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Fuzzy duplex UP-algebras†

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Abstract. Using the concept of a neutrosophic quadruple number to a fuzzy duplex number, we introduce the concept of a fuzzy duplex UP-algebra, and investigate some related properties. Also, we find the necessary condition for a fuzzy duplex UP-set to be a fuzzy duplex UP-algebra. Furthermore, we study the relationship between special subsets of a UP-algebra and special subsets of a fuzzy duplex UP-set.

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1. Introduction

The type of the logical algebra, a UP-algebra was introduced by Iampan [9], and it is known that the class of KU-algebras is a proper subclass of the class of UP-algebras. Later Somjanta et al. [29] studied fuzzy UP-subalgebras, fuzzy UP-ideals and fuzzy UP-filters of UP-algebras. Guntasow et al. [7] studied fuzzy translations of a fuzzy set in UP-algebras. Kesorn et al. [16] studied intuitionistic fuzzy sets in UP-algebras. Kaijae et al. [15] studied anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras. Tanamoon et al. [35] and Sripaeng et al. [34] introduced the concept of Q-fuzzy sets in UP-algebras, and studied anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras. Dokkhamdang et al. [6] introduced the concept of fuzzy UP-subalgebras (fuzzy UP-filters, fuzzy UP-ideals, fuzzy strong UP-ideals) with thresholds of UP-algebras.

Ansari et al. [3] introduced the concept of graphs associated with commutative UP-algebras and defined a graph of equivalence classes of commutative UP-algebras. Songsaeng and Iampan [31–33] studied $\mathcal{N}$-fuzzy sets, fuzzy proper UP-filters, and neutrosophic sets in UP-algebras. Senapati et al. [26, 27] studies applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras.

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More concepts on UP-algebras are discussed in [4, 5, 17].

A fuzzy set f in a nonempty set S is a function from S to the closed interval $[0, 1]$. The concept of a fuzzy set in a nonempty set was first considered by Zadeh [36] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

The concept of a neutrosophic set was introduced by Smarandache [28] in 1999. Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in [11, 12, 19, 21, 30]. Neutrosophic quadruple algebraic structures and hyperstructure are discussed in [1, 2]. Neutrosophic quadruple algebraic structures in BCK/BCI-algebras are discussed in [13, 14, 18, 20, 22].

In this paper, we apply the concept of a neutrosophic quadruple number to a fuzzy duplex number, introduce the concept of a fuzzy duplex set base on a UP-algebra, which is called a fuzzy duplex UP-set, and investigate some related properties. We find the necessary conditions that a fuzzy duplex UP-set form a UP-algebra, which is called a fuzzy duplex UP-algebra. Furthermore, we study the relationship between special subsets of a UP-algebra and the same special subsets of a fuzzy duplex UP-set.

2. Basic concepts and preliminary notes on UP-algebras

Before we begin our study, we will give the definition and useful properties of UP-algebras.

Definition 1. *[9] An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra, where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X (i.e., a nullary operation) if it satisfies the following axioms:*

(UP-1) $(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,

(UP-2) $(\forall x \in X)(0 \cdot x = x)$,

(UP-3) $(\forall x \in X)(x \cdot 0 = 0)$, *and*

(UP-4) $(\forall x, y \in X)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$.

From [9], we know that the concept of UP-algebras is a generalization of KU-algebras (see [23]).

For more examples of UP-algebras, see [6, 10, 24–27].

In a UP-algebra $X = (X, \cdot, 0)$, the following assertions are valid (see [9, 10]).

$$(\forall x \in X)(x \cdot x = 0), \tag{1}$$
$$(\forall x, y, z \in X)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0), \tag{2}$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0), \tag{3}$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0), \tag{4}$$
$$(\forall x, y \in X)(x \cdot (y \cdot x) = 0), \tag{5}$$

$$(\forall x, y \in X)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x), \tag{6}$$
$$(\forall x, y \in X)(x \cdot (y \cdot y) = 0), \tag{7}$$
$$(\forall a, x, y, z \in X)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0), \tag{8}$$
$$(\forall a, x, y, z \in X)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0), \tag{9}$$
$$(\forall x, y, z \in X)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0), \tag{10}$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0), \tag{11}$$
$$(\forall x, y, z \in X)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0), \text{ and} \tag{12}$$
$$(\forall a, x, y, z \in X)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0). \tag{13}$$

From [9], the binary relation $\leq$ on a UP-algebra $X = (X, \cdot, 0)$ defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 0). \tag{14}$$

Definition 2. *[7–9, 29] A nonempty subset S of a UP-algebra $X = (X, \cdot, 0)$ is called*

(1) *a UP-subalgebra of X if $(\forall x, y \in S)(x \cdot y \in S)$.*

(2) *a near UP-filter of X if*

  (i) *the constant 0 of X is in S, and*
  (ii) *$(\forall x, y \in X)(y \in S \Rightarrow x \cdot y \in S)$.*

(3) *a UP-filter of X if*

  (i) *the constant 0 of X is in S, and*
  (ii) *$(\forall x, y \in X)(x \cdot y \in S, x \in S \Rightarrow y \in S)$.*

(4) *a UP-ideal of X if*

  (i) *the constant 0 of X is in S, and*
  (ii) *$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$.*

(5) *a strong UP-ideal (renamed from a strongly UP-ideal) of X if*

  (i) *the constant 0 of X is in S, and*
  (ii) *$(\forall x, y, z \in X)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S)$.*

Guntasow et al. [7] and Iampan [8] proved that the concept of UP-subalgebras is a generalization of near UP-filters, near UP-filters is a generalization of UP-filters, UP-filters is a generalization of UP-ideals, and UP-ideals is a generalization of strong UP-ideals. Furthermore, they proved that the only strong UP-ideal of a UP-algebra X is X.

3. Fuzzy duplex UP-algebras

In this section, we introduce the concepts of fuzzy duplex UP-numbers and fuzzy duplex UP-sets, and investigate some properties. We find the necessary conditions that a fuzzy duplex UP-set form a UP-algebra. Furthermore, we study the relationship between special subsets of a UP-algebra and the same special subsets of a fuzzy duplex UP-set.

Definition 3. *Let X and Y be nonempty sets and $T : X \to Y$ be a function. A fuzzy duplex X-number is an ordered pair (x, yT), where $x, y \in X$, and $T(y)$ denoted by yT. The Cartesian product $X \times \mathrm{Im}(T)$ is called the fuzzy duplex set based on X. If X is a UP-algebra, a fuzzy duplex X-number is called a fuzzy duplex UP-number and we say that $X \times \mathrm{Im}(T)$ is the fuzzy duplex UP-set. For any two nonempty subsets A and B of X, we see that $A \times T(B)$ is a nonempty subset of $X \times \mathrm{Im}(T)$. If $(a, yT) \in A \times T(B)$, then $(x, yT) \in A \times T(B)$ for all $x \in A$.*

In what follows, X will denote a UP-algebra $(X, \cdot, 0)$, Y will denote a nonempty set, and $T : X \to Y$ will be a function.

We define the binary operation $\odot$ on the fuzzy duplex UP-set $X \times \mathrm{Im}(T)$ by

$$(\forall (a, xT), (b, yT) \in X \times \mathrm{Im}(T))((a, xT) \odot (b, yT) = (a \cdot b, (x \cdot y)T)). \tag{15}$$

If the algebra $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ is a UP-algebra, then it is called the *fuzzy duplex UP-algebra*. We denote by $\tilde{a}$ the fuzzy duplex UP-number, that is, $\tilde{a} = (a_1, a_2T)$ for some $a_1, a_2 \in X$, and the zero fuzzy duplex UP-number $(0, 0T)$ is denoted by $\tilde{0}$. We define the binary relation $\ll$ and the equality $\doteq$ on $X \times \mathrm{Im}(T)$ as follows:

$$(\forall (a, xT), (b, yT) \in X \times \mathrm{Im}(T)) \begin{pmatrix} (a, xT) \ll (b, yT) \Leftrightarrow a \leq b, x \leq y \\ (a, xT) \doteq (b, yT) \Leftrightarrow (a, xT) \ll (b, yT), (b, yT) \ll (a, xT) \end{pmatrix}.$$

Then we can easily prove that the binary relation $\ll$ is an order relation on $X \times \mathrm{Im}(T)$ and

$$(\forall (a, xT), (b, yT) \in X \times \mathrm{Im}(T)) \begin{pmatrix} (a, xT) \ll (b, yT) \Leftrightarrow (a, xT) \odot (b, yT) = \tilde{0} \\ (a, xT) \doteq (b, yT) \Leftrightarrow a = b, x = y \end{pmatrix}.$$

Hence, $\doteq \subseteq =$ on $X \times \mathrm{Im}(T)$.

Example 1. *Let $X = \{0, a, b, c\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	b	b
b	0	a	0	b
c	0	a	0	0

Let $T : X \to \{0.5, 1\}$ be a function defined by

$$0T = aT = bT = 0.5, cT = 1.$$

Then the axiom (UP-4) is not satisfied. Indeed, there are $(0, aT), (0, cT) \in X \times \{0.5, 1\}$ such that $(0, aT) = (0, 0.5) \neq (0, 1) = (0, cT)$ but

$$(0, aT) \odot (0, cT) = (0 \cdot 0, (a \cdot c)T) = (0, bT) = (0, 0T) = \tilde{0}$$

and

$$(0, cT) \odot (0, aT) = (0 \cdot 0, (c \cdot a)T) = (0, aT) = (0, 0T) = \tilde{0}.$$

Hence, the algebra $(X \times \{0.5, 1\}, \odot, \tilde{0})$ is not a UP-algebra.

Theorem 1. *The algebra $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ satisfies the axioms (UP-1), (UP-2), and (UP-3).*

Proof. (UP-1) Let $\tilde{x}, \tilde{y}, \tilde{z} \in X \times \mathrm{Im}(T)$ where $\tilde{x} = (x_1, x_2T), \tilde{y} = (y_1, y_2T)$, and $\tilde{z} = (z_1, z_2T)$. Then

$$\begin{aligned}
&(\tilde{y} \odot \tilde{z}) \odot ((\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot \tilde{z})) \\
&= ((y_1, y_2T) \odot (z_1, z_2T)) \odot (((x_1, x_2T) \odot (y_1, y_2T)) \odot ((x_1, x_2T) \odot (z_1, z_2T))) \\
&= (y_1 \cdot z_1, (y_2 \cdot z_2)T) \odot ((x_1 \cdot y_1, (x_2 \cdot y_2)T) \odot (x_1 \cdot z_1, (x_2 \cdot z_2)T)) \\
&= (y_1 \cdot z_1, (y_2 \cdot z_2)T) \odot ((x_1 \cdot y_1) \cdot (x_1 \cdot z_1), ((x_2 \cdot y_2) \cdot (x_2 \cdot z_2))T) \\
&= ((y_1 \cdot z_1) \cdot ((x_1 \cdot y_1) \cdot (x_1 \cdot z_1)), ((y_2 \cdot z_2) \cdot ((x_2 \cdot y_2) \cdot (x_2 \cdot z_2)))T) \\
&= (0, 0T) && \text{((UP-1))} \\
&= \tilde{0}.
\end{aligned}$$

(UP-2) Let $\tilde{x} \in X \times \mathrm{Im}(T)$ where $\tilde{x} = (x_1, x_2T)$. Then

$$\begin{aligned}
\tilde{0} \odot \tilde{x} &= (0, 0T) \odot (x_1, x_2T) \\
&= (0 \cdot x_1, (0 \cdot x_2)T) \\
&= (x_1, x_2T) && \text{((UP-2))} \\
&= \tilde{x}.
\end{aligned}$$

(UP-3) Let $\tilde{x} \in X \times \mathrm{Im}(T)$ where $\tilde{x} = (x_1, x_2T)$. Then

$$\begin{aligned}
\tilde{x} \odot \tilde{0} &= (x_1, x_2T) \odot (0, 0T) \\
&= (x_1 \cdot 0, (x_2 \cdot 0)T) \\
&= (0, 0T) && \text{((UP-3))} \\
&= \tilde{0}.
\end{aligned}$$

Hence, (UP-1), (UP-2), and (UP-3) are valid.

Proposition 1. *The algebra* $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ *satisfies the following properties:*

(1) $(\forall \tilde{a} \in X \times \mathrm{Im}(T))(\tilde{a} \ll \tilde{a})$,

(2) $(\forall \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T))(\tilde{a} \ll \tilde{b}, \tilde{b} \ll \tilde{c} \Rightarrow \tilde{a} \ll \tilde{c})$,

(3) $(\forall \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T))(\tilde{a} \ll \tilde{b} \Rightarrow \tilde{c} \odot \tilde{a} \ll \tilde{c} \odot \tilde{b})$,

(4) $(\forall \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T))(\tilde{a} \ll \tilde{b} \Rightarrow \tilde{b} \odot \tilde{c} \ll \tilde{a} \odot \tilde{c})$,

(5) $(\forall \tilde{a}, \tilde{b} \in X \times \mathrm{Im}(T))(\tilde{a} \ll \tilde{b} \odot \tilde{a})$,

(6) $(\forall \tilde{a}, \tilde{b} \in X \times \mathrm{Im}(T))(\tilde{a} \ll \tilde{b} \odot \tilde{b})$,

(7) $(\forall \tilde{x}, \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T))(\tilde{a} \odot (\tilde{b} \odot \tilde{c}) \ll \tilde{a} \odot ((\tilde{x} \odot \tilde{b}) \odot (\tilde{x} \odot \tilde{c})))$,

(8) $(\forall \tilde{x}, \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T))(((\tilde{x} \odot \tilde{a}) \odot (\tilde{x} \odot \tilde{b})) \odot \tilde{c} \ll (\tilde{a} \odot \tilde{b}) \odot \tilde{c})$,

(9) $(\forall \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T))((\tilde{a} \odot \tilde{b}) \odot \tilde{c} \ll \tilde{b} \odot \tilde{c})$,

(10) $(\forall \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T))(\tilde{a} \ll \tilde{b} \Rightarrow \tilde{a} \ll \tilde{c} \odot \tilde{b})$,

(11) $(\forall \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T))((\tilde{a} \odot \tilde{b}) \odot \tilde{c} \ll \tilde{a} \odot (\tilde{b} \odot \tilde{c}))$, *and*

(12) $(\forall \tilde{x}, \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T))((\tilde{a} \odot \tilde{b}) \odot \tilde{c} \ll \tilde{b} \odot (\tilde{x} \odot \tilde{c}))$.

Proof. By Theorem 1, the algebra $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ satisfies the axioms (UP-1), (UP-2), and (UP-3).

(1) Let $\tilde{a} \in X \times \mathrm{Im}(T)$. Then

$$\begin{aligned} \tilde{0} &= (\tilde{0} \odot \tilde{a}) \odot ((\tilde{0} \odot \tilde{0}) \odot (\tilde{0} \odot \tilde{a})) && \text{((UP-1))} \\ &= (\tilde{0} \odot \tilde{a}) \odot (\tilde{0} \odot \tilde{a}) && \text{((UP-2))} \\ &= \tilde{a} \odot \tilde{a}. && \text{((UP-2))} \end{aligned}$$

Hence, $\tilde{a} \ll \tilde{a}$.

(2) Let $\tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T)$ be such that $\tilde{a} \ll \tilde{b}$ and $\tilde{b} \ll \tilde{c}$. Then $\tilde{a} \odot \tilde{b} = \tilde{0}$ and $\tilde{b} \odot \tilde{c} = \tilde{0}$. Thus

$$\begin{aligned} \tilde{a} \odot \tilde{c} &= \tilde{0} \odot (\tilde{0} \odot (\tilde{a} \odot \tilde{c})) && \text{((UP-2))} \\ &= (\tilde{b} \odot \tilde{c}) \odot ((\tilde{a} \odot \tilde{b}) \odot (\tilde{a} \odot \tilde{c})) \\ &= \tilde{0}. && \text{((UP-1))} \end{aligned}$$

Hence, $\tilde{a} \ll \tilde{c}$.

(3) Let $\tilde{a}, \tilde{b} \in X \times \mathrm{Im}(T)$ be such that $\tilde{a} \ll \tilde{b}$. Then $\tilde{a} \odot \tilde{b} = \tilde{0}$.

$$(\tilde{c} \odot \tilde{a}) \odot (\tilde{c} \odot \tilde{b}) = \tilde{0} \odot ((\tilde{c} \odot \tilde{a}) \odot (\tilde{c} \odot \tilde{b})) \quad \text{((UP-2))}$$

$$
\begin{aligned}
&= (\tilde{a} \odot \tilde{b}) \odot ((\tilde{c} \odot \tilde{a}) \odot (\tilde{c} \odot \tilde{b})) \\
&= \tilde{0}. && \text{((UP-1))}
\end{aligned}
$$

Hence, $\tilde{c} \odot \tilde{a} \ll \tilde{c} \odot \tilde{b}$.

(4) Let $\tilde{a}, \tilde{b} \in X \times \mathrm{Im}(T)$ be such that $\tilde{a} \ll \tilde{b}$. Then $\tilde{a} \odot \tilde{b} = \tilde{0}$.

$$
\begin{aligned}
(\tilde{b} \odot \tilde{c}) \odot (\tilde{a} \odot \tilde{c}) &= (\tilde{b} \odot \tilde{c}) \odot (\tilde{0} \odot (\tilde{a} \odot \tilde{c})) && \text{((UP-2))} \\
&= (\tilde{b} \odot \tilde{c}) \odot ((\tilde{a} \odot \tilde{b}) \odot (\tilde{a} \odot \tilde{c})) \\
&= \tilde{0}. && \text{((UP-1))}
\end{aligned}
$$

Hence, $\tilde{b} \odot \tilde{c} \ll \tilde{a} \odot \tilde{c}$.

(5) Let $\tilde{a}, \tilde{b} \in X \times \mathrm{Im}(T)$. Then

$$
\begin{aligned}
\tilde{a} \odot (\tilde{b} \odot \tilde{a}) &= (\tilde{0} \odot \tilde{a}) \odot (\tilde{0} \odot (\tilde{b} \odot \tilde{a})) && \text{((UP-2))} \\
&= (\tilde{0} \odot \tilde{a}) \odot ((\tilde{b} \odot \tilde{0}) \odot (\tilde{b} \odot \tilde{a})) && \text{((UP-3))} \\
&= \tilde{0}. && \text{((UP-1))}
\end{aligned}
$$

Hence, $\tilde{a} \ll \tilde{b} \odot \tilde{a}$.

(6) Let $\tilde{a}, \tilde{b} \in X \times \mathrm{Im}(T)$. By (UP-3) and (1), we have $\tilde{a} \odot (\tilde{b} \odot \tilde{b}) = \tilde{a} \odot \tilde{0} = \tilde{0}$. Hence, $\tilde{a} \ll \tilde{b} \odot \tilde{b}$.

(7) Let $\tilde{x}, \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T)$. By (UP-1), we have $(\tilde{b} \odot \tilde{c}) \odot ((\tilde{x} \odot \tilde{b}) \odot (\tilde{x} \odot \tilde{c})) = \tilde{0}$. Thus $\tilde{b} \odot \tilde{c} \ll (\tilde{x} \odot \tilde{b}) \odot (\tilde{x} \odot \tilde{c})$. By (3), we have $\tilde{a} \odot (\tilde{b} \odot \tilde{c}) \ll \tilde{a} \odot ((\tilde{x} \odot \tilde{b}) \odot (\tilde{x} \odot \tilde{c}))$.

(8) Let $\tilde{x}, \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T)$. By (UP-1), we have $(\tilde{a} \odot \tilde{b}) \odot ((\tilde{x} \odot \tilde{a}) \odot (\tilde{x} \odot \tilde{b})) = \tilde{0}$. Thus $\tilde{a} \odot \tilde{b} \ll (\tilde{x} \odot \tilde{a}) \odot (\tilde{x} \odot \tilde{b})$. By (4), we have $((\tilde{x} \odot \tilde{a}) \odot (\tilde{x} \odot \tilde{b})) \odot \tilde{c} \ll (\tilde{a} \odot \tilde{b}) \odot \tilde{c}$.

(9) Let $\tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T)$. Then

$$
\begin{aligned}
\tilde{0} &= (((\tilde{a} \odot \tilde{0}) \odot (\tilde{a} \odot \tilde{b})) \odot \tilde{c}) \odot ((\tilde{0} \odot \tilde{b}) \odot \tilde{c}) && \text{((8))} \\
&= ((\tilde{0} \odot (\tilde{a} \odot \tilde{b})) \odot \tilde{c}) \odot (\tilde{b} \odot \tilde{c}) && \text{((UP-2), (UP-3))} \\
&= ((\tilde{a} \odot \tilde{b}) \odot \tilde{c}) \odot (\tilde{b} \odot \tilde{c}). && \text{((UP-2))}
\end{aligned}
$$

Hence, $(\tilde{a} \odot \tilde{b}) \odot \tilde{c} \ll \tilde{b} \odot \tilde{c}$.

(10) Let $\tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T)$ be such that $\tilde{a} \ll \tilde{b}$. By (3), we have $(\tilde{c} \odot \tilde{a}) \odot (\tilde{c} \odot \tilde{b}) = \tilde{0}$. Thus

$$
\begin{aligned}
\tilde{a} \odot (\tilde{c} \odot \tilde{b}) &= \tilde{0} \odot (\tilde{a} \odot (\tilde{c} \odot \tilde{b})) && \text{((UP-2))} \\
&= ((\tilde{c} \odot \tilde{a}) \odot (\tilde{c} \odot \tilde{b})) \odot (\tilde{a} \odot (\tilde{c} \odot \tilde{b})) \\
&= \tilde{0}. && \text{((9))}
\end{aligned}
$$

Hence, $\tilde{a} \ll \tilde{c} \odot \tilde{b}$.

(11) Let $\tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T)$. By (9), we have $(\tilde{a} \odot \tilde{b}) \odot \tilde{c} \ll \tilde{b} \odot \tilde{c}$. By (5), we have $\tilde{b} \odot \tilde{c} \ll \tilde{a} \odot (\tilde{b} \odot \tilde{c})$. It follows from (2) that $(\tilde{a} \odot \tilde{b}) \odot \tilde{c} \ll \tilde{a} \odot (\tilde{b} \odot \tilde{c})$.

(12) Let $\tilde{x}, \tilde{a}, \tilde{b}, \tilde{c} \in X \times \mathrm{Im}(T)$. By (5), we have $\tilde{b} \ll \tilde{a} \odot \tilde{b}$ and $\tilde{a} \odot \tilde{b} \ll \tilde{x} \odot (\tilde{a} \odot \tilde{b})$. By (2), we have $\tilde{b} \ll \tilde{x} \odot (\tilde{a} \odot \tilde{b})$. By (4), we have

$$(\tilde{x} \odot (\tilde{a} \odot \tilde{b})) \odot (\tilde{x} \odot \tilde{c}) \ll \tilde{b} \odot (\tilde{x} \odot \tilde{c}).$$

By (UP-1), we have $((\tilde{a} \odot \tilde{b}) \odot \tilde{c}) \odot ((\tilde{x} \odot (\tilde{a} \odot \tilde{b})) \odot (\tilde{x} \odot \tilde{c})) = \tilde{0}$. Then

$$(\tilde{a} \odot \tilde{b}) \odot \tilde{c} \ll (\tilde{x} \odot (\tilde{a} \odot \tilde{b})) \odot (\tilde{x} \odot \tilde{c}).$$

It follows from (2) that $(\tilde{a} \odot \tilde{b}) \odot \tilde{c} \ll \tilde{b} \odot (\tilde{x} \odot \tilde{c})$.

Theorem 2. *If $T : X \to Y$ is a constant function, that is, the inverse image $T^{-1}(\{0T\}) = X$, then the algebra $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ is a UP-algebra which is UP-isomorphic to X.*

Proof. (UP-4) Let $\tilde{x}, \tilde{y} \in X \times \mathrm{Im}(T)$ be such that $\tilde{x} \odot \tilde{y} = \tilde{0}$ and $\tilde{y} \odot \tilde{x} = \tilde{0}$ where $\tilde{x} = (x_1, x_2T), \tilde{y} = (y_1, y_2T)$. Then

$$(x_1 \cdot y_1, (x_2 \cdot y_2)T) = (x_1, x_2T) \odot (y_1, y_2T) = (0, 0T)$$

and

$$(y_1 \cdot x_1, (y_2 \cdot x_2)T) = (y_1, y_2T) \odot (x_1, x_2T) = (0, 0T).$$

It follows that $x_1 \cdot y_1 = 0$ and $y_1 \cdot x_1 = 0$. By (UP-4), we have $x_1 = y_1$. Since T is constant, we have $x_2T = y_2T$. Thus $\tilde{x} = (x_1, x_2T) = (y_1, y_2T) = \tilde{y}$, (UP-4) holding. By Theorem 1, we have $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ is a UP-algebra. Finally, X and $X \times \mathrm{Im}(T)$ are UP-isomorphic under the UP-isomorphism sending $x \mapsto (x, 0T)$.

Corollary 1. *If Y is a singleton set, then the algebra $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ is a UP-algebra.*

Proof. If Y is a singleton set, then $T : X \to Y$ is a constant function. By Theorem 2, we have the algebra $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ is a UP-algebra.

Theorem 3. *If $T : X \to Y$ is a function with the inverse image $T^{-1}(\{0T\}) = \{0\}$, then the algebra $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ is a UP-algebra.*

Proof. (UP-4) Let $\tilde{x}, \tilde{y} \in X \times \mathrm{Im}(T)$ be such that $\tilde{x} \odot \tilde{y} = \tilde{0}$ and $\tilde{y} \odot \tilde{x} = \tilde{0}$ where $\tilde{x} = (x_1, x_2T), \tilde{y} = (y_1, y_2T)$. Then

$$(x_1 \cdot y_1, (x_2 \cdot y_2)T) = (x_1, x_2T) \odot (y_1, y_2T) = (0, 0T)$$

and

$$(y_1 \cdot x_1, (y_2 \cdot x_2)T) = (y_1, y_2T) \odot (x_1, x_2T) = (0, 0T).$$

It follows that $x_1 \cdot y_1 = 0$ and $y_1 \cdot x_1 = 0$, and $x_2 \cdot y_2, y_2 \cdot x_2 \in T^{-1}(\{0T\}) = \{0\}$, that is, $x_2 \cdot y_2 = 0$ and $y_2 \cdot x_2 = 0$. By (UP-4), we have $x_1 = y_1$ and $x_2 = y_2$. Thus $x_2T = y_2T$ and so $\tilde{x} = (x_1, x_2T) = (y_1, y_2T) = \tilde{y}$, (UP-4) holding. By Theorem 1, we have $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ is a UP-algebra.

Corollary 2. *If $T : X \to Y$ is an injective function, then the algebra $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ is a UP-algebra.*

Proof. If $T : X \to Y$ is an injective function, then the inverse image $T^{-1}(\{0T\}) = \{0\}$. By Theorem 3, we have the algebra $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ is a UP-algebra.

Theorem 4. *Let A and B be nonempty subsets of a UP-algebra X and $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ be a fuzzy duplex UP-algebra.*

(1) *If A and B are UP-subalgebras of X, then $A \times T(B)$ is a UP-subalgebra of $X \times \mathrm{Im}(T)$.*

(2) *If $A \times T(B)$ is a UP-subalgebra of $X \times \mathrm{Im}(T)$, then A is a UP-subalgebra of X.*

Proof. (1) Assume that A and B are UP-subalgebras of X and let $\tilde{x}, \tilde{y} \in A \times T(B)$ where $\tilde{x} = (a_1, b_1T)$ and $\tilde{y} = (a_2, b_2T)$. Then $a_1 \cdot a_2 \in A$ and $b_1 \cdot b_2 \in B$. Thus $\tilde{x} \odot \tilde{y} = (a_1, b_1T) \odot (a_2, b_2T) = (a_1 \cdot a_2, (b_1 \cdot b_2)T) \in A \times T(B)$. Hence, $A \times T(B)$ is a UP-subalgebra of $X \times \mathrm{Im}(T)$.

(2) Assume that $A \times T(B)$ is a UP-subalgebra of $X \times \mathrm{Im}(T)$. Let $x, y \in A$. Since $(0, 0T) \in A \times T(B)$, we have Then $(x, 0T), (y, 0T) \in A \times T(B)$. Thus $(x \cdot y, 0T) = (x \cdot y, (0 \cdot 0)T) = (x, 0T) \odot (y, 0T) \in A \times T(B)$, so $x \cdot y \in A$. Hence, A is a UP-subalgebra of X.

Theorem 5. *Let A and B be nonempty subsets of a UP-algebra X and $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ be a fuzzy duplex UP-algebra.*

(1) *If A and B are near UP-filters of X, then $A \times T(B)$ is a near UP-filter of $X \times \mathrm{Im}(T)$.*

(2) *If $A \times T(B)$ is a near UP-filter of $X \times \mathrm{Im}(T)$, then A is a near UP-filter of X.*

Proof. (1) Assume that A and B are near UP-filters of X. Since $0 \in A$ and $0 \in B$, we have $\tilde{0} = (0, 0T) \in A \times T(B)$. Let $\tilde{x} \in X \times \mathrm{Im}(T)$ and $\tilde{y} \in A \times T(B)$ where $\tilde{x} = (x_1, x_2T)$ and $\tilde{y} = (a, bT)$. Thus $x_1 \cdot a \in A$ and $x_2 \cdot b \in B$, so $\tilde{x} \odot \tilde{y} = (x_1, x_2T) \odot (a, bT) = (x_1 \cdot a, (x_2 \cdot b)T) \in A \times T(B)$. Hence, $A \times T(B)$ is a near UP-filter of $X \times \mathrm{Im}(T)$.

(2) Assume that $A \times T(B)$ is a near UP-filter of $X \times \mathrm{Im}(T)$. Since $\tilde{0} = (0, 0T) \in A \times T(B)$, we have $0 \in A$. Let $x \in X$ and $a \in A$. Then $(x, 0T) \in X \times \mathrm{Im}(T)$ and $(a, 0T) \in A \times T(B)$. Thus $(x \cdot a, 0T) = (x \cdot a, (0 \cdot 0)T) = (x, 0T) \odot (a, 0T) \in A \times T(B)$, so $x \cdot a \in A$. Hence, A is a near UP-filter of X.

Theorem 6. *Let A and B be nonempty subsets of a UP-algebra X and $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ be a fuzzy duplex UP-algebra. If $A \times T(B)$ is a UP-filter of $X \times \mathrm{Im}(T)$, then A is a UP-filter of X.*

Proof. Assume that $A \times T(B)$ is a UP-filter of $X \times \mathrm{Im}(T)$. Since $\tilde{0} = (0, 0T) \in A \times T(B)$, we have $0 \in A$. Let $x, a \in X$ be such that $a \cdot x \in A$ and $a \in A$. Then $(a, 0T) \odot (x, 0T) = (a \cdot x, (0 \cdot 0)T) = (a \cdot x, 0T) \in A \times T(B)$ and $(a, 0T) \in A \times T(B)$. Thus $(x, 0T) \in A \times T(B)$, so $x \in A$. Hence, A is a UP-filter of X.

Theorem 7. *Let A and B be nonempty subsets of a UP-algebra X and $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ be a fuzzy duplex UP-algebra. If $A \times T(B)$ is a UP-ideal of $X \times \mathrm{Im}(T)$, then A is a UP-ideal of X.*

Proof. Assume that $A \times T(B)$ is a UP-ideal of $X \times \mathrm{Im}(T)$. Since $\tilde{0} = (0, 0T) \in A \times T(B)$, we have $0 \in A$. Let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in A$ and $y \in A$. Then $(x, 0T) \odot ((y, 0T) \odot (z, 0T)) = (x \cdot (y \cdot z), (0 \cdot (0 \cdot 0))T) = (x \cdot (y \cdot z), 0T) \in A \times T(B)$ and $(y, 0T) \in A \times T(B)$. Thus $(x \cdot z, 0T) = (x \cdot z, (0 \cdot 0)T) = (x, 0T) \odot (z, 0T) \in A \times T(B)$, so $x \cdot z \in A$. Hence, A is a UP-ideal of X.

The following example shows that the sentence "if A and B are UP-filters (resp., UP-ideals) of X, then $A \times T(B)$ is a UP-filter (resp., UP-ideal) of $X \times \mathrm{Im}(T)$" does not hold in general.

Example 2. *Let $X = \{0, a, b, c\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	b	b
b	0	a	0	a
c	0	0	0	0

Let $T : X \to \{0.5, 0.7, 1\}$ be a function defined by

$$0T = 0.5, aT = bT = 0.7, cT = 1.$$

Let $A = \{0, a\}$. Then A is a UP-ideal (also a UP-filter) of X and

$$A \times T(A) = \{(0, 0T), (0, aT), (a, 0T), (a, aT)\}.$$

Since $(0, aT) \odot (0, cT) = (0, bT) = (0, aT) \in A \times T(A)$ and $(0, aT) \in A \times T(A)$ but $(0, cT) \notin A \times T(A)$. Hence, $A \times T(A)$ is not a UP-filter (also not a UP-ideal) of X.

Theorem 8. *Let A and B be nonempty subsets of a UP-algebra X and $(X \times \mathrm{Im}(T), \odot, \tilde{0})$ be a fuzzy duplex UP-algebra.*

(1) *If A and B are strong UP-ideals of X, then $A \times T(B)$ is a strong UP-ideal of $X \times \mathrm{Im}(T)$.*

(2) *If $A \times T(B)$ is a strong UP-ideal of $X \times \mathrm{Im}(T)$, then A is a strong UP-ideal of X.*

Proof. (1) Assume that A and B are strong UP-ideals of X. Then $A = B = X$, so $A \times T(B) = X \times \mathrm{Im}(T)$. Hence, $A \times T(B)$ is a strong UP-ideal of $X \times \mathrm{Im}(T)$.

(2) Assume that $A \times T(B)$ is a strong UP-ideal of $X \times \mathrm{Im}(T)$. Then $A \times T(B) = X \times \mathrm{Im}(T)$, so $A = X$. Hence, A is a strong UP-ideal of X.

4. Conclusions

In this paper, we have introduced the concept of a fuzzy duplex set base on a UP-algebra, which is called a fuzzy duplex UP-set, and investigated some related properties. We have found the necessary conditions that a fuzzy duplex UP-set form a UP-algebra, which is called a fuzzy duplex UP-algebra. Furthermore, we have studied the relationship between special subsets of a UP-algebra and the same special subsets of a fuzzy duplex UP-set and have presented conflicting examples for certain relationships.

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