On solvability of $p$-harmonic type equations in grand Sobolev spaces

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Abstract. In this paper with the help of variational method existence and uniqueness of solution of $p$-harmonic type equations in grand Sobolev spaces is studied.

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1. Introduction and preliminary notes

It is well known that the existence and uniqueness of Dirichlet problem for $p$-harmonic equations

$$\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = \text{div} f,$$

$$u|_{\partial G} = 0$$  \hspace{1cm} (1,2)

in Sobolev and grand Sobolev spaces were studied, e.g., in \cite{1, 2} see also \cite{4–7, 10–13}. Namely, in these papers the different problems for $p$-harmonic equations were considered. Similar and various problems of partial differential equations in grand Sobolev, Besov and Morrey type spaces were studied in \cite{8, 9, 14–16, 18–23} and others. Most of these papers were used the variational methods. Evidently, in the above-mentioned papers only $p$-harmonic equations (1) was considered.

In this paper we consider Dirichlet problem for $p$-harmonic type equation has a form

$$\text{div} \left( |\nabla u|^{p-q} \nabla u \right) = \text{div} f,$$

$$u|_{\partial G} = \varphi|_{\partial G}$$  \hspace{1cm} (3,4)

where $1 < p < \infty$; $2 \leq q < \infty$; $\varphi \in W^{1}_{p}(G)$, $f \in L_{(p-\epsilon)'}(G)$, $(p-\epsilon)' = \frac{p-\epsilon}{p-\epsilon-1}$ and $G$ in $\mathbb{R}^n$ is a bounded domain.

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Definition 1. ([6, 17, 23]) Denote by $W^{1,p}_1(G)$ the grand Sobolev space of locally summable functions $u$ on $G$ having the weak partial derivatives $D_{x_i}u$ ($i = 1, 2, \ldots, n$) with the finite norm

$$
\|u\|_{W^{1,p}_1(G)} = \|u\|_{L^p(G)} + \|\nabla u\|_{L^p(G)},
$$

where

$$
\|u\|_{L^p(G)} = \sup_{0<\varepsilon<p-1} \left( \frac{\varepsilon}{|G|} \int_G |u(x)|^{p-\varepsilon} \, dx \right)^{\frac{1}{p-\varepsilon}}
$$

and $|G|$ is the Lebesgue measure of $G$.

We note that the correct choice of space for problem (3)-(4) is the grand Lebesgue space (or grand Sobolev space).

In this paper using the variational method an existence and uniqueness of solution to Dirichlet problem for $p-$ harmonic type equations (3)- (4) in grand Sobolev spaces is studied.

A weak solution for the problem (3)-(4) on $G$ is a function $u(x) \in W^{1,p}_1(G)$, if $u - \varphi \in \overset{\circ}{W}^{1,p}_1(G)$ such that

$$
\sum_{i=1}^n \int_G |\nabla u|^{p-q} u_{x_i} \vartheta_{x_i} \, dx = \sum_{i=1}^n \int_G f \vartheta_{x_i} \, dx,
$$

for every $\vartheta \in \overset{\circ}{W}^{1,p}_1(G)$.

2. Main results

In this section we prove the existence and uniqueness of weak solution (5) for the problem (3)-(4).

Theorem 1. Let $G \subset R^n$ is bounded domain, $1 < p < \infty; 2 \leq q < \infty; g, h \in W^{1,1}_p(G)$, $\varphi \in \overset{\circ}{W}^{1,p}_1(G)$ and $f \in L^{1,(p-\varepsilon)}$. Then the Dirichlet problem for pharmonic type equation (3) has a unique weak solutions in $W^{1,p}_1(G)$.

Proof. Since functions $g$ and $h \in W^{1,1}_p(G)$, then we consider the bilinear functional as the form

$$
F(g, h) = \sum_{i=1}^n \int_G |\nabla \varphi|^{p-q} g \vartheta_{x_i} \, dx - \sum_{i=1}^n \int_G f \vartheta_{x_i} \, dx =
$$

$$
= I(g, h) - \sum_{i=1}^n \int_G f \vartheta_{x_i} \, dx = I(g, h) - (f, h),
$$

(6)
since \( f \in L_{(p - \varepsilon)'}(G) \), \((p - \varepsilon)' = \frac{p - \varepsilon}{p - \varepsilon - 1} \). Consequently, we have

\[
|I(g, g)| = |I(g)| = \left| \sum_{i=1}^{n} \int_{G} |\nabla g|^{p-\varepsilon} g_{x_{i}} g_{x_{i}} \, dx \right| \leq \sum_{i=1}^{n} \int_{G} |\nabla g|^{p-\varepsilon} |g_{x_{i}}| |g_{x_{i}}| \, dx = \sum_{i=1}^{n} \int_{G} |\nabla g|^{p-\varepsilon} |g_{x_{i}}|^2 \, dx = \int_{G} |\nabla g|^{p-(q-2)} \, dx < \infty,
\]

\[
|I(g)| \leq \|\nabla g\|^{p-(q-2)}_{L_{p}^{-1}(G)},
\]

Consequently, for every \( q - 2 < \varepsilon < p - 1 \) function \( g \in W_{p}^{1}(G) \) and

\[
\|g\|_{W_{p}^{1}(G)} \leq C_{1} \|g\|^{p-(q-2)}_{W_{p}^{1}(G)},
\]

and, note that

\[
\|\nabla g\|^{p-\varepsilon}_{L_{p}^{-1}(G)} \leq C_{2} |I(g)|, \tag{7}
\]

where \( C_{1} \) and \( C_{2} \) are constants independent on function \( g \).

The variational problem is stated as follows. Find a function \( g \in W_{p}^{1}(G) \) such that which gives the minimum value to the integral \( F(g) \) and is unique. The Euler-Lagrange equation for the variational problem (6) under consideration is the equation (3). With the help of the inequality (7), we have

\[
|F(g, g)| = |F(g)| = \left| I(g) - \sum_{i=1}^{n} \int_{G} f g_{x_{i}} \, dx \right| \geq |I(g)| - \sum_{i=1}^{n} \int_{G} \left| f g_{x_{i}} \right| \, dx \geq |I(g)| - \sum_{i=1}^{n} \int_{G} \left| f g_{x_{i}} \right| \, dx \geq |I(g)| - \sum_{i=1}^{n} \int_{G} |f| \, |g_{x_{i}}| \, dx \geq C_{3} \|g\|^{p-\varepsilon}_{W_{p}^{1}(G)} - \|g\|^{p-\varepsilon}_{L^{1}_{p}(G)} - M_{0},
\]

where \( C_{3} \) and \( C_{4} \) are constants independent on the function \( g(x) \).

This means that \( F(g) \) is lower bounded on \( W_{p}^{1}(G) \) show that there exists \( g_{0} \in W_{p}^{1}(G) \) such that \( F(g_{0}) = \min_{g \in W_{p}^{1}(G)} F(g) \). Fix some sequence \( \{g_{m}\} \in W_{p}^{1}(G) \) \((m = 1, 2, \ldots)\) such that \( \lim_{m \to \infty} F(g_{m}) = r_{0} \). Let \( \sigma > 0 \) choose \( m_{\sigma} \) so for \( m \geq m_{\sigma} \) and \( s = 1, 2, \ldots \) it holds \( F(g_{m+s}) < r_{0} + \sigma \). Then noting that \( \frac{1}{2}(g_{m+s} + g_{m}) \in W_{p}^{1}(G) \) we have
\[ F\left( \frac{g_{m+1}-g_{m}}{2} \right) \geq r_0. \]  By direct calculations we show that \( I\left( \frac{g_{m+1}-g_{m}}{2} \right) < 4\sigma, \) and we have \( \|g_{m+1}+g_{m}\|_{W^1_p(G)} \leq 2\left( \frac{r}{p} \right)^{\frac{1}{p-1}}. \) This means that the sequence \( \{g_m\} \) is fundamental in the spaces \( W^1_p(G) \), consequently in view of completeness the spaces \( W^1_p(G) \) there exist a function \( g_0 \in W^1_p(G) \) such that \( \lim_{m \to \infty} \|g_m - g_0\|_{W^1_p(G)} = 0. \) By theorem on trace in \( W^1_p(G), (3), \) we get
\[ W^1_p(G) \to W^1_{p-t}(G) \to L_{t-\epsilon}(G_k), \quad G_k = G \cap \mathbb{R}^k, \quad p < t \leq \infty, \quad 1 \leq k \leq n. \]
So
\[ |F(g_m) - F(g_0)| \leq C\|g_m - g_0\|_{W^1_p(G)} \]
and hence it follows that \( r_0 = \lim_{m \to \infty} F(g_m) = F(g_0). \) Show that the function delivering minimum to the functional \( F(g) \) is unique and satisfies equation (3) in the space \( W^1_p(G). \)

Since \( g \in W^1_p(G) \) and \( F(g_0) = r_0, \) we have
\[ 0 \leq I\left( \frac{g - g_0}{2} \right) = \frac{1}{2} F(g) + \frac{1}{2} F(g_0) - F\left( \frac{g + g_0}{2} \right) \leq \frac{r_0}{2} + \frac{r_0}{2} - r_0 = 0, \]
\[ I(g - g_0) = 0. \]
By \( \|g_m - g_0\|_{W^1_p(G)} \to 0, m \to \infty, \) it follows that the function \( g \) coincides with \( g_0 \) as an element of the space \( W^1_p(G). \) Again from the theorem on trace in space \( W^1_p(G), \) we have
\[ \|(g_m - g_0)\|_{\partial G \cap L_{t-\epsilon}(\partial G)} \leq C\|g_m - g_0\|_{W^1_p(G)} \to 0, \quad m \to \infty. \]
Since
\[ \|g_m\|_{\partial G} - \|\varphi\|_{\partial G} \leq 0, \quad m \to \infty, \]
therefore
\[ \|g_0\|_{\partial G} - \|\varphi\|_{\partial G} \to 0 \quad m \to \infty. \]

Taking into account the condition \( \frac{d}{d\mu}(F(g_0 + \mu \varphi))_{\mu=0} = 0, \) show that the function \( g_0 \in W^1_p(G), \) minimizing the integral \( F(g) \) satisfies the following equation
\[ I(g_0, \omega) - (f, \omega) = 0. \quad (8) \]

Now prove that the function \( g_0 \in W^1_p(G) \) minimizing the integral \( F(g) \) is the weak solution of the problem (3)-(4). By \( \theta(t) \) we denote some monotonically decreasing function on the segment \( \frac{1}{2} \leq t \leq 1 \) and having the following properties
\[ \theta\left( \frac{1}{2} + 0 \right) = 1, \quad \theta\left( 1 - 0 \right) = -1, \quad \theta(s)\left( \frac{1}{2} + 0 \right) = \theta(s)\left( 1 - 0 \right) = 0, \quad s = 1, 2, \ldots. \]
The function
\[
\gamma(t) = \begin{cases} 
\theta'(t), & \frac{1}{2} \leq t \leq 1, \\
0, & -\infty < t < \frac{1}{2}, \quad 1 < t < \infty
\end{cases}
\]
is infinitely differentiable and finite on the real line. Note that the function \(\gamma\) satisfy condition
\[
\gamma^{(s)}\left(\frac{1}{2} + 0\right) = \gamma(1 - 0), \quad (s = 1, 2, \ldots).
\]

Let \(\delta > 0\) and let \(G_\delta = \{y : \rho(y, R^n \setminus G) > \delta\}\) be arbitrary point of the domain \(G\), and \(r = \rho(x, x_0)\). There \(\rho(x, x_0)\) is the Euclidean distance between \(x\) and \(x_0\), where \(x \in G\) and \(x_0\) be a fixed point in \(G\). Following Sobolev [24], we introduce the function
\[
\omega(x) = \gamma\left(\frac{r}{l_1}\right) - \gamma\left(\frac{r}{l_2}\right),
\]
for \(0 < l_1 < l_2 < \delta\). It is obvious that \(\omega(x)\) is a infinitely differentiable finite function with a support lying on a annular domain \(\frac{1}{2} < r < l_2\). Therefore \(\omega \in C_0^\infty(G)\) and \(D^{(s)}\omega|_{\partial G} = 0\) for all \(s = 1, 2, \ldots\). Then from (8) by definition of the weak derivative it follows that
\[
\int_G K\left(\frac{r}{l_1}\right) g(x) \, dx = \int_G K\left(\frac{r}{l_2}\right) g(x) \, dx,
\]
where
\[
K\left(\frac{r}{l_i}\right) = \text{div} \left(\left|\nabla\gamma\left(\frac{r}{l_i}\right)\right|^{p-q} \nabla\gamma\left(\frac{r}{l_i}\right)\right) - \text{div} f, \quad i = 1, 2.
\]
Note that the function \(K\left(\frac{r}{l_i}\right)\) having all properties of kernel. Namely, the following properties hold:
1) \(K\) is infinitely differentiable function with support in the ball \(r \leq l_i\);
2) The function \(K\) and all its derivatives on sphere \(R = h\) are zero;
3) \[
\frac{1}{\tau_n l_i^n} \int_G K\left(\frac{r}{l_i}\right) \, dx = 1,
\]
where
\[
\tau_n = \frac{2 \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 \xi^{n-1} K(\xi) \, d\xi.
\]

Then for the function \(g_0(x)\) we can constructed Sobolev’s averaging \(g_{0,l_i}(x), i = 1, 2\) on the ball \(l_i\) \((i = 1, 2)\) with centered at the point \(x\) as
\[
g_{0,l_i}(x) = \frac{1}{\tau_n l_i^n} \int_{R^n} K\left(\frac{|z-x|}{l_i}\right) g_0(z) \, dz, \quad i = 1, 2.
\]
The we can rewrite equality (9) in the form \(g_{0,l_i}(x) = g_{0,l_2}(x)\). Consequently, for \(l < \delta\)
\[
g_{0,l}(x) = g_0(x).
\]
Since the average functions \( g_{0,i}(x) \), \( i = 1, 2 \) are continuous and has continuous derivatives for any order, then \( g_0(x) \) also is a kernel. Integrating by parts in the equality \( I(g_0,\omega) - (f,\omega) = 0 \), whence is the limit case

\[
\sum_{i=1}^{n} \int_{G} \omega(x) \frac{\partial}{\partial x_i} \left( |\nabla g_0|^{p-q} \frac{\partial}{\partial x_i} g_0(x) \right) \, dx = \sum_{i=1}^{n} \int_{G} \omega(x) \frac{\partial}{\partial x_i} f(x) \, dx.
\]

Hence by the arbitrariness of the functions \( \omega(x) \) it follows that

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( |\nabla g_0|^{p-q} \frac{\partial}{\partial x_i} g_0(x) \right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x)
\]

i.e

\[
\text{div} \left( |\nabla g_0|^{p-q} \nabla g_0 \right) = \text{div} f.
\]

Thus, solution of the variational problem (5) from the class \( W^{1,p}_p(G) \) is also solution of Dirichlet problem (3)-(4) and this solution is unique.

### 3. Conclusion

In conclusion, we note that for a \( p \)-harmonic type equation in the grand Sobolev space, a result is obtained on the existence and uniqueness of a weak solution.

### References


