E-J Summability of Orthogonal Series

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Abstract. In this paper we obtain a sufficient condition for the E-J summability of certain orthogonal series. Our results generalize the corresponding theorems for ordinary Hausdorff summability obtained by Kalaivana and Youvaraj.

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1. Preliminaries

The set of all real or complex sequences \( \{x_n\} \) for which \( A_n(x) := \sum_k a_{nk}x_k \) converges is called the convergence domain of \( A \), written \( c_A \), where \( A \) is an infinite matrix. A matrix \( A \) is said to be conservative if it maps each convergent sequence into a convergent sequence, not necessarily with the same limit. If the limit is also preserved, then the matrix is called regular. Silverman and Toeplitz established necessary and sufficient conditions for a matrix to be conservative[4]. They are

(i) \( \|A\|_\infty := \sup_n \sum_k |a_{nk}| < \infty \),
(ii) \( t := \lim_n \sum_k a_{nk} \) exists,
(iii) \( a_k := \lim_n a_{nk} \) exists for each \( k \).

A Hausdorff matrix \( H = (h_{nk}) \) is a lower triangular matrix with nonzero entries

\[
h_{nk} = \binom{n}{k} \Delta^{n-k} \mu_k,
\]

where \( \{\mu_n\} \) is any real sequence and \( \Delta \) is the forward difference operator defied by \( \Delta \mu_k = \mu_k - \mu_{k+1} \) and \( \Delta^{n+1} \mu_k = \Delta (\Delta^n \mu_k) \). For every Hausdorff matrix each row sum is equal to \( \mu_0[3] \).

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F. Hausdorff [2] proved that a Hausdorff matrix is conservative if and only if
\[ \mu_n = \int_0^1 x^n d\chi(x), \]
where the mass function \( \chi \in BV[0,1] \).

The E-J generalized Hausdorff matrices, denoted by \( H_\alpha^\mu = (h_{nk}) \), were defined independently by Endl [1] and Jakimovski [5], with nonzero entries
\[ h_{nk}^{(\alpha)} = \frac{(n + \alpha)}{(n - k)} \Delta^{n-k} \mu_k^{(\alpha)}, \quad 0 \leq k \leq n, \]
for any \( \alpha \geq 0 \). For \( \alpha = 0 \), the E-J matrices reduce to the ordinary Hausdorff matrices.

If the \( \mu_n^{(\alpha)} \) satisfy the condition
\[ \mu_n^{(\alpha)} = \int_0^1 x^{n+\alpha} d\chi(x), \]
where \( \chi \in BV[0,1] \), then the corresponding E-J matrix is conservative.

**Definition 1.** Let \( \gamma : [1, \infty) \rightarrow [0, \infty) \) be a nondecreasing function, \( A = (a_{nk}) \) an infinite matrix. Then a series \( \sum_n b_n \) is said to be \( |A, \gamma|_k \) summable, if
\[ \sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} |\sigma_n - \sigma_{n-1}|^k \]
converges, where \( \sigma_n : \sum_n a_{nk} b_k \).

**Definition 2.** Let \( \gamma := \{\gamma_n\} \) be a positive sequence, \( \beta \) a real positive number. Then \( \gamma \) is called quasi- \( \beta \)-power monotone decreasing if there exists a number \( M = M(\beta, \gamma) \geq 1 \) such that
\[ n^\beta \gamma(n) \leq M \beta^\gamma(m) \]
for each \( m \leq n \).

For any real number \( \beta, \Gamma_\beta \) denotes the set of all increasing functions \( \Gamma_\beta : [1, \infty) \rightarrow [0, \infty) \) such that each \( \{\gamma_n\} \) is a quasi- \( \beta \)-power monotone decreasing sequence.

**2. Main Results**

**Theorem 1.** Let \( \{\varphi_n\}_{n=0}^{\infty} \subset L_2[0,1] \) be an orthonormal system, \( H_\mu^\alpha \) an E-J Hausdorff matrix with \( \chi \) monotone decreasing, \( \gamma \in \Gamma_\beta \) for \( \beta > 1 - 1/k, \quad 1 \leq k \leq 2 \). Then every orthogonal series \( \sum_{n=0}^{\infty} b_n \varphi_n \) is \( |H^\alpha, \gamma| \) summable.

The following lemmas will be needed in the proof of Theorem 1.
Lemma 1. Let $H^\alpha$ be an E-J matrix with entries $(h_{nk})$, where $\chi$ is a monotonically increasing mass function on $[0, 1]$ associated with the $\mu_n$. Then

(i) $a_{mn} = K\binom{n-1+\alpha}{m-1+\alpha} \xi^{m+\alpha}(1-\xi)^{n-m}$ for some $\xi \in (0, 1)$,

(ii) $\sum_{m=0}^{n} |a_{mn}|^2 |b_m|^2 \leq K^2 \sum_{m=0}^{n} |b_m|^2$ for all $b_n \in \mathbb{C}$ and $n \in \mathbb{N}$, where $K = \chi(1) - \chi(0)$

and $a_{mn} = \sum_{k=m}^{n} |h_{nk}^{(\alpha)} - h_{n-1,k}^{(\alpha)}|$.

Here $\mathbb{C}$ = complex numbers and $\mathbb{N}$ = natural numbers.

Proof. (i) We consider

$$h_{nk}^{(\alpha)} - h_{n-1,k}^{(\alpha)} = \int_{0}^{1} \mu^{k+\alpha}(1-\mu)^{n-k} \left( \binom{n+\alpha}{k+\alpha} - \binom{n-1+\alpha}{k+\alpha} \right) d\chi(\mu)$$

where $0 \leq k \leq n$. Since

$$\binom{n+\alpha}{k+\alpha} - \binom{n-1+\alpha}{k+\alpha} = \binom{n-1+\alpha}{k-1+\alpha},$$

from (2),

$$\int_{0}^{1} \mu^{k+\alpha}(1-\mu)^{n-k} \left( \binom{n+\alpha}{k+\alpha} - \binom{n-1+\alpha}{k+\alpha} \right) d\chi(\mu)$$

$$= \int_{0}^{1} \mu^{k+\alpha}(1-\mu)^{n-k} \left[ \binom{n+\alpha}{k+\alpha} - \binom{n-1+\alpha}{k+\alpha} - \mu \binom{n+\alpha}{k+\alpha} \right] d\chi(\mu)$$

$$= \int_{0}^{1} \mu^{k+\alpha}(1-\mu)^{n-k} \left[ \binom{n-1+\alpha}{k-1+\alpha} - \mu \binom{n+\alpha}{k+\alpha} \right] d\chi(\mu).$$

Thus

$$a_{mn} = \sum_{k=m}^{n} (h_{nk}^{(\alpha)} - h_{n-1,k}^{(\alpha)}) = \sum_{k=m}^{n} \int_{0}^{1} \mu^{k+\alpha}(1-\mu)^{n-k} \left[ \binom{n-1+\alpha}{k-1+\alpha} - \mu \binom{n+\alpha}{k+\alpha} \right] d\chi(\mu)$$

$$= \int_{0}^{1} \sum_{k=m}^{n} \mu^{k+\alpha}(1-\mu)^{n-k} \left[ \binom{n-1+\alpha}{k-1+\alpha} - \mu \binom{n+\alpha}{k+\alpha} \right] d\chi(\mu).$$

From the above inequality,

$$\sum_{k=m}^{n} \mu^{k+\alpha}(1-\mu)^{n-k} \left[ \binom{n-1+\alpha}{k-1+\alpha} - \mu \binom{n+\alpha}{k+\alpha} \right]$$
Lemma 2. For $0 < K < 1$, $a_{mn} \leq 1$.

Proof. From (3)

$$a_{mn} = K \int_{0}^{1} \mu^{m+\alpha}(1 - \mu)^{n-m} \left( \frac{n-1+\alpha}{m-1+\alpha} \right) d\mu$$

$$= K \left( \frac{n-1+\alpha}{m-1+\alpha} \right) \int_{0}^{1} \mu^{m+\alpha}(1 - \mu)^{n-m} d\mu$$

$$= K \left( \frac{n-1+\alpha}{m-1+\alpha} \right) \frac{\Gamma(m+\alpha+1) \Gamma(n-m+1)}{\Gamma(n+\alpha+2)}$$

$$= K \frac{\Gamma(n+\alpha)}{(n+\alpha+1)(n+\alpha)}.\Gamma(m+\alpha+1) \Gamma(n-m+1)$$

Since $n \geq m$,

$$a_{mn} \leq K < 1.$$
Using equation (4) we can write
\[ \sum_{m=0}^{n} |a_{mn}|^2 |b_m|^2 \leq K^2 \sum_{m=0}^{n} |b_m|^2 \leq \sum_{m=0}^{n} |b_m|^2, \]  
which is a proof of (ii).

Lemma 3. Let \( \{\varphi_n\}_{n=0}^{\infty} \subset L^2[0,1] \) be an orthonormal system, \( H^\alpha_\mu \) be an E-J Hausdorff matrix with monotonically increasing function \( \chi \) on \( [0,1] \). Then, for \( n \in \mathbb{N} \) and

\[ K = \int_{0}^{1} d\chi(\mu), \]

(i) there exists \( \xi \in (0,1) \) such that
\[ \int_{0}^{1} |\sigma_n(x) - \sigma_{n-1}(x)|^2 \, dx = K^2 \sum_{m=0}^{n} \xi^{2m+2\alpha} (1 - \xi)^{2n-2m} \left( \frac{n - 1 + \alpha}{m - 1 + \alpha} \right)^2 |b_m|^2 \]
and
(ii) \[ \int_{0}^{1} |\sigma_n(x) - \sigma_{n-1}(x)|^2 \, dx = K^2 \sum_{m=0}^{n} |b_m|^2, \]
for all \( b_m \in \mathbb{C} \) where, for \( n \in \mathbb{N} \), \( \sigma_n(x) = \sum_{k=0}^{n} h_{nk} S_k(x) \), where \( S_k \) denotes the \( k \)-th partial sum of the orthogonal series \( \sum_{m=0}^{\infty} b_m \varphi_m \).

Proof.

\[ \sigma_n(x) - \sigma_{n-1}(x) = \sum_{k=0}^{n} (h_{nk}^{(\alpha)} - \alpha h_{n-1,k}^{(\alpha)}) S_k(x) \]
\[ = \sum_{k=0}^{n} (h_{nk}^{(\alpha)} - \alpha h_{n-1,k}^{(\alpha)}) \sum_{m=0}^{k} b_m \varphi_m \]
\[ = \sum_{m=0}^{n} \sum_{k=m}^{n} (h_{nk}^{(\alpha)} - \alpha h_{n-1,k}^{(\alpha)}) b_m \varphi_m \]
\[ = \sum_{m=0}^{n} a_{mn} b_m \varphi_m. \]

Since \( \{\varphi_n\}_{n=0}^{\infty} \) is an orthonormal system, using Parseval’s identity,
\[ \int_{0}^{1} |\sigma_n(x) - \sigma_{n-1}(x)|^2 \, dx = \sum_{m=0}^{n} |a_{mn}|^2 |b_m|^2 \]
\[ = K^2 \sum_{m=0}^{n} \xi^{2m+2\alpha} (1 - \xi)^{2n-2m} \left( \frac{n - 1 + \alpha}{m - 1 + \alpha} \right)^2 |b_m|^2. \]
From Lemma 2,
\[ \int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)|^2 dx = \sum_{m=0}^{n} |a_{mn}|^2 |b_m|^2 \leq K^2 \sum_{m=0}^{n} |b_m|^2. \]

**Proof.** To prove Theorem 1, from Definition 1 we need to show that
\[ \sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} |\sigma_n - \sigma_{n-1}|^k \]
converges for \(1 \leq k < 2\), where, for \(n \in \mathbb{N}\), \(\sigma_n(x) = \sum_{k=0}^{n} h(n)_k \varphi_k(x)\).

Using Lemma 3, and Hölder’s inequality with \(p = 2/k\), for any \(1 \leq k \leq 2\), and for all \(b \in \ell_2(\mathbb{Z}^+)\), we have
\[ \sum_{n=1}^{\infty} \gamma(n)^k n^{k} \int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)|^k dx \leq \sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} \{ K^2 \|b\|_2^2 \}^{k/2} \]
\[ \leq \{ K \|b\|_2^k \} \sum_{n=1}^{\infty} \gamma(n)^k n^{k-1}. \]

Here \(\{\gamma(n)\}\) is a quasi \(\beta\)-power monotone decreasing sequence with \(\beta > 1 - 1/k\), and, since for \(\epsilon = \beta - 1 + 1/k\), the sequence \(\{n^{k-1}\gamma(n)^k\}\) is quasi \(k\epsilon\)-power monotone decreasing. Using Lemma 1 of [6], we have
\[ \leq \{ K \|b\|_2^k \} \sum_{n=1}^{\infty} \gamma(2^{n})^k (2^n)^{k-1} \]
\[ \leq \{ K \|b\|_2^k \} \gamma(2)^k (2)^{k-1}, \]
where \(B \geq 1\).

**Theorem 2.** Let \(\{\varphi\}_{n=0}^{\infty} \subset L_2[0,1]\) be an orthogonal system and \(H^\alpha\) the corresponding E-J Hausdorff matrix. For \(1 \leq k \leq 2\) and \(\gamma \in \Gamma(\beta)\) with \(\beta > 1 - 1/k\), every orthogonal series \(\sum_{n=0}^{\infty} b_n \varphi_n\) is \(|H^\alpha, \gamma|_k\) summable.

**Proof.** Let \(\chi \in BV[0,1]\) be the mass function corresponding to the E-J matrix \(H^\alpha\). By the Jordan decomposition theorem, \(\chi = \chi_1 - \chi_2\), where \(\chi_1\) and \(\chi_2\) are monotone increasing functions. To prove the theorem we apply Theorem 1 to \(\chi_1\) and \(\chi_2\). Theorems 1 and 2 are generalizations of Theorems 1 and 2, respectively, in [6].

**Theorem 3.** Let \(\{\varphi\}_{n=0}^{\infty} \subset L_2[0,1]\) be an orthogonal system and \(H^\alpha\) an E-J Hausdorff matrix with \(\chi \in [0,1]\) and monotone increasing. For \(1 \leq k \leq 2\) and \(\gamma \in \Gamma(\beta)\) with \(\beta > 1 - 1/k\), a sufficient condition for the orthogonal series \(\sum_{n=0}^{\infty} b_n \varphi_n\) to be \(|H^\alpha, \gamma|_k\) summable is
\[ \sum_{s=0}^{\infty} \gamma(2^s)^k \left\{ \sum_{m=2^s+1}^{2^{s+1}} \sqrt{m + \alpha} |b_m|^2 \right\}^{k/2} < \infty. \]
Proof. Let $\chi \in BV[0, 1]$ and monotonically increasing on $[0, 1]$. By Lemma 1(i) there exists a $\xi \in (0, 1)$ such that
\[
\int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)| \, dx \leq K^2 \sum_{m=0}^{n} \left( \frac{n + \alpha - 1}{m + \alpha - 1} \right)^2 \xi^{2m+2\alpha} (1 - \xi)^{2n-2m} |b_m|^2,
\] (7)
where
\[
\sigma_n(x) = \sum_{k=0}^{n} h_{nk}^{(\alpha)} s_k.
\]
For $1 \leq k \leq 2$, by using Hölder’s inequality and equation (1),
\[
\sum_{n=2}^{\infty} \gamma(n)^k n^{k-1} \left\{ \int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)| \, dx \right\}^k \leq \sum_{n=2}^{\infty} \gamma(n)^k n^{k-1} \left( \sum_{m=0}^{n} \left( \frac{n + \alpha - 1}{m + \alpha - 1} \right)^2 \xi^{2m+2\alpha} (1 - \xi)^{2n-2m} |b_m|^2 \right)^{k/2}.
\] (8)
Replacing $\xi$ by $1/(1 + q)$ in (8), we obtain
\[
= K^k \sum_{r=0}^{\infty} \sum_{n=2r+1}^{2r+1} \gamma(n)^k n^{k-1} \left\{ \sum_{m=0}^{n} \left( \frac{n + \alpha}{m + \alpha} \right)^2 \left( \frac{m + \alpha}{n + \alpha} \right)^{2q} q^{2n-2m} (1 + q)^{-2n-2\alpha} |b_m|^2 \right\}^{k/2}.
\] (9)
O.A. Ziza [7] proved that, for $q > 0$, there exists a constant $C_q > 0$ such that
\[
\max_{0 \leq k \leq n} \left( \frac{n}{k} \right)^q \leq C_q \frac{(1 + q)^n}{\sqrt{n}}, \quad n = 1, 2, \ldots.
\]
We shall generalize this Lemma for E-J matrices.

Lemma 4. For $q > 0$ there exists a $C_q > 0$ such that
\[
\max_{0 \leq k \leq n} \left( \frac{n + \alpha}{k + \alpha} \right)^{q^k + \alpha} \leq C_q \frac{(1 + q)^n}{\sqrt{n + \alpha}}, \quad n = 1, 2, \ldots.
\]
Proof.
\[
\frac{(n+\alpha+1)}{(k+\alpha)^{k+\alpha}} = \frac{n + \alpha + 1}{k + \alpha}.
\]
Let
\[
d_k = \left( \frac{n + \alpha + 1}{k + \alpha} - 1 \right)\varphi.
\]
The $d_k$ are decreasing in $k$. Let $k_n$ denote the largest value of $k + \alpha$ for which $d_k \geq 1$. Then $d_{k_n+1} < 1$, and
\[
\max_{0 \leq k \leq n} \left( \frac{n + \alpha}{k + \alpha} \right)^{q^k + \alpha} = \left( \frac{n + \alpha}{k_n} \right)^q k_n.
\]
It then follows that one can write
\[ k_n = \frac{q}{1+q}(n + \alpha) + \nu_n, \]
where \( 0 < \nu_n < 1 \).

Then
\[ \left( \frac{n + \alpha}{k_n} \right)^{n + \alpha} \leq C_1 \frac{(n + \alpha)!}{k_n!(n + \alpha - k_n)!} = \frac{(n + \alpha)^{n+\alpha}e^{-(n+\alpha)}\sqrt{n + \alpha}}{k_n^{k_n}e^{-(k_n)}\sqrt{k_n(n + \alpha - k_n)}e^{-(n+\alpha-k_n)}\sqrt{n + \alpha - k_n}}. \tag{10} \]

With \( p = q/(1+q) \), the right hand side of (10) equals
\[
\frac{(n + \alpha)^{n+\alpha}e^{-(n+\alpha)}\sqrt{n + \alpha}}{(p(n + \alpha) + \nu_n)(p(n+\alpha)+\nu_n)e^{-(p(n+\alpha)+\nu_n)}\sqrt{(p(n + \alpha) + \nu_n)}} \\
\times \frac{1}{(n + \alpha - p(n + \alpha) - \nu_n)(n+\alpha-p(n+\alpha)-\nu_n)e^{-(n+\alpha-p(n+\alpha)-\nu_n)}\sqrt{(n + \alpha - p(n + \alpha) - \nu_n)}}.
\]

Note that
\[
\frac{(n + \alpha)}{(p(n + \alpha) + \nu_n)(n + \alpha - p(n + \alpha) - \nu_n)} = \frac{(n + \alpha)}{(n + \alpha)^2(p + \frac{\nu_n}{n+\alpha})(1 - (p + \frac{\nu_n}{n+\alpha}))}.
\]

Set \( a = p + \nu_n/(n + \alpha) \) and define a function \( f \) by \( f(a) = a(1-a) \). Then \( f(a) \) has a minimum value of 1/4 at \( a = 1/2 \). Therefore \( 1/\sqrt{f(a)} \leq 2 \). From (11)
\[
\left( \frac{n + \alpha}{k_n} \right)^{n+\alpha} \leq 2 \frac{1}{(p + \frac{\nu_n}{n+\alpha})^{p(n+\alpha)+\nu_n}} \times \frac{1}{(1 - p - \frac{\nu_n}{n+\alpha})^{(1-p)(n+\alpha)-\nu_n}} \times \frac{1}{\sqrt{n + \alpha}} \times \frac{1}{\sqrt{n + \alpha}}.
\]

Since \( (1-p)/p = (1/p)-1 \) and \( p \) is a fixed positive constant between 0 and 1, \( (p/(1-p))^{-\nu_n} \) is clearly bounded. So also is \( (1 + \nu_n/(p(n + \alpha)))^{-p(n+\alpha)-\nu_n} \).

Let \( g(p) = 1 - \nu_n/(1 - p)(n + \alpha) \). Then
\[
g'(p) = \frac{\nu_n}{(1-p)^2(n + \alpha)},
\]
and \( g \) is decreasing in \( p \). Since \( 0 < p < 1 \) and fixed, \( g(p) \) is bounded. Using the above facts,

\[
\left( \frac{1-p}{p} \right)^{-\nu_n} \quad \text{and} \quad \left( \frac{p}{1-p} \right)^{-\nu_n} \left( 1 + \frac{\nu_n}{p(n+\alpha)} \right)^{-(p(n+\alpha)+\nu_n)} \left( 1 - \frac{\nu_n}{(1-p)n+\alpha} \right)^{-(1-p)(n+\alpha)+\nu_n}
\]

are bounded. Also,

\[
\left( \frac{n+\alpha}{k_n} \right) \leq C_3 \frac{1}{p^{\theta(n+\alpha)}} \times \frac{1}{(1-p)^{(1-p)(n+\alpha)}} \times \frac{1}{\sqrt{n+\alpha}}.
\]

We can write (12) as

\[
\left( \frac{n+\alpha}{k_n} \right) \leq C_3 \frac{q^{k_n}}{(1+q)^{(n+\alpha)}} \times \frac{1}{(1+q)^{(1+q)(n+\alpha)}} \times \frac{1}{\sqrt{n+\alpha}}.
\]

From (13)

\[
\left( \frac{n+\alpha}{k_n} \right)^{q^{k_n}} \leq C_3 \frac{q^{k_n}}{(1+q)^{(n+\alpha)}} \times \frac{1}{(1+q)^{(1+q)(n+\alpha)}} \times \frac{1}{\sqrt{n+\alpha}} q^{n\alpha} \leq C_q q^{(1+q)^{n+\alpha}} \sqrt{n+\alpha}.
\]

Thus

\[
\max_{0 \leq k \leq n} \left( \frac{n+\alpha}{k+\alpha} \right)^{q^{k+\alpha}} \leq C_q \frac{(1+q)^{n+\alpha}}{\sqrt{n+\alpha}}.
\]

From (9)

\[
K^k \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}-1} \gamma(n) k_n^{k-1} \left\{ \sum_{m=0}^{n} \binom{n+\alpha}{m+\alpha} \left( \frac{m+\alpha}{n+\alpha} \right)^2 q^{2n-2m(1+q)^{-2n-2\alpha} |b_m|^2} \right\}^{k/2}
\]

\[
= K^k \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}-1} \gamma(n) k_n^{k-1} \left\{ \sum_{m=0}^{n} \binom{n+\alpha}{m+\alpha} \left( \frac{m+\alpha}{n+\alpha} \right)^2 \left( \frac{n+\alpha}{n-m} \right) q^{n-m} q^{-m(1+q)^{-2n-2\alpha} |b_m|^2} \right\}^{k/2}.
\]

Using Lemma 4, equation (14) can be written as

\[
\leq K^k C_q^{k/2} \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}-1} \gamma(n) k_n^{k-1} \left\{ \sum_{m=0}^{n} \binom{n+\alpha}{m+\alpha} \left( \frac{m+\alpha}{n+\alpha} \right)^2 \frac{1}{\sqrt{n+\alpha}} q^{n-m} q^{-m(1+q)^{-2n-2\alpha} |b_m|^2} \right\}^{k/2}.
\]
\[ \Omega = \frac{1}{\alpha} \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}} \frac{(n+\alpha)(m+\alpha)^2}{(n+\alpha)^{5/2}q^{-m}(1+q)^{-2n-2\alpha}|b_m|^2} \] 

(15)

For \( k = 2 \), the above inequality becomes

\[ \Omega \leq K^2C_q \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}} \frac{(n+\alpha)^2n(n+\alpha)^{-5/2}}{(n+\alpha)^{5/2}q^{-m}(1+q)^{-2n-2\alpha}|b_m|^2} \] 

(16)

**Lemma 5.** There exists a \( D_q > 0 \) such that

\[ \sum_{n=m}^{\infty} \frac{(n+\alpha)}{(m+\alpha)}q^{-m}(1+q)^{-n-\alpha} \leq D_q \]

for all \( m \in \mathbb{Z}^+ \) and \( 1 \leq k \leq 2 \).

**Proof.** The proof of the lemma is easy to verify and it is a generalization of Theorem B of [6] to E-J matrices. Using Theorem B of [6], we can write (16) as

\[ \Omega \leq K^2C_qD_q \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}} \frac{(n+\alpha)^2(2^r + \alpha)^{-3/2}}{(m+\alpha)^{5/2}q^{-m}(1+q)^{-2n-2\alpha}|b_m|^2}, \quad \alpha > 0. \]

Let \( p = 2/k \). By Hölder’s inequality, for \( \alpha > 0 \) and \( 1 \leq k \leq 2 \),

\[ \Omega = K^k C_q^{k/2} \sum_{r=0}^{\infty} \left( \sum_{n=2^r+1}^{2^{r+1}} \frac{(n+\alpha)^k nq^{-k}\beta^{-1}}{q^{-k}\beta^{-1}} \right)^{1/q} \left\{ \sum_{r=0}^{2^r+1} \sum_{m=0}^{n} \frac{(n+\alpha)(m+\alpha)^2}{(m+\alpha)^{5/2}q^{-m}(1+q)^{-n-\alpha}|b_m|^2} \right\}. \]

(17)

Since \( \gamma \in \Gamma(\beta) \) and, for \( \beta \in \mathbb{R} \), by Theorem A in [6], we can write the expression in the first bracket in (17) as

\[ \left( \sum_{n=2^r+1}^{2^{r+1}} \frac{(n+\alpha)^k nq^{-k}\beta^{-1}}{q^{-k}\beta^{-1}} \right)^{1/q} \leq K^{1/q} \gamma(2^r + 1)^k(2^r + 1)^{(-\frac{2}{q} - 1)} \]

\[ \leq K^{1/q} \gamma(2^r + 1)^k(2^r)^{(-\frac{2}{q} - 1)}. \]

Thus, from (17),

\[ \Omega \leq K^{k+1/4} C_q^{k/2} \sum_{r=0}^{\infty} \frac{(2^r + 1)^k(2^r)^{(-\frac{2}{q} - 1)}}{r=0} \left\{ \sum_{n=2^r+1}^{2^{r+1}} \sum_{m=0}^{n} \frac{(n+\alpha)(m+\alpha)^2}{(m+\alpha)^{5/2}q^{-m}(1+q)^{-n-\alpha}|b_m|^2} \right\}^{k/2}. \]

(18)
Changing the order of summation inside the brackets in the above inequality, (18) is equal to

\begin{align*}
&= K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \left\{ \gamma(2^{r+1}) k q (2^r)^{-\frac{q}{2} - 1} \right\} \left\{ \sum_{m=0}^{2^{r+1}} \sum_{n=2^r+1}^{2^{r+1}} \left( \frac{n}{m+\alpha} \right)^2 q^{n-m(1+q)} (1 + q)^{-n-\alpha} |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} \sum_{n=2^r+1}^{2^{r+1}} \left( \frac{n}{m+\alpha} \right)^2 q^{n-m(1+q)} (1 + q)^{-n-\alpha} |b_m|^2 \right\}^{k/2} \\
&+ K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=2^r+1}^{2^{r+1}} \sum_{n=2^r+1}^{2^{r+1}} \left( \frac{n}{m+\alpha} \right)^2 q^{n-m(1+q)} (1 + q)^{-n-\alpha} |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} D_q (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&+ K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=2^r+1}^{2^{r+1}} D_q (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} D_q (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} \left( \frac{n}{m+\alpha} \right)^2 q^{n-m(1+q)} (1 + q)^{-n-\alpha} |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} D_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} (m+\alpha)^2 |b_m|^2 \right\}^{k/2} .
\end{align*}

Using Lemma 5,

\begin{align*}
\Omega &\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} D_q (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&+ K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=2^r+1}^{2^{r+1}} D_q (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} D_q (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} D_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} (m+\alpha)^2 |b_m|^2 \right\}^{k/2} .
\end{align*}

For \(1 \leq k \leq 2\),

\begin{equation}
\Omega \leq L \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \sum_{m=0}^{2^{r+1}} (m+\alpha)^2 |b_m|^2 \right\}^{k/2} ,
\end{equation}

where \(L = K^{k+\frac{1}{q}} C_q^{k/2} D_q^{k/2}\). From (19),

\begin{equation}
\Omega \leq L \sum_{r=0}^{\infty} \gamma(2^{r+1}) k (2^r)^{-\frac{1}{q} - 1} \left\{ \alpha^2 |b_0|^2 + \sum_{s=0}^{r} \sum_{m=2^s+1}^{2^{s+1}} (m+\alpha)^2 |b_m|^2 \right\}^{k/2}.
\end{equation}
\[-L \sum_{s=0}^{\infty} \sum_{r=0}^{s} \gamma(2^{r+1}k(2^r)^{-\frac{4}{3}}) \left\{ \sum_{m=2^r+1}^{2^{r+1}} (m + \alpha)^2 |b_m|^2 \right\}^{k/2} + L \sum_{r=0}^{\infty} \gamma(2^{r+1}k(2^r)^{-\frac{4}{3}}) \alpha^k |b_0|^k.\]

Here \(\{\gamma(n)\}\) is quasi \(\beta\)-power monotone decreasing and \(\{n^{-3/4} \gamma(n)\}\) is quasi \(\epsilon\)-power monotone decreasing, where \(\beta > 3/4\) and \(\epsilon = \beta + 3/4\). Thus, by using Lemma 1 of [6],

\[\sum_{n=m}^{\infty} \gamma(n^k)(2^n)^{-3k/4} \leq M \gamma(2^m)^k(2^m)^{-3k/4}, M \in \mathbb{Z}^+.\]

Therefore

\[\Omega \leq L \sum_{s=0}^{\infty} \gamma(2^{s+1}k(2^s)^{-3k/4}) \left\{ \sum_{m=2^s+1}^{2^{s+1}} (m + \alpha)^2 |b_m|^2 \right\}^{k/2} + L \gamma(2)^k \alpha^k |b_0|^k\]

\[\leq 2^k L \sum_{s=0}^{\infty} \gamma(2^sk(2^s)^{-3k/4}) \left\{ \sum_{m=2^s+1}^{2^{s+1}} (m + \alpha)^2 |b_m|^2 \right\}^{k/2} + L \gamma(2)^k \alpha^k |b_0|^k\]

\[\leq 2^k L \sum_{s=0}^{\infty} \gamma(2^sk) \left\{ \sum_{m=2^s+1}^{2^{s+1}} (m + \alpha)^2 |b_m|^2 \right\}^{k/2} + L \gamma(2)^k \alpha^k |b_0|^k\]

and thus

\[\sum_{n=2}^{\infty} \gamma(n)^k n^{k-1} \int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)| dx\]

\[\leq 2^{(-\frac{4k}{3})} L \sum_{s=0}^{\infty} \gamma(2^sk) \left\{ \sum_{m=2^s+1}^{2^{s+1}} (m + \alpha)^2 |b_m|^2 \right\}^{k/2} + L \gamma(2)^k \alpha^k |b_0|^k.\]

The following Corollaries can be verified by taking \(\alpha = 0\) in the above theorems.

**Corollary 1.** Every orthogonal series \(\sum_{n=0}^{\infty} c_n \psi_n\), \(c_n \in \ell_2(\mathbb{Z}^+)\) is \(|H, \psi|_k\) summable for \(1 \leq k \leq 2\) and \(\gamma \in \Gamma_\beta\) with \(\beta > 1 - l/k\), where \(\{\psi_n\}_{n=0}^{\infty} \subset L_2[0,1]\) and \(H\) is a Hausdorff matrix with entries \((h_{nk})_{n,k} \in \mathbb{Z}^+\).

This is Theorem 2 of [6].

**Corollary 2.** Let \(1 \leq k \leq 2\) and \(\gamma \in \Gamma_\beta\) with \(\beta > -3/4\), where \(\{\phi_n\}_{n=0}^{\infty} \subset L_2[0,1]\) and \(H\) is a Hausdorff matrix. Then, for any \(c_n \in \ell_2(\mathbb{Z}^+)\), a sufficient condition for the orthogonal series \(\sum_{n=0}^{\infty} c_n \psi_n\) to be \(|H, \gamma|_k\) summable is

\[\sum_{m=0}^{\infty} \gamma(2^m)^k \left\{ \sum_{n=2^m+1}^{2^{m+1}} \sqrt{n} |c_n|^2 \right\}^{k/2} < \infty.\] (20)

This includes the results of Theorem 3 of [6].
References


