Almost Bi-Γ-Ideals and Fuzzy Almost Bi-Γ-Ideals of Γ-Semigroups

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Abstract. In this paper, we introduce the notions of almost bi-Γ-ideals and fuzzy almost bi-Γ-ideals of Γ-semigroups and give properties of them. Moreover, we investigate relationships between almost bi-Γ-ideals and fuzzy almost bi-Γ-ideals.

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1. Introduction and Preliminaries

Ideal theory in semigroups, like all other algebraic structures, plays an important role in studying them. Good and Hughes [8] introduced the notion of bi-ideals of semigroups in 1952. An introductory definition of left, right, two-sided almost ideals of semigroups was launched by Grosek and Satko [9] in 1980. They gave the characterization of these ideals when a semigroup S contains no proper left, right, two-sided almost ideals in [9], and afterwards, they discovered the minimal almost ideals and the smallest almost ideals of semigroups in [10] and [11], respectively. In 1981, Bogdanovic [3] introduced the definition of almost bi-ideals in semigroups by using the definitions of almost ideals and bi-ideals in semigroups. In [5], Wattanatripop, Chinram and Changphas gave the properties of quasi-almost-ideals and first defined the concept of fuzzy almost ideals in semigroups. Moreover, they provided the relationships between almost ideals and their fuzzification. Furthermore, they investigated fuzzification of almost bi-ideals in semigroups in [4]. Almost (m,n)-ideals and their fuzzification in semigroups were studied by Suebsung, Wattanatripop and Chinram in [23]. Moreover, the idea of almost ideals and their fuzzification were extended to n-ary semigroups in [21].

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The notion of Γ-semigroups has been first studied by Sen [18] in 1981. In 1986, Sen and Saha [19] improved more general definition as follows:

**Definition 1.** ([19]) Let \( M \) and \( \Gamma \) be non-empty sets. \((M, \Gamma)\) is called a **Γ-semigroup** if it satisfies the following laws:

1. \( a \alpha b \in M \) for all \( a, b \in M \) and \( \alpha \in \Gamma \).
2. \( M \) is associative under \( \Gamma \), that is

\[(a \alpha b) \beta c = a \alpha (b \beta c)\]

for all \( a, b, c \in M \) and all \( \alpha, \beta \in \Gamma \).

Every semigroup \((S, \cdot)\) can be considered as a Γ-semigroup \( S \) by choosing \( \Gamma = \{\cdot\} \). Then a Γ-semigroup is one of the generalizations of semigroups. The investigation on Γ-semigroups was done by certain mathematicians which are parallel to some results of semigroups, for example, one may see [6, 7, 17–19]. Similar to semigroups, ideal theory in Γ-semigroups plays an important role (for example, we can see in [1, 6, 7, 12–14, 20]).

Let \( M \) be a Γ-semigroup. For nonempty subsets \( A \) and \( B \) of \( M \), let

\[A \Gamma B = \{a \alpha b \mid a \in A, b \in B, \alpha \in \Gamma\}\]

If \( m \in M \), we let \( A \Gamma m = A \Gamma \{m\} \) and \( m \Gamma A = \{m\} \Gamma A \). If \( \alpha \in \Gamma \), we let

\[A \alpha B = \{a \alpha b \mid a \in A, b \in B\}\]

**Definition 2.** (see [7]) Let \( M \) be a Γ-semigroup.

1. A nonempty subset \( T \) of \( M \) is called a **sub Γ-semigroup** of \( M \) if \( T \Gamma T \subseteq T \).
2. A sub Γ-semigroup \( B \) of \( M \) is called a **bi-Γ-ideal** of \( M \) if \( B \Gamma M \Gamma B \subseteq B \).

A bi-Γ-ideal in Γ-semigroups was sometimes called a bi-ideal (see [14]). Some generalizations of this ideal were studied in [2] and [16]. Recently, Wattanatripop and Changphas first studied the concept of almost ideals in Γ-semigroups. In [22], they defined the definitions of left [right] almost ideals in Γ-semigroups. Moreover, a Γ-semigroup containing no proper left [right] almost ideals was characterized.

In 1965, Zadeh [24] introduced the concept of fundamental fuzzy sets. Since then, fuzzy sets have been studied in various fields. A function from a set \( M \) into the closed unit interval \([0, 1]\) is called a fuzzy subset of \( M \). Let \( f \) and \( g \) be any two fuzzy subsets of a set \( M \).

1. A fuzzy subset \( f \cap g \) of \( M \) is defined by

\[(f \cap g)(m) = \min\{f(m), g(m)\}\]

for all \( m \in M \).
(2) A fuzzy subset \( f \cup g \) of \( M \) is defined by
\[
(f \cup g)(m) = \max\{f(m), g(m)\}
\]
for all \( m \in M \).

(3) If \( f(m) \leq g(m) \) for all \( m \in M \), we say that \( f \) is a subset of \( g \), and use the notation \( f \subseteq g \) and sometimes we will say that \( f \) is contained in \( g \).

For a fuzzy subset \( f \) of any set \( M \), the support of \( f \) is the set of points in \( M \) defined by
\[
\text{supp}(f) = \{ m \in M \mid f(m) \neq 0 \}.
\]
For a subset \( A \) of any set \( M \), the characteristic function \( \chi_A \) of \( A \) is a fuzzy subset of \( M \) defined by
\[
\chi_A(m) = \begin{cases} 
1 & m \in A, \\
0 & m \notin A.
\end{cases}
\]
For any element \( m \) of any set \( M \) and \( t \in (0, 1] \), a fuzzy point \( m_t \) of \( M \) is a fuzzy subset of \( M \) defined by
\[
m_t(x) = \begin{cases} 
t & x = m, \\
0 & x \neq m
\end{cases}
\]
(see [15]).

2. Almost bi-\( \Gamma \)-ideals

First, we define almost bi-\( \Gamma \)-ideals of \( \Gamma \)-semigroups as follows:

**Definition 3.** A non-empty subset \( B \) of a \( \Gamma \)-semigroup \( M \) is called an almost bi-\( \Gamma \)-ideal of \( S \) if
\[
B \Gamma m \Gamma B \cap B \neq \emptyset
\]
for all \( m \in M \).

**Example 1.** Let \( B \) be any bi-\( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( M \). Then \( B \Gamma M \Gamma B \subseteq B \). This implies that for any \( m \in M \), \( B \Gamma m \Gamma B \subseteq B \Gamma M \Gamma B \subseteq B \). So \( B \Gamma m \Gamma B \cap B = B \Gamma m \Gamma B \neq \emptyset \) for all \( m \in M \). Then \( B \) is an almost bi-\( \Gamma \)-ideal of \( M \).

By Example 1, we conclude that every bi-\( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( M \) is an almost bi-\( \Gamma \)-ideal of \( M \).

**Example 2.** Consider the \( \Gamma \)-semigroup \( \mathbb{Z}_8 \) with \( \Gamma = \{0, 1, 2\} \) under the usual addition. Let \( B = \{4, 6\} \). We see that
\[(B + \Gamma + \Gamma + \Gamma + B) \cap B = \mathbb{Z}_8 \cap \{4, 5\} \neq \emptyset,\]
\[(B + \Gamma + \Gamma + \Gamma + B) \cap B = \mathbb{Z}_8 \cap \{4, 5\} \neq \emptyset,\]
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\[(B + \Gamma + \Gamma + \Gamma + B) \cap B = \mathbb{Z}_8 \cap \{4, 5\} \neq \emptyset,\]
\[(B + \Gamma + \Gamma + \Gamma + B) \cap B = \mathbb{Z}_8 \cap \{4, 5\} \neq \emptyset,\]

Therefore, \(B\) is an almost bi-\(\Gamma\)-ideal of \(\mathbb{Z}_8\). However, \(B\) is not a bi-\(\Gamma\)-ideal of \(\mathbb{Z}_8\) because \(B + \Gamma + \mathbb{Z}_8 + \Gamma + B = \mathbb{Z}_8 \not\subseteq B\).

From Example 2, we see that an almost bi-\(\Gamma\)-ideal of \(\Gamma\)-semigroup \(S\) need not be a bi-\(\Gamma\)-ideal of \(S\).

**Example 3.** Consider the \(\Gamma\)-semigroup \(M = \{a, b, c, d\}\) with \(\Gamma = \{\alpha, \beta\}\) and the multiplication table:

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Let \(B = \{a, c\}\). Then

\[B \Gamma a \Gamma B \cap B = \{a, c\} \cap \{a, c\} = \{a, c\} \neq \emptyset,\]
\[B \Gamma b \Gamma B \cap B = \{a, c\} \cap \{a, c\} = \{a, c\} \neq \emptyset,\]
\[B \Gamma c \Gamma B \cap B = \{a, c\} \cap \{a, c\} = \{a, c\} \neq \emptyset,\]
\[B \Gamma d \Gamma B \cap B = \{a, c\} \cap \{a, c\} = \{a, c\} \neq \emptyset.\]

Therefore, \(B\) is an almost bi-\(\Gamma\)-ideal of \(M\).

**Theorem 1.** Assume that \(B\) is an almost bi-\(\Gamma\)-ideal of a \(\Gamma\)-semigroup \(M\). If \(A\) is any subset of \(M\) containing \(B\), then \(A\) is also an almost bi-\(\Gamma\)-ideal of \(M\).

**Proof.** Since \(B\) is an almost bi-\(\Gamma\)-ideal of \(M\) and \(B \subseteq A\), we have \(B \Gamma m \Gamma B \cap B \neq \emptyset\) and \(B \Gamma m \Gamma B \cap B \subseteq A \Gamma m \Gamma A \cap A\) for all \(m \in M\), respectively. This implies that \(A \Gamma m \Gamma A \cap A \neq \emptyset\) for all \(m \in M\). Therefore, \(A\) is an almost bi-\(\Gamma\)-ideal of \(M\).

**Corollary 1.** The union of any two almost bi-\(\Gamma\)-ideals of a \(\Gamma\)-semigroup \(M\) is also an almost bi-\(\Gamma\)-ideal of \(M\).

**Proof.** Let \(A\) and \(B\) be any two almost bi-\(\Gamma\)-ideals of \(M\). Since \(A \subseteq A \cup B \subseteq M\), it follows from Theorem 1 that \(A \cup B\) is an almost bi-\(\Gamma\)-ideal of \(M\).
**Example 4.** Consider the $\Gamma$-semigroup $\mathbb{Z}_8$ with $\Gamma = \{0, 1, 2\}$ under the usual addition. Let $A = \{2, 3\}$ and $B = \{4, 6\}$. Clearly, $A$ and $B$ are almost bi-$\Gamma$-ideals of $\mathbb{Z}_8$ but $A \cap B = \emptyset$, so it is not an almost bi-$\Gamma$-ideal of $\mathbb{Z}_8$.

By Example 4, we have the following remark.

**Remark 1.** The intersection of any two almost bi-$\Gamma$-ideals of a $\Gamma$-semigroup $M$ need not be an almost bi-$\Gamma$-ideal of $M$.

**Theorem 2.** A $\Gamma$-semigroup $M$ contains a proper almost bi-$\Gamma$-ideal if and only if there exists an element $m$ of $M$ such that $M \setminus \{m\}$ is an almost bi-$\Gamma$-ideal of $M$.

**Proof.** Assume that a $\Gamma$-semigroup $M$ contains a proper almost bi-$\Gamma$-ideal $B$ and let $m \in M \setminus B$. Then $B \subseteq M \setminus \{m\} \subseteq M$. By Theorem 1, $M \setminus \{m\}$ is an almost bi-$\Gamma$-ideal of $M$.

Conversely, let $m \in M$ be such that $M \setminus \{m\}$ is an almost bi-$\Gamma$-ideal of $M$. Since $M \setminus \{m\} \subseteq M$, we get $M \setminus \{m\}$ is a proper almost bi-$\Gamma$-ideal of $M$.

**Theorem 3.** Let $M$ be a $\Gamma$-semigroup such that $|M| > 1$. Then $M$ has no proper almost bi-$\Gamma$-ideals if and only if for all $m \in M$ there exists $a \in M$ such that

$$(M \setminus \{m\})\Gamma_a \Gamma (M \setminus \{m\}) = \{m\}.$$ 

**Proof.** Assume that $M$ has no proper almost bi-$\Gamma$-ideals and let $m \in M$. By Theorem 2, $M \setminus \{m\}$ is not an almost bi-$\Gamma$-ideal of $M$. Thus there exists an element $a$ of $M$ such that $(M \setminus \{m\})\Gamma_a \Gamma (M \setminus \{m\}) \cap (M \setminus \{m\}) = \emptyset$. Hence, $(M \setminus \{m\})\Gamma_a \Gamma (M \setminus \{m\}) = \{m\}$.

Conversely, suppose $M$ contains a proper almost bi-$\Gamma$-ideal $B$. Let $m \in M \setminus B$. By assumption, we have $(M \setminus \{m\})\Gamma_a \Gamma (M \setminus \{m\}) = \{m\}$ for some element $a$ in $M$. Since $B \subseteq M \setminus \{m\} \subseteq M$, we get $M \setminus \{m\}$ is an almost bi-$\Gamma$-ideal of $M$ by Theorem 1. This implies that $\emptyset = \{m\} \cap (M \setminus \{m\}) = (M \setminus \{m\})\Gamma_a \Gamma (M \setminus \{m\}) \cap (M \setminus \{m\}) \neq \emptyset$, which is a contradiction. Therefore, $M$ has no proper almost bi-$\Gamma$-ideals.

### 3. Fuzzy almost bi-$\Gamma$-ideals

For a $\Gamma$-semigroup $M$, let $\mathcal{F}(M)$ be the set of all fuzzy subsets of $M$. For each $\alpha \in \Gamma$, define a binary operation $\circ_\alpha$ on $\mathcal{F}(M)$ by

$$(f \circ_\alpha g)(m) = \begin{cases} \sup_{m = \alpha a b} \{\min\{f(a), g(b)\}\} & \text{if } m \in M\alpha M, \\ 0 & \text{otherwise}. \end{cases}$$

Let $\Gamma^* := \{\circ_\alpha \mid \alpha \in \Gamma\}$. Then $(\mathcal{F}(M), \Gamma^*)$ is a $\Gamma$-semigroup.

**Proposition 1.** For fuzzy subsets $f$ and $g$ of a $\Gamma$-semigroup $M$ such that $f \subseteq g$ and $\alpha \in \Gamma$, if $h$ is any fuzzy subset of $M$, then $h \circ_\alpha f \subseteq h \circ_\alpha g$ and $f \circ_\alpha h \subseteq g \circ_\alpha h$. 
We define fuzzification of almost bi-$\Gamma$-ideals in $\Gamma$-semigroups as follows:

**Definition 4.** A fuzzy subset $f$ of a $\Gamma$-semigroup $M$ is called a fuzzy almost bi-$\Gamma$-ideal of $M$ if for all fuzzy points $m_t$ of $M$, there exist $\alpha, \beta \in \Gamma$ such that $(f \circ \alpha m_t \circ \beta f) \cap f \neq 0$.

**Theorem 4.** Assume that $f$ and $g$ are fuzzy subsets of a $\Gamma$-semigroup $M$ such that $f \subseteq g$. If $f$ is a fuzzy almost bi-$\Gamma$-ideal of $M$, then $g$ is also a fuzzy almost bi-$\Gamma$-ideal of $M$.

**Proof.** Since $f$ is a fuzzy almost bi-$\Gamma$-ideal of $M$, for each fuzzy point $m_t$ of $M$, there exist $\alpha, \beta \in \Gamma$ such that $(f \circ \alpha m_t \circ \beta f) \cap f \neq 0$. We have that $(f \circ \alpha m_t \circ \beta f) \cap f \subseteq (g \circ \alpha m_t \circ \beta g) \cap g$, this implies that $(g \circ \alpha m_t \circ \beta g) \cap g \neq 0$. Hence, $g$ is also a fuzzy almost bi-$\Gamma$-ideal of $M$.

**Corollary 2.** If $f$ and $g$ are fuzzy almost bi-$\Gamma$-ideals of a $\Gamma$-semigroup $M$, then $f \cup g$ is also a fuzzy almost bi-$\Gamma$-ideal of $M$.

**Proof.** It follows by Theorem 4 because of $f \subseteq f \cup g$.

**Example 5.** Consider the $\Gamma$-semigroup $\mathbb{Z}_5$ where $\Gamma = \{\overline{0}\}$ and $\overline{\pi \gamma \delta} := \pi + \gamma + \delta$. Let $f$ and $g$ be fuzzy subsets of $\mathbb{Z}_5$ defined by

$$f(\overline{0}) = 0, f(\overline{1}) = 0.5, f(\overline{2}) = 0, f(\overline{3}) = 0.1, f(\overline{4}) = 0.4$$

and

$$g(\overline{0}) = 0, g(\overline{1}) = 0.3, g(\overline{2}) = 0.7, g(\overline{3}) = 0, g(\overline{4}) = 0.2.$$ 

It is easy to check that $[(f \circ \alpha m_t \circ \beta f) \cap f] (\overline{4}) \neq 0$ and $[(g \circ \alpha m_t \circ \beta g) \cap g] (\overline{4}) \neq 0$ for all $\alpha, \beta \in \Gamma, m \in \mathbb{Z}_5$ and $t \in (0, 1]$. So $f$ and $g$ are fuzzy almost bi-$\Gamma$-ideals of $\mathbb{Z}_5$.

From the definition of the intersection of two fuzzy subsets, we have

$$(f \cap g)(\overline{0}) = 0, (f \cap g)(\overline{1}) = 0.3, (f \cap g)(\overline{2}) = 0, (f \cap g)(\overline{3}) = 0, (f \cap g)(\overline{4}) = 0.2.$$ 

We can easily to check that $[(f \cap g) \circ \alpha m_t \circ \beta (f \cap g)](a) \cap (f \cap g)](a) = 0$ for all $\alpha, \beta \in \Gamma, t \in (0, 1]$ and $a \in \mathbb{Z}_5$, so $f \cap g$ is not a fuzzy almost bi-$\Gamma$-ideal of $\mathbb{Z}_5$.

The following remark follows from Example 5.

**Remark 2.** The intersection of two fuzzy almost bi-$\Gamma$-ideals of a $\Gamma$-semigroup $M$ need not be a fuzzy almost bi-$\Gamma$-ideal of $M$.

### 4. Relationships between almost bi-$\Gamma$-ideals and their fuzzification

**Theorem 5.** A non-empty subset $B$ of a $\Gamma$-semigroup $M$ is an almost bi-$\Gamma$-ideal of $M$ if and only if $\chi_B$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. 
Proof. Assume that $B$ is an almost bi-$\Gamma$-ideal of a $\Gamma$-semigroup $M$ and let $m_t$ be any fuzzy point of $M$. Then $B\Gamma m\Gamma B \cap B \neq \emptyset$. Thus there exists $b \in B$ such that $b \in B\alpha m\beta B$ for some $\alpha, \beta \in \Gamma$. This implies that $(\chi_B \circ \alpha m_t \circ \beta \chi_B)(b) \neq 0$ and $\chi_B(b) \neq 0$. Hence, $(\chi_B \circ \alpha m_t \circ \beta \chi_B) \cap \chi_B \neq 0$. Therefore, $\chi_B$ is a fuzzy almost bi-$\Gamma$-ideal of $M$.

To prove the converse, we assume that $\chi_B$ is a fuzzy almost bi-$\Gamma$-ideal of $M$ and let $m \in M$. Then there exist $\alpha, \beta \in \Gamma$ such that $(\chi_B \circ \alpha m_t \circ \beta \chi_B) \cap \chi_B \neq 0$, so $[(\chi_B \circ \alpha m_t \circ \beta \chi_B) \cap \chi_B](y) \neq 0$ for some $y \in M$. Hence, $y \in B$ and $y = a\alpha m\beta b$ for some $a,b \in B$ and $\alpha, \beta \in \Gamma$. Therefore, $y \in B\Gamma m\Gamma B \cap B$. So $B\Gamma m\Gamma B \cap B \neq \emptyset$. Consequently, $B$ is an almost bi-$\Gamma$-ideal of $M$.

**Theorem 6.** A fuzzy subset $f$ of a $\Gamma$-semigroup $M$ is a fuzzy almost bi-$\Gamma$-ideal of $M$ if and only if $\supp(f)$ is an almost bi-$\Gamma$-ideal of $M$.

Proof. Assume that $f$ is a fuzzy almost bi-$\Gamma$-ideal of a $\Gamma$-semigroup $M$ and let $m \in M$ and $t \in (0,1]$. Then there exist $\alpha, \beta \in \Gamma$ such that $(f \circ \alpha m_t \circ \beta f) \cap f \neq 0$. Hence, $[(f \circ \alpha m_t \circ \beta f) \cap f](x) \neq 0$ for some $x \in M$. So there exist $y_1, y_2 \in S$ such that $x = y_1 \alpha m_t \beta y_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. That is, if $x \in \supp(f)$. Thus $(\chi_{\supp(f)} \circ \alpha m_t \circ \beta \chi_{\supp(f)})(x) \neq 0$ and $\chi_{\supp(f)}(x) \neq 0$. Therefore, $(\chi_{\supp(f)} \circ \alpha m_t \circ \beta \chi_{\supp(f)}) \cap \chi_{\supp(f)} \neq 0$. Hence, $\chi_{\supp(f)}$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. By Theorem 5, $\supp(f)$ is an almost bi-$\Gamma$-ideal of $M$.

On the other hand, we assume that $\supp(f)$ is an almost bi-$\Gamma$-ideal of $M$. It follows from Theorem 5 that $\chi_{\supp(f)}$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. Let $m_t$ be any fuzzy point of $M$. Thus, $(\chi_{\supp(f)} \circ \alpha m_t \circ \beta \chi_{\supp(f)}) \cap \chi_{\supp(f)} \neq 0$ for some $\alpha, \beta \in \Gamma$. Then there exists an element $x \in M$ such that $[(\chi_{\supp(f)} \circ \alpha m_t \circ \beta \chi_{\supp(f)}) \cap \chi_{\supp(f)}](x) \neq 0$. Therefore, $(\chi_{\supp(f)} \circ \alpha m_t \circ \beta \chi_{\supp(f)})(x) \neq 0$ and $\chi_{\supp(f)}(x) \neq 0$. Then there exist $y_1, y_2 \in M$ such that $x = y_1 \alpha m_t \beta y_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. This means that $(f \circ \alpha m_t \circ \beta f) \cap f \neq 0$.

We conclude that $f$ is a fuzzy almost bi-$\Gamma$-ideal of $M$.

Next, we will study the minimality of fuzzy almost bi-$\Gamma$-ideals.

**Definition 5.** A fuzzy almost bi-$\Gamma$-ideal $f$ of a $\Gamma$-semigroup $M$ is called minimal if for all fuzzy almost bi-$\Gamma$-ideal $g$ of $M$ contained in $f$, we must have $\supp(g) = \supp(f)$.

Now, we provide the relationship between minimal almost bi-$\Gamma$-ideals and their fuzzification.

**Theorem 7.** A non-empty subset $A$ of a $\Gamma$-semigroup $M$ is a minimal almost bi-$\Gamma$-ideal of $M$ if and only if $\chi_A$ is a minimal fuzzy almost bi-$\Gamma$-ideal of $M$.

Proof. Let $A$ be a minimal almost bi-$\Gamma$-ideal of a $\Gamma$-semigroup $M$. By Theorem 5, we have that $\chi_A$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. Assume that $g$ is a fuzzy almost bi-$\Gamma$-ideal of $M$ contained in $\chi_A$. Thus, $\supp(g) \subseteq \supp(\chi_A) = A$. Because of $g \subseteq \chi_{\supp(g)}$, we have $(g \circ \alpha m_t \circ \beta g) \cap g \subseteq (\chi_{\supp(g)} \circ \alpha m_t \circ \beta \chi_{\supp(g)}) \cap \chi_{\supp(g)}$ for all fuzzy points $m_t$ of $M$. Thus $\chi_{\supp(g)}$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. By Theorem 5, $\supp(g)$ is an almost bi-$\Gamma$-ideal of $M$. Because of $A$ is a minimal, then $\supp(g) = A = \supp(\chi_A)$. Therefore, $\chi_A$ is minimal.

To prove the converse, assume that $\chi_A$ is a minimal fuzzy almost bi-$\Gamma$-ideal of $M$ and $B$ is an almost bi-$\Gamma$-ideal of $M$ contained in $A$. Then $\chi_B$ is a fuzzy almost bi-$\Gamma$-ideal of $M$ and $\chi_B \subseteq \chi_A$. Thus, $B = \supp(\chi_B) = \supp(\chi_A) = A$. We conclude that $A$ is minimal.
Corollary 3. A \( \Gamma \)-semigroup \( M \) has no proper almost bi-\( \Gamma \)-ideals if and only if for all fuzzy almost bi-\( \Gamma \)-ideal \( f \) of \( M \), \( \text{supp}(f) = M \).

Proof. Assume that \( M \) has no proper almost bi-\( \Gamma \)-ideals and let \( f \) be a fuzzy almost bi-\( \Gamma \)-ideal of \( M \). By Theorem 6, we have \( \text{supp}(f) \) is almost bi-\( \Gamma \)-ideal of \( M \). Thus \( \text{supp}(f) = M \).

To prove the converse, we let \( B \) be any almost bi-\( \Gamma \)-ideal of \( M \). Follow by Theorem 5, we have that \( \chi_B \) is a fuzzy almost bi-\( \Gamma \)-ideal of \( M \). By assumption, we get \( B = \text{supp}(\chi_B) = M \). This implies that \( M \) has no proper almost bi-\( \Gamma \)-ideals.

Definition 6. Let \( M \) be a \( \Gamma \)-semigroup and \( \alpha \in \Gamma \).

(1) An almost bi-\( \Gamma \)-ideal \( B \) of \( M \) is called \( \alpha \)-prime if

\[
x \alpha y \in B \Rightarrow x \in B \text{ or } y \in B
\]

for any \( x, y \in M \).

(2) A fuzzy almost bi-\( \Gamma \)-ideal \( f \) of \( M \) is called \( \alpha \)-prime if

\[
f(x \alpha y) \leq \max\{f(x), f(y)\}
\]

for any \( x, y \in M \).

Next, we investigate relationship between \( \alpha \)-prime almost bi-\( \Gamma \)-ideals and their fuzzification.

Theorem 8. A nonempty subset \( A \) of a \( \Gamma \)-semigroup \( M \) is an \( \alpha \)-prime almost bi-\( \Gamma \)-ideal of \( M \) if and only if \( \chi_A \) is an \( \alpha \)-prime fuzzy almost bi-\( \Gamma \)-ideal of \( M \).

Proof. Let \( A \) be any \( \alpha \)-prime almost bi-\( \Gamma \)-ideal of \( M \). Then \( \chi_A \) is a fuzzy almost bi-\( \Gamma \)-ideal of \( M \) by Theorem 5. Let \( x \) and \( y \) be elements in \( M \). If \( x \alpha y \in A \), then \( x \in A \) or \( y \in A \). This implies that

\[
\chi_A(x \alpha y) = 1 \leq \max\{\chi_A(x), \chi_A(y)\}.
\]

If \( x \alpha y \notin A \), then

\[
\chi_A(x \alpha y) = 0 \leq \max\{\chi_A(x), \chi_A(y)\}.
\]

We conclude that \( \chi_A(x \alpha y) \leq \max\{\chi_A(x), \chi_A(y)\} \) for all \( x, y \in M \). Therefore, \( \chi_A \) is an \( \alpha \)-prime fuzzy almost bi-\( \Gamma \)-ideal of \( M \).

To prove the converse, suppose that \( \chi_A \) is an \( \alpha \)-prime fuzzy almost bi-\( \Gamma \)-ideal of \( M \). By Theorem 5, we have that \( A \) is an almost bi-\( \Gamma \)-ideal of \( M \). Let \( x \) and \( y \) be elements in \( M \) such that \( x \alpha y \in A \). Thus, \( \chi_A(x \alpha y) = 1 \). By assumption, we have that \( \chi_A(x \alpha y) \leq \max\{\chi_A(x), \chi_A(y)\} \). Therefore, \( \max\{\chi_A(x), \chi_A(y)\} = 1 \). We can conclude that \( x \in A \) or \( y \in A \). Hence, \( A \) is an \( \alpha \)-prime almost bi-\( \Gamma \)-ideal of \( M \).
**Definition 7.** Let $M$ be a $\Gamma$-semigroup and $\alpha \in \Gamma$.

(1) An almost bi-$\Gamma$-ideal $A$ of $M$ is called $\alpha$-semiprime if

$$m\alpha m \in A \Rightarrow m \in A$$

for all $m \in M$.

(2) A fuzzy almost bi-$\Gamma$-ideal $f$ of $M$ is called $\alpha$-semiprime if

$$f(m\alpha m) \leq f(m)$$

for all $m \in M$.

Finally, we give relationship between $\alpha$-semiprime almost bi-$\Gamma$-ideals and their fuzzification.

**Theorem 9.** A nonempty subset $A$ of a $\Gamma$-semigroup $M$ is an $\alpha$-semiprime almost bi-$\Gamma$-ideal of $M$ if and only if $\chi_A$ is an $\alpha$-semiprime fuzzy almost bi-$\Gamma$-ideal of $M$.

**Proof.** Let $A$ be an $\alpha$-semiprime almost bi-$\Gamma$-ideal of $M$. By Theorem 5, $\chi_A$ is a fuzzy almost bi-$\Gamma$-ideal of $M$. Let $m \in M$. If $m\alpha m \in A$, then $m \in A$. So, $\chi_A(m) = 1$. Hence, $\chi_A(m\alpha m) \leq \chi_A(m)$. If $m\alpha m \notin A$, then $\chi_A(m\alpha m) = 0 \leq \chi_A(m)$. By both cases, we conclude that $\chi_A(m\alpha m) \leq \chi_A(m)$ for all $m \in M$. Thus, $\chi_A$ is an $\alpha$-semiprime fuzzy almost bi-$\Gamma$-ideal of $M$.

Conversely, assume that $\chi_A$ is an $\alpha$-semiprime fuzzy almost bi-$\Gamma$-ideal of $M$. By Theorem 5, we have that $A$ is an almost bi-$\Gamma$-ideal of $M$. Let $m \in M$ be such that $m\alpha m \in A$. Thus $\chi_A(m\alpha m) = 1$. By assumption, we have that $\chi_A(m\alpha m) \leq \chi_A(m)$. Since $\chi_A(m\alpha m) = 1$, it follows that $\chi_A(m) = 1$. Therefore, $m \in A$. Consequently, $A$ is an $\alpha$-semiprime almost bi-$\Gamma$-ideal of $M$.

5. Conclusion

In this paper, we define almost bi-$\Gamma$-ideals and their fuzzification of $\Gamma$-semigroups. Every bi-$\Gamma$-ideal is an almost bi-$\Gamma$-ideal but the converse is not true in general. We show that the union of two almost bi-$\Gamma$-ideals is also an almost bi-$\Gamma$-ideal. However, it is not generally true in case the intersection. Similarly, we have that the union of two fuzzy almost bi-$\Gamma$-ideals is also a fuzzy almost bi-$\Gamma$-ideal but it is not generally true in case the intersection. Moreover, the relationships between almost bi-$\Gamma$-ideals and their fuzzification were shown in Section 4.

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