], have obtained error estimation of functions in various Lipschitz and Hölder classes using different product summability operators.

After reviewing the above mentioned works, we observe that these works cannot provide the best error estimation of a function in the function spaces considered in their works. This fact strongly motivates us to consider a more advanced class of function, which provide the best approximation of a function using summability operator.

Therefore, in the present work, we establish a theorem on the best error approximation of a function g in the generalized Hölder class $H_r^{(\eta)}$ $(r \geq 1)$ using Matrix-Hausdorff $(T\Delta_H)$

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DOI: https://doi.org/10.29020/nybg.ejpam.v13i2.3667

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product operator of its Fourier series. Our main theorem generalizes tweleve previously known results. Thus, the results of [4, 5, 11–16, 18, 26, 29, 30] become the particular cases of our theorem.

2. Preliminaries

Let $\sum_{l=0}^{\infty} c_l$ be an infinite series having l^{th} partial sum $s_l = \sum_{\nu=0}^{l} c_\nu$.

Let $T \equiv (b_{l,j})$ be an infinite triangular matrix satisfying the conditions of regularity [24] i.e.

$$\begin{cases} \sum_{j=0}^{l} b_{l,j} = 1 \quad \text{as} \quad l \to \infty; \\ \forall \quad j \geq 0, \quad b_{l,j} = 0 \quad \text{as} \quad l \to \infty; \\ \exists \quad M > 0 \quad \forall \quad l \geq 0, \quad \sum_{j=0}^{\infty} |b_{l,j}| < M. \end{cases} \tag{1}$$

The sequence-to-sequence transformation

$$\begin{aligned} t_l^T &:= \sum_{j=0}^{l} b_{l,j} s_j \\ &= \sum_{j=0}^{l} b_{l,l-j} s_{l-j} \end{aligned}$$

defines the sequence t_l^T of triangular matrix means of the sequence $\{s_l\}$ generated by the sequence of coefficients $(b_{l,j})$.

If $t_l^T \to s$ as $l \to \infty$, then the infinite series $\sum_{l=0}^{\infty} c_l$ or the sequence $\{s_l\}$ is summable to s by triangular matrix (T) [1].

A Hausdorff matrix $H \equiv (h_{l,j})$ is an infinite lower triangular matrix [8] defined by

$$h_{l,j} \equiv \begin{cases} \begin{pmatrix} l \\ j \end{pmatrix} \Delta^{l-j} \mu_j, & 0 \leq j \leq l; \\ 0, & j > l, \end{cases}$$

where the operator Δ is defined $\Delta \mu_j \equiv \mu_j - \mu_{j+1}$ and $\Delta^{l+1} \mu_j \equiv \Delta^l (\Delta \mu_j)$.
If $t_l^{\Delta_H} = \sum_{m=0}^{l} h_{l,m} s_m \to s$ as $l \to \infty$ then the series or the sequence $\{s_l\}$ is summable to the sum s by the Hausdorff method (Δ_H method).

A Hausdorff matrix H is regular, i.e., H preserves the limit of each convergent sequence iff

$$\int_0^1 |d\xi(z)| < \infty,$$

where the mass function $\xi \in BV[0,1]$, $\xi(0+) = \xi(0) = 0$, and $\xi(1) = 1$. In this case, μ_l has the representation

$$\mu_l = \int_0^1 z^l d\xi(z) \quad [17].$$

Superimposing T- method on Δ_H method, $(T\Delta_H)$ is obtained. $T\Delta_H$ mean of the sequence $\{s_l\}$ is given by

$$\begin{aligned} t_l^{T\Delta_H} &:= \sum_{j=0}^{l} b_{l,j} t_j^{\Delta_H} \\ &= \sum_{j=0}^{l} b_{l,j} \sum_{v=0}^{j} h_{j,v} s_v. \end{aligned}$$

If $t_l^{T\Delta_H} \to s$ as $l \to \infty$, then $\{s_l\}$ is summable by the $T\Delta_H$ means to the limit s.

Since T and Δ_H method are regular, then $T\Delta_H$ method is also regular. This can be shown as

$$\begin{aligned} s_l \to s \quad &\Rightarrow \quad t_l^{\Delta_H} \to s, \text{ as } l \to \infty, \text{ since the } \Delta_H \text{ method is regular,} \\ &\Rightarrow \quad T(t_l^{\Delta_H}) = t_l^{T\Delta_H} \to s, \text{ as } l \to \infty, \text{ since the } T \text{ method is regular,} \\ &\Rightarrow \quad T\Delta_H \text{ method is regular.} \end{aligned}$$

Remark 1. *$T\Delta_H$ means reduces to*

(i) $(C,\alpha)\Delta_H$ *or* $C_\alpha\Delta_H$ *means when* $b_{l,j} = \frac{\binom{l-j+\alpha-1}{\alpha-1}}{\binom{l+\alpha}{\alpha}}$ *for all* $\alpha \geq -1$.

(ii) $\left(H, \frac{1}{l+1}\right)\Delta_H$ *or* $H_{1/l+1}\Delta_H$ *means if* $b_{l,j} = \frac{1}{(l-j+1)\log(l+1)}$.

(iii) $(N, p_l, q_l)\Delta_H$ *or* $N_{p,q}\Delta_H$ *means if* $b_{l,j} = \frac{p_{l-j}q_j}{R_l}$, $R_l = \sum_{j=0}^{l} p_j q_{l-j}$.

(iv) $(N, p_l)\Delta_H$ *or* $N_p\Delta_H$ *means if* $b_{l,j} = \frac{p_{l-j}}{P_l}$ *where* $P_l = \sum_{j=0}^{l} p_j$, $q_l = 1$.

(v) $(\tilde{N}, p_l)\Delta_H$ *or* $\tilde{N}_p\Delta_H$ *means if* $b_{l,j} = \frac{p_j}{P_l}$, $q_l = 1 \ \forall \quad l$.

(vi) $(E, q_l)\Delta_H$ *or* $E_q\Delta_H$ *means if* $b_{l,j} = \frac{1}{(1+q)^l}\binom{l}{j} q^{l-j}$.

(vii) $T(C,\alpha)$ *or* TC_α *means if* $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \geq 1$.

(viii) $T(E, q_l)$ *or* TE_q *means if* $h_{l,j} = \binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0 \leq j \leq l$.

In above Remark 1 (iii), (iv) and (v), $\{p_l\}$ *and* $\{q_l\}$ *are two non-negative monotonic non-decreasing sequence of real constants.*

Remark 2.

(i) $(C,\alpha)\Delta_H$ *or* $C_\alpha\Delta_H$ *means further reduces to*

(a) $(C,\alpha)(C,\alpha)$ *or* $C_\alpha C_\alpha$ *means if* $\xi(z)=\prod_{j=1}^{\alpha} z^j$, $\alpha\geq 1$.

(b) $(C,\alpha)(E,q_l)$ *or* $C_\alpha E_q$ *means if* $h_{l,j}=\binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0\leq j\leq l$.

(c) $(C,1)\Delta_H$ *or* $C_1\Delta_H$ *means if* $\alpha=1$.

(ii) $\left(H,\frac{1}{l+1}\right)\Delta_H$ *or* $H_{1/l+1}\Delta_H$ *means further reduces to*

(a) $\left(H,\frac{1}{l+1}\right)(C,\alpha)$ *or* $H_{1/l+1}C_\alpha$ *means if* $\xi(z)=\prod_{j=1}^{\alpha} z^j$, $\alpha\geq 1$.

(b) $\left(H,\frac{1}{l+1}\right)(E,q_l)$ *or* $H_{1/l+1}E_q$ *if* $h_{l,j}=\binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0\leq j\leq l$.

(iii) $(N,p_l,q_l)\Delta_H$ *or* $N_{p,q}\Delta_H$ *means further reduces to*

(a) $(N,p_l,q_l)(C,\alpha)$ *or* $N_{p,q}C_\alpha$ *means if* $\xi(z)=\prod_{j=1}^{\alpha} z^j$, $\alpha\geq 1$.

(b) $(N,p_l,q_l)(E,q_l)$ *or* $N_{p,q}E_q$ *means if* $h_{l,j}=\binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0\leq j\leq l$.

(iv) $(N,p_l)\Delta_H$ *or* $N_p\Delta_H$ *means further reduces to*

(a) $(N,p_l)(C,\alpha)$ *or* N_pC_α *means if* $\xi(z)=\prod_{j=1}^{\alpha} z^j$, $\alpha\geq 1$.

(b) $(N,p_l)(E,q_l)$ *or* N_pE_q *means if* $h_{l,j}=\binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0\leq j\leq l$.

(v) $(\tilde{N},p_l)\Delta_H$ *or* $\tilde{N}_p\Delta_H$ *means further reduces to*

(a) $(\tilde{N},p_l)(C,\alpha)$ *or* $\tilde{N}_pC_\alpha$ *means if* $\xi(z)=\prod_{j=1}^{\alpha} z^j$, $\alpha\geq 1$.

(b) $(\tilde{N},p_l)(E,q_l)$ *or* $\tilde{N}_pE_q$ *means if* $h_{l,j}=\binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0\leq j\leq l$.

(vi) $(E,q_l)\Delta_H$ *or* $E_q\Delta_H$ *means further reduces to*

(a) $(E,q_l)(C,\alpha)$ *or* E_qC_α *means if* $\xi(z)=\prod_{j=1}^{\alpha} z^j$, $\alpha\geq 1$.

(b) $(E,q_l)(E,q_l)$ *or* E_qE_q *means if* $h_{l,j}=\binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0\leq j\leq l$.

(vii) $T(C,\alpha)$ *or* TC^α *means further reduces to*

(a) $T(C,1)$ *or* TC^1 *means if* $\alpha=1$.

(viii) $T(E,q_l)$ *or* TE_q *means further reduces to*

(a) $T(E,1)$ *or* TE_1 *means if* $q_l=1\ \forall\ l$.

Remark 3.

(i) *Above particular case (i)(b) in Remark 2 is further reduced to* C_1E_q, $C_\alpha E_1$ *and* C_1E_1 *means for* $\alpha=1$, $q_l=1\ \forall\ l$ *and* $\alpha=1$, $q_l=1\ \forall\ l$ *respectively.*

(ii) *Above particular cases (ii)(a) and (b) in Remark 2 are further reduced to* $H_{1/l+1}C_1$ *and* $H_{1/l+1}E_1$ *means for* $\alpha=1$ *and* $q_l=1\ \forall\ l$ *respectively.*

(iii) *Above particular cases (iii)(a) and (b) in Remark 2 are further reduced to* $(N, p_l, q_l)(C, 1)$ *and* $(N, p_l, q_l)(E, 1)$ *means for* $\alpha = 1$ *and* $q_l = 1\ \forall\ l$ *respectively.*

(iv) *Above particular cases (iv)(a) and (b) in Remark 2 are further reduced to* $(N, p_l)(C, 1)$ *and* $(N, p_l)(E, 1)$ *means for* $\alpha = 1$ *and* $q_l = 1\ \forall\ l$ *respectively.*

(v) *Above particular cases (v)(a) and (b) in Remark 2 are further reduced to* $(\tilde{N}, p_l)(C, 1)$ *and* $(\tilde{N}, p_l)(E, 1)$ *means for* $\alpha = 1$ *and* $q_l = 1\ \forall\ l$ *respectively.*

(vi) *Above particular cases (vi)(a) in Remark 2 is further reduced to* E_qC_1, E_1C_α *and* E_1C_1 *means for* $\alpha = 1$, $q_l = 1\ \forall\ l$ *and* $q_l = 1\ \forall\ l$, $\alpha = 1$ *respectively.*

The space of the functions L^r is given by

$$L^r[0, 2\pi] = \left\{ g : [0, 2\pi] \mapsto \mathbb{R} : \int_0^{2\pi} |g(x)|^r dx < \infty, r \geq 1 \right\}.$$

The norm $\| \cdot \|_{(r)}$ by

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(x)|^r dx \right\}^{1/r}, \quad r \geq 1.$$

As defined in [1], $\eta : [0, 2\pi] \mapsto \mathbb{R}$ is an arbitrary function with $\eta(s) > 0$ for $0 < s \leq 2\pi$ and $\lim_{s \to 0^+} \eta(s) = \eta(0) = 0$.

Now, we define

$$H_r^{(\eta)} := \left\{ g \in L^r[0, 2\pi] : \sup_{s \neq 0} \frac{\|g(\cdot, +s) - g(\cdot)\|_r}{\eta(s)} < \infty, r \geq 1 \right\}$$

and

$$\| \cdot \|_r^{(\eta)} = \|g\|_r^{(\eta)} = \|g\|_r + \sup_{s \neq 0} \frac{\|g(\cdot, +s) - g(\cdot)\|_r}{\eta(s)}; r \geq 1.$$

Clearly, $\| \cdot \|_r^{(\eta)}$ is a norm on $H_r^{(\eta)}$.

Note 1. *$\eta(s)$ and $\chi(s)$ denote moduli of continuity of order two such that $\frac{\eta(s)}{\chi(s)}$ is positive, non-decreasing and*

$$\|g\|_r^{(\chi)} \leq \max\left(1, \frac{\eta(2\pi)}{\chi(2\pi)}\right) \|g\|_r^{(\eta)} < \infty.$$

Thus,

$$H_r^{(\eta)} \subset H_r^{(\chi)} \subset L^r; \quad r \geq 1 \quad [1].$$

Remark 4.

(i) *If* $\eta(s) = s^\alpha$ *in* $H^{(\eta)}$, $H^{(\eta)}$ *implies* $H^{(\alpha)}$ *class.*

(ii) *If* $\eta(s) = s^\alpha$ *in* $H_r^{(\eta)}$, $H^{(\eta)}$ *implies* $H_{\alpha,r}$ *class.*

(iii) If $r \to \infty$ in $H_r^{(\eta)}$, $H_r^{(\eta)}$ implies $H^{(\eta)}$ class and $H_{\alpha,r}$ implies H_α class.

We denote the l^{th} partial sum of the Fourier series as

$$s_l(g;x) - g(x) = \frac{1}{2\pi}\int_0^\pi \phi(x,s)\frac{\sin\left(l+\frac{1}{2}\right)s}{\sin\frac{s}{2}}ds \quad [1].$$

The l-order error estimation of function g is given by

$$E_l(g) = \min \|g - t_l\|_r,$$

where t_l is a trigonometric polynomial of degree l [1].

If $E_l(g) \to 0$ as $l \to \infty$, then $E_l(g)$ is said to be the best approximation of g [1].

We write

$$\begin{aligned}
\phi(x,s) &= g(x+s) + g(x-s) - 2g(x);\\
\Delta b_{l,j} &= b_{l,j} - b_{l,j+1};\\
K_l^{T\Delta_H}(s) &= \frac{1}{2\pi}\sum_{j=0}^{l} b_{l,j}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}\ d\xi(z)\frac{\sin\left(a+\frac{1}{2}\right)s}{\sin\frac{s}{2}}.
\end{aligned}$$

3. Main Theorem

Theorem 1. *If $g \in H_r^{(\eta)}$ class, $r \geq 1$, then the error estimation of g using $T\Delta_H$ product means of its Fourier series is given by*

$$\|t_l^{T\Delta_H} - g\|_r^{(\chi)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right),$$

where $T \equiv (b_{l,j})$ is an infinite triangular matrix satisfying (1) and η, χ are as defined in Note 1, provided

$$\sum_{j=0}^{l-1}|\Delta b_{l,j}| = O\left(\frac{1}{l+1}\right) \tag{2}$$

and

$$(l+1)b_{l,l} = O(1). \tag{3}$$

4. Lemmas

Lemma 1. *Under the conditions of regularity of matrix $T \equiv (b_{l,j})$,*

$$K_l^{T\Delta_H}(s) = O(l+1) \quad \textit{for} \quad 0 < s < \frac{1}{l+1}.$$

Proof. For $0 \leq s \leq \frac{1}{l+1}$, $\sin\frac{s}{2} \geq \frac{s}{\pi}$, $\sin ls \leq ls$, we have

$$
\begin{aligned}
K_l^{T\Delta_H}(s) &= \frac{1}{2\pi}\sum_{j=0}^{l} b_{l,j}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}\ d\xi(z)\frac{\sin\left(a+\frac{1}{2}\right)s}{2\sin\frac{s}{2}} \\
&= \frac{1}{2\pi}\sum_{j=0}^{l} b_{l,j}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}\ d\xi(z)\frac{(2a+1)\frac{s}{2}}{\frac{s}{\pi}} \\
&= \frac{1}{4}\sum_{j=0}^{l} b_{l,j}\left\{\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}\ d\xi(z)(2a+1)\right\} \\
&= \frac{1}{4}\sum_{j=0}^{l} b_{l,j}\left[2\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}\ a\ d\xi(z)\right] \\
&+ \frac{1}{4}\sum_{j=0}^{l} b_{l,j}\left[\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}\ d\xi(z)\right]. \qquad (4)
\end{aligned}
$$

First, we solve

$$
\begin{aligned}
2\sum_{a=0}^{j}\binom{j}{a} z^a(1-z)^{j-a}a &= 2(1-z)^j\sum_{a=0}^{j}\binom{j}{a}\left(\frac{z}{1-z}\right)^a a \\
&= 2(1-z)^j\sum_{a=0}^{j}\binom{j}{a} d^a a, \qquad (5) \\
&\quad \text{where } \frac{z}{1-z} = d.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\sum_{a=0}^{j}\binom{j}{a} d^a a &= \binom{j}{0}d^0 0 + \binom{j}{1}d^1 1 + \binom{j}{2}d^2 2 + \cdots + \binom{j}{j}d^j j \\
&= \binom{j}{1}d + 2\binom{j}{2}d^2 + 3\binom{j}{3}d^3\cdots + j\binom{j}{j}d^j. \qquad (6)
\end{aligned}
$$

We observe that

$$
\begin{aligned}
(1+d)^j &= \binom{j}{0}1^{j-0}\cdot d^0 + \binom{j}{1}1^{j-1}\cdot d^1 + \binom{j}{2}1^{j-2}\cdot d^2 + \cdots + \binom{j}{j}1^{j-j}\cdot d^j \\
(1+d)^j &= \binom{j}{0} + \binom{j}{1}d + \binom{j}{2}d^2 + \cdots + \binom{j}{j}d^j \\
j(1+d)^{j-1} &= 0 + \binom{j}{1} + 2\binom{j}{2}d + 3\binom{j}{3}d^2 + \cdots + j\binom{j}{j}d^{j-1} \\
&\quad \text{(by differentiating w.r.t } d\text{)}
\end{aligned}
$$

$$jd(1+d)^{j-1} = \binom{j}{1}d + 2\binom{j}{2}d^2 + 3\binom{j}{3}d^3 + \cdots + j\binom{j}{j}d^j \tag{7}$$

(multiplying both side by d).

Now, from (6) and (7), we get

$$\begin{aligned}\sum_{a=0}^{j}\binom{j}{a}d^a a &= jd(1+d)^{j-1}\\ &= j\left(\frac{z}{1-z}\right)\left(\frac{1}{(1-z)^{j-1}}\right)\\ &= \frac{jz}{(1-z)^j}.\end{aligned} \tag{8}$$

Thus, from (5) and (8), we get

$$\begin{aligned}2\sum_{a=0}^{j}\binom{j}{a}z^a(1-z)^{j-a}a &= 2(1-z)^j\sum_{a=0}^{j}\binom{j}{a}d^a a\\ &= 2(1-z)^j\frac{jz}{(1-z)^j}\\ &= 2jz.\end{aligned} \tag{9}$$

Now,

$$\begin{aligned}\sum_{a=0}^{j}\binom{j}{a}z^a(1-z)^{j-a} &= \binom{j}{0}z^0(1-z)^j + \binom{j}{1}z^1(1-z)^{j-1} + \cdots + \binom{j}{j}z^j(1-z)^{j-j}\\ &= (1-z+z)^j\\ &= 1.\end{aligned} \tag{10}$$

Thus, from (4), (9) and (10), we get

$$\begin{aligned}K_l^{T\Delta_H}(s) &= \frac{1}{4}\sum_{j=0}^{l}b_{l,j}\left[\sum_{a=0}^{j}\int_0^1\binom{j}{a}z^a(1-z)^{j-a}(2a+1)\ d\xi(z)\right]\\ &= \frac{1}{4}\sum_{j=0}^{l}b_{l,j}\int_0^1(2jz+1)\ dz\\ &= \frac{1}{4}\sum_{j=0}^{l}b_{l,j}(j+1).\\ &= O(l+1)\sum_{j=0}^{l}b_{l,j}\\ &= O(l+1).\end{aligned}$$

Lemma 2. *Under the conditions of regularity of matrix $T \equiv (b_{l,j})$,*

$$K_l^{T\Delta_H}(s) = O\left(\frac{1}{s^2(l+1)}\right) \quad \textit{for} \quad \frac{1}{l+1} \leq s \leq \pi.$$

Proof. For $\frac{1}{l+1} \leq s \leq \pi$, $\sin\frac{s}{2} \geq \frac{s}{\pi}$, $\sin^2 ls \leq 1$ and $\sup_{0\leq z\leq 1}|\xi'(z)| = N$, we have

$$\begin{aligned}
K_l^{T\Delta_H}(s) &= \frac{1}{\pi}\sum_{j=0}^{l} b_{l,j}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}\ d\xi(z)\frac{\sin\left(a+\frac{1}{2}\right)s}{2\sin\frac{s}{2}} \\
&= \frac{1}{2\pi}\sum_{j=0}^{l} b_{l,j}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}\ d\xi(z)\frac{\sin\left(a+\frac{1}{2}\right)s}{\frac{s}{\pi}} \\
&= \frac{1}{2s}\sum_{j=0}^{n} b_{l,j}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}\ d\xi(z)\sin\left(a+\frac{1}{2}\right)s \\
&\leq \frac{N}{2s}\left|\sum_{j=0}^{l} b_{l,j}\mathrm{Im}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}e^{i\left(a+\frac{1}{2}\right)s}\ d\xi(z)\right|. \qquad (11)
\end{aligned}$$

Now, first we solve

$$\begin{aligned}
\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}\sin\left(a+\frac{1}{2}\right)s\ d\xi(z) &= (1-z)^j\sum_{a=0}^{j}\int_0^1\binom{j}{a}\left(\frac{z}{1-z}\right)^a \mathrm{Im}\left\{e^{i\left(a+\frac{1}{2}\right)s}\right\}\ d\xi(z) \\
&= (1-z)^j\sum_{a=0}^{j}\int_0^1\binom{j}{a}\left(\frac{z}{1-z}\right)^a \mathrm{Im}\left\{e^{ias}\cdot e^{\frac{is}{2}}\right\}\ d\xi(z) \\
&= (1-z)^j\mathrm{Im}\left[e^{\frac{is}{2}}\sum_{a=0}^{j}\int_0^1\binom{j}{a}\left(\frac{ze^{is}}{1-z}\right)^a d\xi(z)\right] \\
&= \mathrm{Im}\left[e^{\frac{is}{2}}\int_0^1(1-z+ze^{is})^j dz\right] \\
&= \mathrm{Im}\left[e^{\frac{is}{2}}\int_0^1\left\{1+z(e^{is}-1)\right\}^j dz\right] \\
&= \mathrm{Im}\left[\frac{e^{i(j+1)s}-1}{(1+j)(e^{\frac{is}{2}}-e^{\frac{-is}{2}})}\right] \\
&= \mathrm{Im}\left[\frac{e^{i(j+1)s}-1}{(j+1)2i\sin\frac{s}{2}}\right] \\
&= \mathrm{Im}\left[\frac{\cos(j+1)s+i\sin(j+1)s-1}{2i(j+1)\sin\frac{s}{2}}\right] \\
&= \frac{\sin^2(j+1)\frac{s}{2}}{(j+1)\sin\frac{s}{2}}. \qquad (12)
\end{aligned}$$

Now, from (11) and (12), we get

$$
\begin{aligned}
K_l^{T\Delta_H}(s) &\leq \frac{N}{2s}\Bigg|\sum_{j=0}^{l} b_{l,j}\frac{\sin^2(j+1)\frac{s}{2}}{(j+1)\sin\frac{s}{2}}\Bigg| \\
&\leq \frac{N}{2s}\Bigg|\sum_{j=0}^{l} b_{l,j}\frac{1}{(j+1)\frac{s}{\pi}}\Bigg| \\
&= \frac{N\pi}{2s^2}\Bigg|\sum_{j=0}^{l} b_{l,j}\frac{1}{j+1}\Bigg|
\end{aligned}
$$

Using Abel's Lemma, we have

$$
\begin{aligned}
K_l^{T\Delta_H}(s) &= \frac{N\pi}{2s^2}\Bigg|\sum_{j=0}^{l-1}(b_{l,j}-b_{l,j+1})\sum_{k=0}^{j}\frac{1}{k+1}+b_{l,l}\sum_{j=0}^{l}\frac{1}{j+1}\Bigg| \\
&\leq \frac{N\pi}{2s^2}\Bigg|\sum_{j=0}^{l-1}\Delta b_{l,j}\sum_{k=0}^{j}\frac{1}{k+1}\Bigg| + b_{l,l}\Bigg|\sum_{j=0}^{l}\frac{1}{j+1}\Bigg| \\
&\leq \frac{N\pi}{2s^2}\left[\sum_{j=0}^{l-1}|\Delta b_{l,j}|+b_{l,l}\right]\max_{0\leq j\leq p}\Bigg|\sum_{j=0}^{p}\frac{1}{j+1}\Bigg| \\
&= \frac{N\pi}{2s^2}\left[O\left(\frac{1}{l+1}\right)+O\left(\frac{1}{l+1}\right)\right] \\
&= O\left(\frac{1}{s^2(l+1)}\right).
\end{aligned}
$$

Lemma 3. [*28*] *Let* $g \in H_r^{(\eta)}$, *then for* $0 < s \leq \pi$:

(i) $\|\phi(\cdot, s)\|_r = O(\eta(s))$;

(ii) $\|\phi(\cdot + z, s) - \phi(\cdot, s)\|_r = \begin{cases} O(\eta(s)) \\ O(\eta(z)). \end{cases}$

(iii) If $\eta(s)$ *and* $\chi(s)$ *are as defined in Note 1, then* $\|\phi(\cdot+z, s)-\phi(\cdot, s)\|_r = O\left(\chi(|z|)\left(\frac{\eta(s)}{\chi(s)}\right)\right)$.

5. Proof of the main theorem

5.1. Proof of Theorem 1

Proof. Following [7], $s_l(g; x)$ of Fourier series

$$
s_l(g; x) - g(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(x, s)\frac{\sin\left(l+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds.
$$

The Hausdorff matrix mean of $s_l(x)$, denoted by $t_l^{\Delta_H}(x)$, we get

$$\begin{aligned} t_l^{\Delta_H}(x)-g(x) &= \sum_{j=0}^{l} h_{l,j}(s_j(x)-g(x)) \\ &= \sum_{j=0}^{l}\binom{l}{j}\Delta^{l-j}\mu_j\left\{\frac{1}{2\pi}\int_0^{\pi}\phi(x,s)\frac{\sin\left(j+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\ ds\right\} \\ &= \frac{1}{2\pi}\int_0^{\pi}\phi(x,s)\sum_{j=0}^{l}\binom{l}{j}\Delta^{l-j}\left(\int_0^1 z^j\ d\xi(z)\right)\frac{\sin\left(j+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\ ds \\ &= \frac{1}{2\pi}\int_0^{\pi}\phi(x,s)\sum_{j=0}^{l}\int_0^1\binom{l}{j}z^j(1-z)^{l-j}\ d\xi(z)\frac{\sin\left(j+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\ ds. \end{aligned}$$

The T transform of $t_l^{\Delta_H}(x)$ denoted by $t_l^{T\Delta_H}(x)$, is given by

$$\begin{aligned} t_l^{T\Delta_H}(x)-g(x) &= \sum_{j=0}^{l} b_{l,j}\left(\frac{1}{2\pi}\int_0^{\pi}\phi(x,s)\sum_{a=0}^{j}\int_0^1\binom{j}{a}z^a(1-z)^{j-a}\ d\xi(z)\frac{\sin\left(a+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\ ds\right) \\ &= \frac{1}{2\pi}\int_0^{\pi}\phi(x,s)\sum_{j=0}^{l}b_{l,j}\sum_{a=0}^{j}\int_0^1\binom{j}{a}z^a(1-z)^{j-a}\ d\xi(z)\frac{\sin\left(a+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\ ds \\ &= \int_0^{\pi}\phi(x,s)K_l^{T\Delta_H}(s)\ ds. \end{aligned}$$

Let

$$T_l(x)=t_l^{T\Delta_H}(x)-g(x)=\int_0^{\pi}\phi(x,s)K_l^{T\Delta_H}(s)\ ds.$$

Then

$$T_l(x+z)-T_l(x)=\int_0^{\pi}\left(\phi(x+z,s)-\phi(x,s)\right)K_l^{T\Delta_H}(s)\ ds.$$

Using generalized Minkowski's inequality [6], we obtain

$$\begin{aligned} \|T_l(\cdot,+z)-T_l(\cdot)\|_r &\le \int_0^{\pi}\|\phi(\cdot+z,s)-\phi(\cdot,s)\|_r K_l^{T\Delta_H}(s)\ ds \\ &= \int_0^{\frac{1}{l+1}}\|\phi(\cdot+z,s)-\phi(\cdot,s)\|_r K_l^{T\Delta_H}(s)\ ds \\ &+ \int_{\frac{1}{l+1}}^{\pi}\|\phi(\cdot+z,s)-\phi(\cdot,s)\|_r K_l^{T\Delta_H}(s)\ ds \\ &= I_1+I_2. \end{aligned} \tag{13}$$

Using Lemmas 1 and 3 (iii), we get

$$I_1 = \int_0^{\frac{1}{l+1}}\|\phi(\cdot+z,s)-\phi(\cdot,s)\|_r K_l^{T\Delta_H}(s)\ ds$$

$$\begin{aligned} &= O(l+1)\left(\chi(|z|)\int_0^{\frac{1}{l+1}} \frac{\eta(s)}{\chi(s)}\,ds\right) \\ &= \left(\chi(|z|)\frac{\eta(\frac{1}{l+1})}{\chi(\frac{1}{l+1})}\right). \end{aligned} \tag{14}$$

Also, using Lemmas 2 and 3 (iii), we get

$$\begin{aligned} I_2 &= \int_{\frac{1}{l+1}}^{\pi} \|\phi(\cdot+z,s)-\phi(\cdot,s)\|_r K_l^{T\Delta_H}(s)\,ds \\ &= O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \chi(|z|)\frac{\eta(s)}{s^2\chi(s)}\,ds\right). \end{aligned} \tag{15}$$

From (13), (14) and (15), we have

$$\sup_{z\neq 0}\frac{\|T_l(\cdot,+z)-T_l(\cdot)\|_r}{\chi(|z|)} = O\left(\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)}\right) + O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}\,ds\right). \tag{16}$$

Again applying Minkowski's inequality and using Lemmas 1, 2 and 3 (i), we obtain

$$\begin{aligned} \|T_l(\cdot)\|_r &= \|t_l^{T\Delta_H}-g\|_r \\ &\leq \left(\int_0^{\frac{1}{l+1}}+\int_{\frac{1}{l+1}}^{\pi}\right)\|\phi(\cdot,s)\|_r K_l^{T\Delta_H}(s)\,ds \\ &= O\left((l+1)\int_0^{\frac{1}{l+1}}\eta(s)\,ds\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2}\,ds\right) \\ &= O\left(\eta\left(\frac{1}{l+1}\right)\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2}\,ds\right). \end{aligned} \tag{17}$$

We know that

$$\|T_l(\cdot)\|_r^{(\chi)} = \|T_l(\cdot)\|_r + \sup_{z\neq 0}\frac{\|T_l(\cdot,+z)-T_l(\cdot)\|_r}{\chi(|z|)}. \tag{18}$$

Now, using (16), (17) and (18), we get

$$\begin{aligned} \|T_l(\cdot)\|_r^{(\chi)} &= O\left(\eta\left(\frac{1}{l+1}\right)\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2}\,ds\right) \\ &+ O\left(\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)}\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}\,ds\right). \end{aligned} \tag{19}$$

By the monotonicity of $\chi(s)$, $\eta(s) = \frac{\eta(s)}{\chi(s)}\chi(s) \leq \chi(\pi)\frac{\eta(s)}{\chi(s)}$ for $0 < s \leq \pi$, we get

$$\|T_l(\cdot)\|_r^{(\chi)} = O\left(\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)}\right) + O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\,ds\right). \tag{20}$$

Since η and χ are as defined in Note 1, therefore

$$\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\,ds \geq \frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)}\left(\frac{1}{l+1}\right)\int_{\frac{1}{l+1}}^{\pi} \frac{1}{s^2}\,ds \geq \frac{\eta\left(\frac{1}{l+1}\right)}{2\chi\left(\frac{1}{l+1}\right)}.$$

Then,

$$\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\,ds\right). \tag{21}$$

From (20) and (21), we get

$$\begin{aligned}
\|T_l(\cdot)\|_r^{(\chi)} &= O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\,ds\right), \\
\|t_l^{T\Delta_H} - g\|_r^{(\chi)} &= O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\,ds\right).
\end{aligned} \tag{22}$$

6. Corollaries

Corollary 1. *Let $g \in H_{(\alpha),r}$; $r \geq 1$ and $0 \leq \beta < \alpha \leq 1$, then*

$$\|t_l^{T\Delta_H} - g\|_{(\beta),r} = \begin{cases} O((l+1)^{\beta-\alpha}) & \text{if} \quad 0 \leq \beta < \alpha < 1 \\ O\left(\frac{\log \pi(l+1)}{l+1}\right) & \text{if} \quad \beta = 0, \alpha = 1. \end{cases}$$

Proof. The proof is obtained by putting $\eta(s) = s^{\alpha}$, $\chi(s) = s^{\beta}$, $0 \leq \beta < \alpha \leq 1$ in Theorem 1.

Corollary 2. *Following the Remark 1(i), we obtain*

$$\|t_l^{C_\alpha\Delta_H} - g\|_r^{(\chi)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\,ds\right).$$

Corollary 3. *Following the Remark 1(ii), we obtain*

$$\|t_l^{H_{1/l+1}\Delta_H} - g\|_r^{(\chi)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\,ds\right).$$

Corollary 4. *Following the Remark 1(iii), we obtain*

$$\|t_l^{N_{p,q}\Delta_H} - g\|_r^{(\chi)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\ ds\right).$$

Corollary 5. *Following the Remark 1(iv), we obtain*

$$\|t_l^{N_p\Delta_H} - g\|_r^{(\chi)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\ ds\right).$$

Corollary 6. *Following the Remark 1(v), we obtain*

$$\|t_l^{\tilde{N}_p\Delta_H} - g\|_r^{(\chi)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\ ds\right).$$

Corollary 7. *Following the Remark 1(vi), we obtain*

$$\|t_l^{E_q\Delta_H} - g\|_r^{(\chi)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\ ds\right).$$

Corollary 8. *Following the Remark 1(vii), we obtain*

$$\|t_l^{TC_\alpha} - g\|_r^{(\chi)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\ ds\right).$$

Corollary 9. *Following the Remark 1(viii), we obtain*

$$\|t_l^{TE_q} - g\|_r^{(\chi)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}\ ds\right).$$

Remark 5.

(i) Corollary 2 can be further reduced for $C_\alpha E_q$ and $C_1\Delta_H$ means in view of Remark 2 (i)(b) and (c) respectively.

(ii) Corollary 3 can be further reduced for $H_{1/l+1}C_\alpha$ and $H_{1/l+1}E_q$ means in view of Remark 2 (ii)(a) and (b) respectively.

(iii) Corollary 4 can be further reduced for $N_{p,q}C_\alpha$ and $N_{p,q}E_q$ in view of Remark 2 (iii)(a) and (b) respectively.

(iv) Corollary 5 can be further reduced for N_pC_α and N_pE_q means in view of Remark 2 (iv)(a) and (b) respectively.

(v) Corollary 6 can be further reduced for $\tilde{N}_pC_\alpha$ and $\tilde{N}_pE_q$ means in view of Remark 2 (v)(a) and (b) respectively.

(vi) *Corollary 7 can be further reduced for E_qC_α means in view of Remark 2 (vi)(a).*

(vii) *Corollaries 8 can be further reduced for TC_1 means in view of Remark 2 (vii)(a).*

(viii) *Corollaries 9 can be further reduced for TE_1 means in view of Remark 2 (viii)(a).*

Remark 6.

(i) *In our Theorem 1, if $r \to \infty$ in $H_r^{(\eta)}$ class, then this turns down to $H^{(\eta)}$ class. Also putting $\eta(s) = s^\alpha$ and $\chi(s) = s^\beta$ in our Theorem 1, $H^{(\eta)}$ class then this turns down to H_α class. Then for $\beta = 0$ in H_α class, this turns down to Lipα class.*

(ii) *In our Theorem 1, by putting $\eta(s) = s^\alpha$, $\chi(s) = s^\beta$ in $H_r^{(\eta)}$ class, $H_r^{(\eta)}$ class then this turns down to $H_{\alpha,r}$ class. Then for $\beta = 0$ in $H_{\alpha,r}$ class, this turns down to Lip(α, r) class.*

Remark 7.

(i) *If $\zeta(s) = s^\alpha$ and $r \to \infty$ then Lip$(\zeta(s), r)$ class turns down to Lipα class. Thus, the results of [12], [15], [16] and [30] reduces to Lipα class.*

(ii) *If $\beta = 0$, $\zeta(s) = s^\alpha$ and $r \to \infty$ then $W(L_r, \zeta(s))$ class turns down to Lipα class. Thus, the results of [11], [13] and [14] reduces to Lipα class.*

7. Particular cases

(i) Using Remark 6(i) and putting $h_{l,j} = \frac{1}{l+1}$, $0 \le j \le l$ in our Theorem 1, the result of Dhakal [4] follows.

(ii) Using Remark 6(i) , putting $b_{l,j} = \frac{p_{l-j}q_j}{R_l}$, $R_l = \sum_{j=0}^{l} p_j q_{l-j}$ and $h_{l,j} = \frac{1}{l+1}$, $0 \le j \le l$ in our Theorem 1, the result of Dhakal [5] follows.

(iii) Using Remark 6(i), putting $b_{l,j} = \frac{1}{2^l}\binom{l}{j}$ and $h_{l,j} = \frac{1}{l+1}$, $0 \le j \le l$ in our Theorem 1, then in view of Remark 7(ii), the result of Nigam [11] follows.

(iv) Using Remark 6(i), putting $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \ge 1$ and $h_{l,j} = \frac{1}{l+1}$, $0 \le j \le l$ in our Theorem 1, then in view of Remark 7(i), the result of Nigam [12] follows.

(v) Using Remark 6(i), putting $b_{l,j} = \frac{1}{l+1}$ and $h_{l,j} = \frac{1}{(1+q)^l}\binom{l}{j} q^{l-j}$ in our Theorem 1, in view of Remark 7(ii), the result of Nigam [13] follows.

(vi) Using Remark 6(i) and 6(ii), putting $b_{l,j} = \frac{p_{l-j}}{P_j}$, $\sum_{j=0}^{l} p_j \ne 0$, $q_l = 1\ \forall\ l$ and $h_{l,j} = \frac{1}{l+1}$, $0 \le j \le l$ in our Theorem 1 then in view of Remark 7(ii), the result of Nigam and Sharma [14] follows.

(vii) Using Remark 6(i), putting $b_{l,j} = \frac{1}{l+1}$ and $h_{l,j} = \frac{1}{(1+q)^l}\binom{l}{j} q^{l-j}$ in our Theorem 1, then in view of Remark 7(i), the result of Nigam and Sharma [15] follows.

(viii) Using Remark 6(i), putting $b_{l,j} = \frac{1}{2^l}\binom{l}{j}$ and $h_{l,j} = \frac{1}{l+1}$, $0 \leq j \leq l$ in our Theorem 1, then in view of Remark 7(i), the result of Nigam and Sharma [16] follows.

(ix) Using Remark 6(ii), putting $b_{l,j} = \frac{p_{l-j}q_j}{R_l}$, $R_l = \sum_{j=0}^{l} p_j q_{l-j}$ and $h_{l,j} = \frac{1}{l+1}$, $0 \leq j \leq l$ in our Theorem 1, the result of Kushwaha and Dhakal [18] follows.

(x) Using Remark 6(i), putting $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \geq 1$ and $h_{l,j} = \frac{1}{l+1}$, $0 \leq j \leq l$ in our Theorem 1, the result of Tiwari and Bariwal [26] follows.

(xi) Using Remark 6(i), putting $b_{l,j} = \frac{1}{l+1}$ and $h_{l,j} = \frac{1}{(1+q)^l}\binom{l}{j} q^{l-j}$ in our Theorem 1, the result of Lal [29] follows.

(xii) Using Remark 6(i), putting $h_{l,j} = \frac{1}{l+1}$, $0 \leq j \leq l$ in our Theorem 1, then in view of Remark 7(i), the result of Shrivastava, Rathore and Shukla [30] follows.

8. Conclusion

In this paper, we obtain the error estimation of the function g in the Hölder space $H_r^{(\eta)}$ $(r \geq 1)$ by Matrix-Hausdorff $(T\Delta_H)$ product means of its Fourier series. Since, in view of Remark 1, the product summability means $C_\alpha\Delta_H$, $H_{1/l+1}\Delta_H$, $N_{p,q}\Delta_H$, $N_p\Delta_H$, $\tilde{N}_p\Delta_H$, $E_q\Delta_H$, TC_α and TE_q are the particular cases of $T\Delta_H$ product means. Some useful results are also deduced in the form of corollaries from our theorem.

Some other studies regarding modulus of continuity (smoothness) of functions using more generalized functional spaces may be the future interest of a few investigators in the direction of this work.

Acknowledgements

The first author expresses his gratitude towards his mother for her blessings. The first author also expresses his gratitude towards his father in heaven, whose soul is always guiding and encouraging him. The first author is also thankful to Council of Scientific and Industrial Research, Government of India for support under the scheme 25/(0225)/13/EMR-II. The second author also expresses her gratitude towards her parents for their blessings.

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