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Identities on Generalized Apostol-Genocchi Numbers and Polynomials Involving Binomial Coefficients

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Abstract. In [11], Jolany et al. defined generalizations of Apostol-Genocchi numbers and polynomials. Most identities on classical or generalized Apostol-Genocchi numbers and polynomials are related to the well-known Bernoulli and Euler numbers and polynomials. However, in this paper, identities on generalized Apostol-Genocchi numbers and polynomials which are not associated with the Bernoulli- and Euler-types are introduced. Specifically, identities involving binomial coefficients and some integral identities which only relate generalized Apostol-Genocchi numbers and polynomials are established.

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1. Introduction

The long history of the Genocchi numbers and polynomials can be traced back to Angelo Genocchi (1817-1889). The classical Genocchi numbers are a sequence of integers that satisfy the exponential generating function

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi.$$

The first few Genocchi numbers are

$$G_0 = 0, G_1 = 1, G_2 = -1, G_3 = 0, G_4 = 1, G_5 = 0, G_6 = -3.$$

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The classical Genocchi polynomials are usually defined by means of the exponential generating function

$$\frac{2t}{e^t+1}\cdot e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\quad |t|<\pi.$$

It can be seen that $G_n(0)=G_n$. Nowadays, Genocchi numbers and kinds of Genocchi polynomials have been widely studied and extensive studies have linked these numbers and polynomials in many branches of mathematics such as in analytic number theory, p-adic number theory, special functions and mathematical analysis, numerical analysis, combinatorics [1–7], and others.

Many researchers introduced generalizations to the classical Genocchi numbers and polynomials. For instance, Araci et.al [7] and Kim et al. [12] explored the Genocchi polynomials of higher order arising from Genocchi basis, which were defined by

$$\left(\frac{2t}{e^t+1}\right)^k\cdot e^{xt}=\sum_{n=0}^{\infty}G_n^{(k)}(x)\frac{t^n}{n!},\ (|t|<\pi, k\in\mathbb{N}\cup\{0\})$$

and established interesting identities. Moreover, He et al.[9] defined the Apostol-Genocchi polynomials as an extension of the classic Genocchi polynomials, which were given by

$$\frac{2t}{\lambda e^t+1}\cdot e^{xt}=\sum_{n=0}^{\infty}G_n^{\lambda}(x)\frac{t^n}{n!}\ (|t+\log\lambda|<\pi,\lambda\neq 0).$$

In [11], Jolany et al. generalized Apostol-Genocchi numbers and polynomials using the following generating functions: For $a,b,c>0$ and $\lambda\neq 0$,

$$\frac{2t}{\lambda b^t+a^t} = \sum_{n=0}^{\infty}G_n^{\lambda}(a,b)\frac{t^n}{n!}\quad\left(|t\log(b/a)+\log\lambda|<\pi\right), \tag{1}$$

$$\frac{2t}{\lambda b^t+a^t}e^{xt} = \sum_{n=0}^{\infty}G_n^{\lambda}(x;a,b)\frac{t^n}{n!}\quad\left(|t\log(b/a)+\log\lambda|<\pi\right), \tag{2}$$

$$\frac{2t}{\lambda b^t+a^t}c^{xt} = \sum_{n=0}^{\infty}G_n^{\lambda}(x;a,b,c)\frac{t^n}{n!}\quad\left(|t\log(b/a)+\log\lambda|<\pi\right). \tag{3}$$

For similar generalizations and applications of Genocchi polynomials and other type of polynomials involving parameters a,b and c, see [8, 13, 15].

The generating function of the Apostol-Genocchi numbers (polynomials) is similar to those of the Bernoulli numbers(polynomials) and the Euler numbers(polynomials), so it may be expected that the Apostol-Genocchi numbers(polynomials) satisfy similar identities as those established for Euler and Bernoulli numbers and polynomials. In fact, most literature on Apostol-Genocchi numbers and polynomials provide the associations of these three kinds of numbers (polynomials) (e.g.[10]). In [14], Ozden unified the generating functions of the Bernoulli, Euler and Genocchi numbers and polynomials and gave some new relations on these numbers.

In [16], Zou obtained identities which associate only the classical Genocchi numbers G_n and polynomials $G_n(x)$. This motivates us to establish identities which concern only the multiparameter generalized Apostol-Genocchi numbers $G_n^{\lambda}(a,b)$ and generalized Apostol-Genocchi polynomials $G_n^{\lambda}(x;a,b,c)$. Obviously, $G_n^{\lambda}(a,b)$ and $G_n^{\lambda}(x;a,b,c)$ reduce to G_n and $G_n(x)$ when $\lambda=1, b=c=e$, and $a=1$. Hence, results here are generalizations of the results obtained in [16].

2. Identities on Generalized Apostol-Genocchi Numbers and Polynomials

In this section, we establish some identities involving the generalized Apostol-Genocchi numbers and generalized Apostol-Genocchi polynomials using their generating functions with the aid of binomial inversion formula and summation transform techniques.

Theorem 1. *For $n \geq 2$,*

(i) $$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_k^{\lambda}(x;a,b,c)\left[\lambda\ln b\cdot G_{n-k+1}^{\lambda}(\ln b;a,b)+\ln a\cdot G_{n-k+1}^{\lambda}(\ln a;a,b)\right]}{n-k+1}=x\ln c\cdot G_n^{\lambda}(x;a,b,c)-\frac{n}{n+1}G_{n+1}^{\lambda}(x;a,b,c).$$

(ii) $$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{k+1}^{\lambda}(x;a,b,c)\left[\lambda\ln b\cdot G_{n-k}^{\lambda}(\ln b;a,b)+\ln a\cdot G_{n-k}^{\lambda}(\ln a;a,b)\right]}{k+1}=x\ln c\cdot G_n^{\lambda}(x;a,b,c)-\frac{n}{n+1}G_{n+1}^{\lambda}(x;a,b,c).$$

Proof. Taking the partial derivatives of the left side of equation (3) with respect to t yields

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{2t}{\lambda b^t+a^t}c^{xt}\right) &= \frac{2c^{xt}}{\lambda b^t+a^t}+\frac{x\ln c\cdot 2tc^{xt}}{\lambda b^t+a^t}-\frac{2tc^{xt}(\lambda\ln b\cdot b^t+\ln a\cdot a^t)}{(\lambda b^t+a^t)^2}\\
&= \frac{1}{t}\cdot\frac{2tc^{xt}}{\lambda b^t+a^t}+\frac{x\ln c\cdot 2tc^{xt}}{\lambda b^t+a^t}-\frac{2tc^{xt}}{\lambda b^t+a^t}\cdot\frac{1}{2t}\left[\frac{\lambda\ln b\cdot 2te^{t\ln b}+\ln a\cdot 2te^{t\ln a}}{\lambda b^t+a^t}\right] \qquad (4)\\
&= \sum_{n=0}^{\infty}G_n^{\lambda}(x;a,b,c)\frac{t^{n-1}}{n!}+x\ln c\cdot\sum_{n=0}^{\infty}G_n^{\lambda}(x;a,b,c)\frac{t^n}{n!}\\
&\quad -\frac{1}{2}\sum_{n=0}^{\infty}G_n^{\lambda}(x;a,b,c)\frac{t^n}{n!}\cdot\sum_{n=0}^{\infty}\left[\lambda\ln b\cdot G_n^{\lambda}(\ln b;a,b)+\ln a\cdot G_n^{\lambda}(\ln a;a,b)\right]\frac{t^{n-1}}{n!}\\
&= \sum_{n=1}^{\infty}G_n^{\lambda}(x;a,b,c)\frac{t^{n-1}}{n!}+x\ln c\cdot\sum_{n=0}^{\infty}G_n^{\lambda}(x;a,b,c)\frac{t^n}{n!}\\
&\quad -\frac{1}{2}\sum_{n=0}^{\infty}G_n^{\lambda}(x;a,b,c)\frac{t^n}{n!}\cdot\sum_{n=1}^{\infty}\left[\lambda\ln b\cdot G_n^{\lambda}(\ln b;a,b)+\ln a\cdot G_n^{\lambda}(\ln a;a,b)\right]\frac{t^{n-1}}{n!}.
\end{aligned}$$

The last equation follows from the fact that $G_0^{\lambda}(x;a,b)=G_0^{\lambda}(x;a,b,c)=0$. Reindexing and using Cauchy product for series, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{2t}{\lambda b^t+a^t}c^{xt}\right) &= \sum_{n=0}^{\infty}\left[\frac{G_{n+1}^{\lambda}(x;a,b,c)}{n+1}+x\ln c\cdot G_n^{\lambda}(x;a,b,c)\right.\\
&\quad \left.-\frac{1}{2}\cdot\sum_{k=0}^{n}\binom{n}{k}G_k^{\lambda}(x;a,b,c)\left(\frac{\lambda\ln b\cdot G_{n-k+1}^{\lambda}(\ln b;a,b)+\ln a\cdot G_{n-k+1}^{\lambda}(\ln a;a,b)}{n-k+1}\right)\right]\frac{t^n}{n!}. \qquad (5)
\end{aligned}$$

On the other hand, taking the partial derivative of the right hand side of equation (3) gives us

$$\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty} G_n^{\lambda}(x;a,b,c)\frac{t^n}{n!}\right] = \sum_{n=0}^{\infty} G_{n+1}^{\lambda}(x;a,b,c)\frac{t^n}{n!}. \tag{6}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in equations (5) and (6), we obtain Theorem 1(i).

For the second part, we note that (4) can also be expressed as

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{2t}{\lambda b^t + a^t}c^{xt}\right) &= \frac{1}{t}\cdot\frac{2tc^{xt}}{\lambda b^t + a^t} + \frac{x\ln c\cdot 2tc^{xt}}{\lambda b^t + a^t} - \frac{1}{2t}\cdot\frac{2tc^{xt}}{\lambda b^t + a^t}\left[\frac{\lambda \ln b\cdot 2te^{t\ln b} + \ln a\cdot 2te^{t\ln a}}{\lambda b^t + a^t}\right]\\ &= \sum_{n=0}^{\infty} G_n^{\lambda}(x;a,b,c)\frac{t^{n-1}}{n!} + x\ln c\cdot\sum_{n=0}^{\infty} G_n^{\lambda}(x;a,b,c)\frac{t^n}{n!}\\ &\quad -\frac{1}{2}\sum_{n=0}^{\infty} G_n^{\lambda}(x;a,b,c)\frac{t^{n-1}}{n!}\cdot\sum_{n=0}^{\infty}\left[\lambda\ln b\cdot G_n^{\lambda}(\ln b;a,b) + \ln a\cdot G_n^{\lambda}(\ln a;a,b)\right]\frac{t^n}{n!}\end{aligned}$$

Reindexing and grouping, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{2t}{\lambda b^t + a^t}c^{xt}\right) &= \sum_{n=0}^{\infty}\frac{G_{n+1}^{\lambda}(x;a,b,c)}{n+1}\frac{t^n}{n!} + x\ln c\cdot\sum_{n=0}^{\infty} G_n^{\lambda}(x;a,b,c)\frac{t^n}{n!}\\ &\quad -\frac{1}{2}\sum_{n=0}^{\infty}\frac{G_{n+1}^{\lambda}(x;a,b,c)}{n+1}\frac{t^n}{n!}\cdot\sum_{n=0}^{\infty}[\lambda\ln b\cdot G_n^{\lambda}(\ln b;a,b) + \ln a\cdot G_n^{\lambda}(\ln a;a,b)]\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left[\frac{G_{n+1}^{\lambda}(x;a,b,c)}{n+1} + x\ln c\cdot G_n^{\lambda}(x;a,b,c)\right.\\ &\quad \left.-\frac{1}{2}\cdot\sum_{k=0}^{n}\binom{n}{k}\frac{G_{k+1}^{\lambda}(x;a,b,c)}{k+1}[\lambda\ln b\cdot G_{n-k}^{\lambda}(\ln b;a,b) + \ln a\cdot G_{n-k}^{\lambda}(\ln a;a,b)]\right]\frac{t^n}{n!}. \end{aligned} \tag{7}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in equations (6) and (7), we obtain Theorem 1(ii). □

When k goes from 0 to n, $n-k$ also goes from 0 to n. Hence, replacing k by $n-k$ in Theorem 1, we have the following remark.

Remark 1. *For $n \geq 2$,*

(i) $\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{n-k}^{\lambda}(x;a,b,c)\left[\lambda\ln b\cdot G_{k+1}^{\lambda}(\ln b;a,b) + \ln a\cdot G_{k+1}^{\lambda}(\ln a;a,b)\right]}{k+1} = x\ln c\cdot G_n^{\lambda}(x;a,b,c) - \frac{n}{n+1}G_{n+1}^{\lambda}(x;a,b,c).$

(ii) $\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{n-k+1}^{\lambda}(x;a,b,c)\left[\lambda\ln b\cdot G_{k}^{\lambda}(\ln b;a,b) + \ln a\cdot G_{k}^{\lambda}(\ln a;a,b)\right]}{n-k+1} = x\ln c\cdot G_n^{\lambda}(x;a,b,c) - \frac{n}{n+1}G_{n+1}^{\lambda}(x;a,b,c).$

In the case when $c = 1$ or $x = 0$ in Theorem 1 yields the following corollary.

Corollary 1. *For $n \geq 2$,*

(i) $$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_k^\lambda(a,b)\left[\lambda \ln b \cdot G_{n-k+1}^\lambda(\ln b;a,b)+\ln a \cdot G_{n-k+1}^\lambda(\ln a;a,b)\right]}{n-k+1} = -\frac{n}{n+1}G_{n+1}^\lambda(a,b).$$

(ii) $$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{k+1}^\lambda(a,b)\left[\lambda \ln b \cdot G_{n-k}^\lambda(\ln b;a,b)+\ln a \cdot G_{n-k}^\lambda(\ln a;a,b)\right]}{k+1} = -\frac{n}{n+1}G_{n+1}^\lambda(a,b).$$

At this point, we now take a look on the $G_n^\lambda(\ln b;a,b)$. Differentiating both sides of equation (2) with respect to t, and evaluating it at $t=0$, we obtain $G_1^\lambda(x;a,b) = \dfrac{2}{\lambda+1}$. Also, we note that

$$\sum_{n=0}^{\infty}\left[\lambda G_n^\lambda(\ln b;a,b)+G_n^\lambda(\ln a;a,b)\right]\frac{t^n}{n!} = \frac{\lambda 2t}{\lambda b^t+a^t}b^t + \frac{2t}{\lambda b^t+a^t}a^t = 2t. \tag{8}$$

Hence, evaluating the n^{th} derivative of (8) at $t=0$ for $n \geq 2$, we obtain

$$\lambda G_n^\lambda(\ln b;a,b)+G_n^\lambda(\ln a;a,b) = 0.$$

Thus, we have the following lemma.

Lemma 1.

$$G_n^\lambda(\ln b;a,b) = \begin{cases} \dfrac{2}{\lambda+1}, & \text{if } n=1 \\ -\dfrac{1}{\lambda}G_n^\lambda(\ln a;a,b), & \text{if } n \geq 2. \end{cases} \tag{9}$$

By applying Lemma 1 and using the fact that $G_0^\lambda(\ln b;a,b)=0$, Theorem 1 reduces to the next corollary.

Corollary 2. *For $n \geq 2$,*

(i) $$\frac{1}{2}\ln\left(\frac{b}{a}\right)\sum_{k=0}^{n-1}\binom{n}{k}\frac{G_k^\lambda(x;a,b,c)G_{n-k+1}^\lambda(\ln a;a,b)}{n-k+1} = \left(\frac{\lambda \ln b+\ln a}{\lambda+1} - x\ln c\cdot\right)G_n^\lambda(x;a,b,c) + \frac{n}{n+1}G_{n+1}^\lambda(x;a,b,c).$$

(ii) $$\frac{1}{2}\ln\left(\frac{b}{a}\right)\sum_{k=0}^{n-2}\binom{n}{k}\frac{G_{k+1}^\lambda(x;a,b,c)G_{n-k}^\lambda(\ln a;a,b)}{k+1} = \left(\frac{\lambda \ln b+\ln a}{\lambda+1} - x\ln c\cdot\right)G_n^\lambda(x;a,b,c) + \frac{n}{n+1}G_{n+1}^\lambda(x;a,b,c).$$

Proof. For the first part, we replace $G_1^\lambda(\ln a;a,b)$ and $G_1^\lambda(\ln b;a,b)$ by $\frac{2}{\lambda+1}$, $G_{n-k+1}^\lambda(\ln b;a,b)$ by $-\frac{1}{\lambda}G_{n-k+1}^\lambda(\ln a;a,b)$ for $k \neq n$ in Theorem 1 (i) to obtain

$$\begin{aligned} x\ln c\cdot G_n^\lambda(x;a,b,c) - \frac{n}{n+1}G_{n+1}^\lambda(x;a,b,c) &= G_n^\lambda(x;a,b,c)\left(\frac{\lambda \ln b+\ln a}{\lambda+1}\right) \\ &\quad -\frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}\frac{G_k^\lambda(x;a,b,c)G_{n-k+1}^\lambda(\ln a;a,b)[\ln b-\ln a]}{n-k+1} \end{aligned}$$

Arranging and grouping the terms will give the desired result.

For the second part, we replace $G^{\lambda}_{n-k}(\ln b; a, b)$ by $-\frac{1}{\lambda}G^{\lambda}_{n-k}(\ln a; a, b)$ for $k \neq n$ and $k \neq n-1$ in Theorem 1(ii) to obtain

$$\begin{aligned} x \ln c \cdot G^{\lambda}_n(x; a, b, c) - \frac{n}{n+1} G^{\lambda}_{n+1}(x; a, b, c) \quad &= \quad G^{\lambda}_n(x; a, b)\left(\frac{\lambda \ln b + \ln a}{\lambda + 1}\right) \\ &\quad -\frac{1}{2}\sum_{k=0}^{n-2}\binom{n}{k}\frac{G^{\lambda}_{k+1}(x; a, b, c)G^{\lambda}_{n-k}(\ln a; a, b)[\ln b - \ln a]}{k+1}. \quad \square \end{aligned}$$

It can be seen that $G^{\lambda}_n(\ln a; a, b)$ can be expressed in terms of the generalized Apostol-Genocchi numbers. Indeed,

$$\sum_{n=0}^{\infty} G^{\lambda}_n(\ln a; a, b)\frac{t^n}{n!} = \frac{2t \cdot a^t}{\lambda b^t + a^t} = \frac{2t}{\lambda\left(\frac{b}{a}\right)^t + 1^t} = \sum_{n=0}^{\infty} G^{\lambda}_n\left(1, b/a\right)\frac{t^n}{n!}.$$

Thus, we have

$$G^{\lambda}_n(\ln a; a, b) = G^{\lambda}_n\left(1, b/a\right). \tag{10}$$

Taking $x = 0$ in Corollary 2 and using identity (10), we obtain the following identities involving generalized Apostol-Genocchi numbers only.

Corollary 3. *For $n \geq 2$,*

(i) $$\frac{1}{2}\ln\left(\frac{b}{a}\right)\sum_{k=0}^{n-1}\binom{n}{k}\frac{G^{\lambda}_k(a, b)G^{\lambda}_{n-k+1}\left(1, b/a\right)}{n-k+1} = \left(\frac{\lambda \ln b + \ln a}{\lambda + 1}\right)G^{\lambda}_n(a, b) + \frac{n}{n+1}G^{\lambda}_{n+1}(a, b).$$

(ii) $$\frac{1}{2}\ln\left(\frac{b}{a}\right)\sum_{k=0}^{n-2}\binom{n}{k}\frac{G^{\lambda}_{k+1}(a, b)G^{\lambda}_{n-k}\left(1, b/a\right)}{k+1} = \left(\frac{\lambda \ln b + \ln a}{\lambda + 1}\right)G^{\lambda}_n(a, b) + \frac{n}{n+1}G^{\lambda}_{n+1}(a, b).$$

Now, we consider the generalized Apostol-Genocchi polynomials $G^{\lambda}_n(x+y; a, b, c)$ that involve sum of two variables.

Theorem 2. *For $n \geq 2$ and $y \neq 0$,*

$$G^{\lambda}_n(x + y; a, b, c) = \sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}G^{\lambda}_k(x; a, b, c)y^{n-k}. \tag{11}$$

Proof. By definition,

$$\sum_{n=0}^{\infty} G^{\lambda}_n(x + y; a, b, c)\frac{t^n}{n!} = \frac{2t}{\lambda b^t + a^t}c^{(x+y)t} = \frac{2t}{\lambda b^t + a^t}c^{xt} \cdot c^{yt}.$$

Hence, for $y \neq 0$,

$$\sum_{n=0}^{\infty} G^{\lambda}_n(x + y; a, b, c)\frac{t^n}{n!} \quad = \quad \sum_{n=0}^{\infty} G^{\lambda}_n(x; a, b, c)\frac{t^n}{n!}\sum_{n=0}^{\infty}(\ln c)^n y^n\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}G_k^{\lambda}(x;a,b,c)(\ln c)^{n-k}y^{n-k}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired identity. □

Theorem 3. *For $n \geq 2$ and $y \neq 0$,*

$$G_n^{\lambda}(x;a,b,c) = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\ln c)^{n-k}G_k^{\lambda}(x+y;a,b,c)y^{n-k}. \tag{12}$$

Proof. In this case, we need the binomial inversion formula

$$r_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k s_k \Leftrightarrow s_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k r_k.$$

Note that equation (11) can written as

$$\frac{G_n^{\lambda}(x+y;a,b,c)}{(\ln c)^n y^n} = \sum_{k=0}^{n}\binom{n}{k}\frac{G_k^{\lambda}(x;a,b,c)}{(\ln c)^k y^k}.$$

Taking

$$r_k = \frac{G_k^{\lambda}(x+y;a,b,c)}{(\ln c)^k y^k} \quad \text{and} \quad (-1)^k s_k = \frac{G_k^{\lambda}(x;a,b,c)}{(\ln c)^k y^k}$$

gives us

$$(-1)^n\frac{G_n^{\lambda}(x;a,b,c)}{(\ln c)^n y^n} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{G_k^{\lambda}(x+y;a,b,c)}{(\ln c)^k y^k}.$$

That is,

$$G_n^{\lambda}(x;a,b,c) = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\ln c)^{n-k}G_k^{\lambda}(x+y;a,b,c)y^{n-k}. \quad \square$$

Symmetrically, we obtain the following:

Corollary 4. *For $n \geq 2$ and $x \neq 0$,*

(i) $G_n^{\lambda}(x+y;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}G_k^{\lambda}(y;a,b,c)x^{n-k}$;

(ii) $G_n^{\lambda}(y;a,b,c) = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\ln c)^{n-k}G_k^{\lambda}(x+y;a,b,c)x^{n-k}$.

Replacing k by $n-k$ in Theorem 3 and Corollary 4 (ii), we obtain a more beautiful expressions given in the next corollary.

Corollary 5. *For $n \geq 2$,*

(i) $G_n^\lambda(x;a,b,c) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}(\ln c)^k G_{n-k}^\lambda(x+y;a,b,c)y^k, \quad y \neq 0;$

(ii) $G_n^\lambda(y;a,b,c) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}(\ln c)^k G_{n-k}^\lambda(x+y;a,b,c)x^k, \quad x \neq 0.$

By taking $y = (p-1)x$, equation (11) reduces to the multiplication formula of the generalized Apostol-Genocchi polynomials as shown in the following corollary.

Corollary 6. *For $p \neq 1$ and $x \neq 0$,*

$$G_n^\lambda(px;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k} G_k^\lambda(x;a,b,c)(p-1)^{n-k}x^{n-k}.$$

Theorem 4. *For $n \geq 2$,*

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}(\ln b)^{1-k}(\ln c)^{n-k}G_n^\lambda(1,b/a) = (-1)^n\lambda(\ln b)^{1-n}G_n^\lambda(a,b) + 2n.$$

Proof. Note that

$$c^{xt} = \frac{1}{2t}\left[\frac{2t\lambda b^t c^{xt} + 2ta^t c^{xt}}{\lambda b^t + a^t}\right] = \frac{1}{2t}\left[\frac{\lambda 2tc^{(x+\log_c b)t} + 2tc^{(x+\log_c a)t}}{\lambda b^t + a^t}\right].$$

Consequently,

$$\sum_{n=0}^{\infty}(\ln c)^n x^n \frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\frac{\lambda G_{n+1}^\lambda(x+\log_c b;a,b,c) + G_{n+1}^\lambda(x+\log_c a;a,b,c)}{2(n+1)}\right]\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain

$$\frac{\lambda G_{n+1}^\lambda(x+\log_c b;a,b,c) + G_{n+1}^\lambda(x+\log_c a;a,b,c)}{2(n+1)} = (\ln c)^n x^n,$$

or equivalently

$$\lambda G_n^\lambda(x+\log_c b;a,b,c) + G_n^\lambda(x+\log_c a;a,b,c) = 2n(\ln c)^{n-1}x^{n-1}. \tag{13}$$

Taking $x = -\log_c b$ in equation (13) yields

$$\lambda G_n^\lambda(a,b) + G_n^\lambda(\log_c a - \log_c b;a,b,c) = 2n(-\ln b)^{n-1}. \tag{14}$$

Moreover, letting $x = \log_c a$ and $y = -\log_c b$ in Theorem 2 results to

$$G_n^\lambda(\log_c a - \log_c b; a, b, c) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (\ln b \cdot \ln c)^{n-k} G_k^\lambda(\log_c a; a, b, c). \tag{15}$$

Plugging (18) in (14) and using the fact that $G_n^\lambda(\log_c a; a, b, c) = G_n^\lambda(1, b/a)$, we get the desired result. $\square$

Now, we express $G_n^\lambda(1, b/a)$ as linear combination of the generalized Apostol-Genocchi numbers $G_k^\lambda(a, b)$.

Corollary 7. *For $n \geq 2$,*

$$G_n^\lambda(1, b/a) = -\lambda \sum_{k=0}^{n} \binom{n}{k} (\ln b)^{n-k} G_k^\lambda(a, b).$$

Proof. Taking $x = \log_c b$ and $y = 0$ in Corollary 4 (i),we obtain

$$G_n^\lambda(\log_c b; a, b, c) = \sum_{k=0}^{n} \binom{n}{k} (\ln b)^{n-k} G_k^\lambda(a, b) \tag{16}$$

Utilizing Lemma 1, we get

$$G_n^\lambda(\log_c b; a, b, c) = G_n^\lambda(\ln b; a, b) = -\frac{1}{\lambda} G_n^\lambda(\ln a; a, b).$$

Combining (10) and (16) proves this corollary. $\square$

Now, let us see some identities involving definite integrals of generalized Apostol-Genocchi polynomials.

Differentiating both sides of the exponential generating function for $G_n^\lambda(x; a, b, c)$ in (3) with respect to x gives

$$\frac{d}{dx} G_n^\lambda(x; a, b, c) = n \ln c \cdot G_{n-1}^\lambda(x; a, b, c) \quad \text{and} \quad \deg G_{n+1}^\lambda(x; a, b, c) = n.$$

Consequently,

$$\int_{u_1}^{u_2} G_n^\lambda(x; a, b, c) dx = \frac{G_{n+1}^\lambda(u_2; a, b, c) - G_{n+1}^\lambda(u_1; a, b, c)}{\ln c \cdot (n+1)}. \tag{17}$$

Theorem 5.

$$\int_{\log_c a}^{\log_c b} G_n^\lambda(x; a, b, c) dx = \begin{cases} 0, & n = 0 \\ -\left(\dfrac{\lambda+1}{\lambda \ln c}\right) \dfrac{G_{n+1}^\lambda(\ln a; a, b)}{(n+1)}, & n \geq 1. \end{cases} \tag{18}$$

Proof. This follows from (17) and Lemma 1. □

Note that when $a = 1, b = c = e$ and $\lambda = 1$, (18) reduces to the known identity for classical Genocci numbers and polynomials,

$$\int_0^1 G_n(x)dx = \begin{cases} 0, & n = 0 \\ -2\dfrac{G_{n+1}}{n+1}, & n \geq 1. \end{cases}$$

The next corollary shows that the definite integral in the left-hand side of equation (18) can be expressed as linear combination of generalized Apostol-Genocchi numbers $G_k^\lambda(a,b)$.

Corollary 8. *For* $n \geq 2$*,*

$$\int_{\log_c a}^{\log_c b} G_{n-1}^\lambda(x; a, b, c)dx = \frac{\lambda+1}{n \ln c} \sum_{k=0}^{n} \binom{n}{k} (\ln b)^{n-k} G_k^\lambda(a,b).$$

Proof. This follows from Theorem 5, identity (10), and Corollary 7. □

Using (17), we obtain the double integral of $G_n^\lambda(x+y; a, b, c)$ in the next corollary.

Theorem 6.

$$\begin{aligned} \int_{v_1}^{v_2} \int_{u_1}^{u_2} G_n^\lambda(x+y; a,b,c)dxdy \quad &= \quad \frac{G_{n+2}^\lambda(u_2+v_2; a,b,c) - G_{n+2}^\lambda(u_2+v_1; a,b,c)}{(\ln c)^2(n+1)(n+2)} \\ &\quad - \left[\frac{G_{n+2}^\lambda(u_1+v_2; a,b,c) - G_{n+2}^\lambda(u_1+v_1; a,b,c)}{(\ln c)^2(n+1)(n+2)} \right]. \end{aligned}$$

Remark 2. *In the case when $a = 1, b = c = e$ and and $\lambda = 1$, the obtained results here reduce to old (or new) identities of classical Genocchi polynomials.*

Conclusion

A significant result of this paper is that we have established relationships between generalized Apostol-Genocchi numbers and generalized Apostol-Genocchi polynomials involving binomial coefficients even without associating these numbers (polynomials) to the Bernoulli, Euler and Stirling-type numbers (polynomials). However, combining these new identities with the existing identities between Genocchi, Bernoulli and Euler numbers (polynomials), one can obtain other further identities.

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