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More on Perfect Roman Domination in Graphs

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Abstract. A *perfect Roman dominating function* on a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \{0, 1, 2\}$ for which each $u \in V(G)$ with $f(u) = 0$ is adjacent to exactly one vertex $v \in V(G)$ with $f(v) = 2$. The *weight* of a perfect Roman dominating function f is the value $\omega_G(f) = \sum_{v \in V(G)} f(v)$. The *perfect Roman domination number* of G is the minimum weight of a perfect Roman dominating function on G. In this paper, we study the perfect Roman domination numbers of graphs under some binary operations.

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1. Introduction

Throughout this paper, all graphs considered are finite, simple and undirected. Let $G = (V(G), E(G)$ be a graph. The sets $V(G)$ and $E(G)$ are the *vertex set* and *edge set*, respectively, of G. For $S \subseteq V(G)$, $|S|$ is the cardinality of S. In particular, $|V(G)|$ is called the *order* of G. For notation and terminology not given here, see [5].

Vertices u and v of G are *neighbors* if $uv \in E(G)$. The *open neighborhood* of v refers to the set $N_G(v)$ consisting of all neighbors of v. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v, denoted $deg_G(v)$, refers to the value $|N_G(v)|$, and we define $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. Vertex v is an *endvertex* if $deg_G(v) = 1$, and $End(G)$ is the set of all endvertices of G. Vertex v is an *isolated vertex* if $deg_G(v) = 0$. We denote by $Iso(G)$ the set of all isolated vertices of G. For $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$, and $N_G[S] = S \cup N_G(S)$.

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Let G and H be graphs with disjoint vertex sets. The *disjoint union* of G and H is the graph $G \cup H$ with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The *join* of G and H is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. The *edge corona* of G and H is the graph $G \diamond H$ obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H. The *composition* $G[H]$ of G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. The *complementary prism*, denoted $G\overline{G}$, is the graph formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For the complementary prism, $V(G\overline{G}) = V(G) \cup V(\overline{G})$ and $E(G\overline{G}) = E(G) \cup E(\overline{G}) \cup \{v\overline{v} : v \in V(G)\}$, where $\overline{v}$ is the vertex in $\overline{G}$ corresponding to $v \in V(G)$ in the perfect matching.

A subset $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The minimum cardinality of a dominating set is the *domination number* of G, denoted by $\gamma(G)$. For more details and results on domination number, we refer to [4, 9–11, 13]. In particular, if $\gamma(G) = 1$ and $N_G[v] = V(G)$, then v is said to be a *dominating vertex* of G. In this case, $Dom(G)$ denotes the set of all dominating vertices of G. Any dominating set of G of cardinality $\gamma(G)$ is called γ*-set* of G.

A dominating set S of G is a *perfect dominating set* if for every $v \in V(G) \setminus S$, there exists exactly one $u \in S$ for which $uv \in E(G)$ [16]. The minimum cardinality of a perfect dominating set is the *perfect domination number* of G, which is denoted by $\gamma^P(G)$. Since perfect dominating sets are dominating sets, $\gamma(G) \leq \gamma^P(G)$ for any graph G.

A *Roman dominating function* on G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that for each $u \in V(G)$ for which $f(u) = 0$, there exists $v \in V(G)$ such that $f(v) = 2$ and $uv \in E(G)$. The *weight* of f is the value $\omega_G(f) = \sum_{v \in V(G)} f(v)$. The *Roman domination number* of G, denoted by $\gamma_R(G)$, is the minimum weight of a function f on G. We refer to [2, 3, 7, 8, 12, 17, 18] for the history, introduction, importance and for some of the recent developments of the study of Roman domination in graphs.

Customarily, we write $f = (V_0, V_1, V_2)$ for a Roman dominating function f on G, where $V_k = \{v \in V(G) : f(v) = k\}$. With this convention, $\omega_G(f) = |V_1| + 2|V_2|$ and $V_1 \cup V_2$ is a dominating set of G. In [8], it is known that for any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

A *perfect Roman dominating function* (or *PRD-function*) on G is a Roman domination function $f = (V_0, V_1, V_2)$ on G such that for each $u \in V_0$ there exists exactly one $v \in V_2$ for which $uv \in E(G)$. In other words, a *PRD*-function on G is a colouring of the vertices of G using colours 0, 1 and 2 such that each vertex coloured 0 is adjacent to exactly one vertex coloured 2. The *perfect Roman domination number* of G, denoted by $\gamma_R^P(G)$, is the minimum weight of a *PRD*-function on G. A *PRD*-function f with $\omega_G(f) = \gamma_R^P(G)$ is called γ_R^P*-function* of G.

The perfect Roman domination, a variation of the Roman domination, was introduced and first investigated in 2018 by Henning et al. [15], particularly in trees. It is further studied in [14] for regular graphs. More recent studies on the concept include [1, 19, 20].

In this present paper, we continue the study of perfect Roman domination, specifically on the join, corona, complementary prism, edge corona and composition of graphs.

The following bounds are established in the referred articles above.

Theorem 1.1. (i)[15] *If T is a tree of order $n \geq 3$, then $\gamma_R^P(T) \leq \frac{4}{5}n$;*

(ii) [14] *If G is a k-regular graph of order n with $k \geq 4$, then $\gamma_R^P(G) \leq \left(\frac{k^2+k+3}{k^2+3k+1}\right)n$;*

(iii) [19] *If G is a graph of order n, then $\gamma_R^P(G) \leq n+1-\Delta(G)$.*

(iv) [19] *For paths P_n and cycles C_n on $n \geq 3$ vertices, $\gamma_R^P(P_n) = \gamma_R^P(C_n) = \lceil \frac{2n}{3} \rceil$.*

For convenience, we adapt the symbol $PRD(G)$ to denote the set of all perfect Roman dominating functions on the graph G.

2. Results

The following proposition plays an important role in proving the desired results.

Proposition 2.1. *If $f = (V_0, V_1, V_2)$ is a γ_R^P-function of G, then $|N_G(v) \cap V_2| \neq 1$ for each $v \in V_1$.*

Proof: Suppose that there exists $v \in V_1$ for which $|N_G(v) \cap V_2| = 1$. Consider, in particular, the function $f^* = (V_0^*, V_1^*, V_2^*)$ given by $f^*(v) = 0$ and $f^*(x) = f(x)$ for all $x \neq v$. We have $f^* \in PRD(G)$ with $V_0^* = V_0 \cup \{v\}$, $V_1^* = V_1 \setminus \{v\}$ and $V_2^* = V_2$. Thus, $\omega_G(f^*) = \gamma_R^P(G) - 1$, a contradiction. ■

Proposition 2.2. *For a nontrivial connected graph G of order n,*

$$\max\{2, \gamma(G)\} \leq \gamma_R^P(G) \leq \min\{n+1-\Delta(G), 2\gamma^P(G)\}.$$

Proof: Since a perfect Roman domination is a Roman domination, $\gamma(G) \leq \gamma_R^P(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of G. If $V_0 = \varnothing$, then $\gamma_R^P(G) = n \geq 2$. On the other hand, if $V_0 \neq \varnothing$, then $V_2 \neq \varnothing$ so that $\gamma_R^P(G) \geq 2|V_2| \geq 2$.

By Theorem 1.1(iii), $\gamma_R^P(G) \leq n+1-\Delta(G)$. Now, let $S \subseteq V(G)$ be a γ^P-set of G. Then $f = (V_0, V_1, V_2) \in PRD(G)$, where $V_0 = V(G) \setminus S$, $V_1 = \varnothing$ and $V_2 = S$. Therefore, $\gamma_R^P(G) \leq 2|S| = 2\gamma^P(G)$. ■

Observe that $\gamma_R^P(C_k) = 4 = k+1-\Delta(C_k) < 2\gamma^P(C_k)$ for $k = 5$ and $\gamma_R^P(C_{3n}) = 2n = 2\gamma^P(C_{3n}) < (3n+1) - \Delta(C_{3n})$ for all $n \geq 2$. Therefore, the upper bound of the inequality in Proposition 2.2 is sharp and may be determined by exactly one of $n+1-\Delta(G)$ and $2\gamma^P(G)$. The inequality, however, can also be strict. To see this, note that $\gamma_R^P(C_7) = 5 < \min\{(7+1) - \Delta(C_7), 2\gamma^P(C_7)\}$.

Corollary 2.3. *Let G be a connected graph of order $n \geq 2$. Then*

(i) [19] $\gamma_R^P(G) = 2$ *if and only if* $\gamma(G) = 1$.

(*ii*) *$\gamma_R^P(G) = n$ if and only if $n = 2$.*

(*iii*) [19] *$\gamma_R^P(G) = 3$ if and only if $\Delta(G) = n - 2$.*

(*iv*) *If G is the complete multipartite graph $K_{r_1,r_2,\ldots,r_m}$, where $2 \leq r_1 \leq r_2 \leq \ldots \leq r_m$, then*

$$\gamma_R^P(G) = \begin{cases} \min\{r_1+1, 4\}, & \text{if } m = 2; \\ r_1 + 1, & \text{if } m \geq 3. \end{cases}$$

Proof: Clearly, if $\gamma(G) = 1$, then $\gamma^P(G) = 1$ and the inequalities in Proposition 2.2 imply that $\gamma_R^P(G) = 2$. Now, suppose that $\gamma_R^P(G) = 2$, and let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of G. If $V_2 = \varnothing$, then $V(G) = V_1$ and $\gamma_R^P(G) = n = 2$. Since G is connected, $G = P_2$ and $\gamma(G) = 1$. If $V_2 \neq \varnothing$, then $V_1 = \varnothing$ and $V_2 = \{v\}$ with $N_G[v] = V(G)$. This means that $\gamma(G) = 1$. This proves (*i*).

If $n = 2$, then $G = P_2$ and $\gamma_R^P(G) = 2 = n$. Conversely, suppose that $n \geq 3$. Pick $v \in V(G)$ such that $deg_G(v) = \Delta(G) \geq 2$. Define on G

$$f(x) = \begin{cases} 2, & \text{if } x = v; \\ 0, & \text{if } x \in N_G(v); \\ 1, & \text{else.} \end{cases}$$

Then $f \in PRD(G)$ and $\omega(f) = n - (\Delta(G) - 1) < n$, a contradiction. Thus, if $\gamma_R^P(G) = n$, then $n = 2$. We have proved (*ii*).

If $\Delta(G) = n - 2$, then Proposition 2.2 implies that $2 \leq \gamma_R^P(G) \leq 3$. Since $\gamma(G) \geq 2$, $\gamma_R^P(G) = 3$ by (*i*). Conversely, suppose that $\gamma_R^P(G) = 3$. By (*i*), $\gamma(G) \geq 2$ so that $\Delta(G) \leq n - 2$, and by (*ii*), $n \geq 4$. Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on G. If $V_2 = \varnothing$, then $V_1 = V(G)$ and $\gamma_R^P(G) = n \geq 4$, a contradiction. Thus, $|V_2| = |V_1| = 1$, say $V_1 = \{u\}$ and $V_2 = \{v\}$. This means that $V(G) \setminus \{u, v\} \subseteq V_0$. Further, by Proposition 2.1, $uv \notin E(G)$. Accordingly, $deg_G(v) = n - 2$. Therefore, $\Delta(G) \geq n - 2$. This proves (*iii*).

Suppose that G is the complete multipartite graph described in (*iv*). Then $\Delta(G) = n - r_1$. Suppose first that $m = 2$. Then $\gamma(G) = \gamma^P(G) = 2$. By Proposition 2.2, $\gamma_R^P(G) \leq \min\{r_1 + 1, 4\}$. Also, by (*i*), $\gamma_R^P(G) \geq 3$. If $r_1 = 2$, then $\gamma_R^P(G) = 3 = r_1 + 1$. On the other hand, if $r_1 \geq 3$, then $\gamma_R^P(G) = 4 \geq r_1 + 1$. Now, assume that $m \geq 3$. By (*ii*), $\gamma_R^P(G) < n$. Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on G. Then $|V_2| = 1$, say $V_2 = \{v\}$. Since f is a γ_R^P-function, $v \in U$, where U is the partite set of G with $|U| = r_1$. More precisely, $f(v) = 2$, $f(x) = 1$ for all $x \in U \setminus \{v\}$ and $f(x) = 0$ for all $x \in V(G) \setminus U$. Thus, $\gamma_R^P(G) = \omega_G(f) = r_1 + 1$. This proves (*iv*). ■

Proposition 2.4. [19] *Let G_1, G_2, …, G_k be the components of G. Then $\gamma_R^P(G) = \sum_{j=1}^{k} \gamma_R^P(G_j)$.*

Proposition 2.4 and Corollary 2.3(*ii*) yield the following corollary.

Corollary 2.5. *Let G be a graph of order n. Then $\gamma_R^P(G) = n$ if and only if $G = \cup_{j=1}^{k} G_j$, where $G_j \in \{K_1, K_2\}$ for all $j = 1, 2, \ldots, k$.*

Corollary 2.6. *Let G be a graph of order n. Then $\gamma(G) = \gamma_R^P(G)$ if and only if $G = \overline{K_n}$.*

Proof: If $G = \overline{K_n}$, then $\gamma(G) = n$ and by Corollary 2.5, $\gamma_R^P(G) = n$. Conversely, suppose that $\gamma(G) = \gamma_R^P(G)$, and let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of G. Note that if $V_2 \neq \varnothing$, then $\gamma(G) \leq |V_1| + |V_2| < \gamma_R^P(G)$, a contradiction. Thus, $V_2 = V_0 = \varnothing$ and $\gamma_R^P(G) = n$. This means that $\gamma(G) = n$ and, thus, $G = \overline{K_n}$. ■

2.1. On the join of graphs

By Corollary 2.3(i), $\gamma_R^P(G + K_n) = 2$ for all graphs G and for all $n \geq 1$.

The following theorem characterizes all PRD-functions on the join of nontrivial connected graphs.

Theorem 2.7. *Let G and H be any nontrivial connected graphs and $f = (V_0, V_1, V_2)$. Then $f \in PRD(G + H)$ if and only if one of the following holds:*

(i) $V_2 \subseteq V(G)$ and one of the following holds:

- *(a) $V_0 \subseteq V(G)$, $V(H) \subseteq V_1$ and $(V_0, V_1 \cap V(G), V_2) \in PRD(G)$;*
- *(b) $V_0 \cap V(H) \neq \varnothing$ and $V_2 = \{v\}$ for which $V_0 \cap V(G) \subseteq N_G(v)$.*

(ii) $V_2 \subseteq V(H)$ and one of the following holds:

- *(a) $V_0 \subseteq V(H)$, $V(G) \subseteq V_1$ and $(V_0, V_1 \cap V(H), V_2) \in PRD(H)$;*
- *(b) $V_0 \cap V(G) \neq \varnothing$ and $V_2 = \{v\}$ for which $V_0 \cap V(H) \subseteq N_H(v)$.*

(iii) $A_1 = V_2 \cap V(G) \neq \varnothing$ and $A_2 = V_2 \cap V(H) \neq \varnothing$ and the following holds:

- *(a) If $V_0 \cap V(G) \neq \varnothing$, then $|A_2| = 1$ and $(V_0 \cap V(G)) \cap N_G(A_1) = \varnothing$;*
- *(b) If $V_0 \cap V(H) \neq \varnothing$, then $|A_1| = 1$ and $(V_0 \cap V(H)) \cap N_H(A_2) = \varnothing$.*

Proof: Assume that f is a perfect Roman dominating function on $G + H$. We consider three cases:

Case 1: Suppose that $V_2 \subseteq V(G)$. If $V_0 \subseteq V(G)$, then $V(H) \subseteq V_1$ and the restriction $f|_{V(G)} = (V_0, V_1 \cap V(G), V_2)$ of f on G is a perfect dominating function on G. Suppose that $V_0 \cap V(H) \neq \varnothing$. Then, $|V_2| = 1$, say $V_2 = \{v\}$. Necessarily, $V_0 \cap V(G) \subseteq N_G(v)$.

Case 2: Similarly, if $V_2 \subseteq V(H)$, then either $(ii)(a)$ or $(ii)(b)$ holds.

Case 3: Assume that V_2 intersects both $V(G)$ and $V(H)$, and $A_1 = V_2 \cap V(G)$ and $A_2 = V_2 \cap V(H)$. Suppose that $V_0 \cap V(G) \neq \varnothing$, and let $v \in V_0 \cap V(G)$. Since $A_2 \subseteq N_{G+H}(v)$, $|A_2| = 1$ and $v \notin N_G(A_1)$. Since v is arbitrary, $(iii)(a)$ holds. Similarly, $(iii)(b)$ holds.

Conversely, suppose that $(i)(a)$ holds for f, and let $w \in V_0$. Then $w \in V(G)$ and there exists a unique $u \in V_2$ for which $uw \in E(G)$. Since $V(H) \subseteq V_1$, u is unique in $V(G+H)$ for which $uw \in E(G+H)$. This means that $f \in PRD(G+H)$. Suppose that $(i)(b)$ holds for f, and let $w \in V_0$. Whether $w \in V(G)$ or $w \in V(H)$, v is a unique element in V_2 for which $wv \in E(G+H)$. Thus, $f \in PRD(G+H)$. Similarly, if (ii) holds, the same conclusion is attained for f. Suppose now that (iii) holds for f. Let $v \in V_0$. If $v \in V(G)$, then by condition (a), $A_2 = \{u\}$ for some $u \in V(H)$ and $N_{G+H}(v) = \{u\}$. Similarly, if $v \in V(H)$, then $A_1 = \{u\}$ for some $u \in V(G)$ and $N_{G+H}(v) = \{u\}$. Accordingly, $f \in PRD(G+H)$. ■

We now use Theorem 2.7 to prove the following result which is also provided in [19].

Corollary 2.8. [19] *Let Gand H be nontrivial connected graphs of orders m and n, respectively. Then*

$$\gamma_R^P(G+H) = min\{4 + \delta(G) + \delta(H), m + 1 - \Delta(G), n + 1 - \Delta(H)\}.$$

Proof: Let $\alpha = min\{4 + \delta(G) + \delta(H), m + 1 - \Delta(G), n + 1 - \Delta(H)\}$. Let $v \in V(G)$ for which $deg_G(v) = \Delta(G)$. Define $f = (V_0, V_1, V_2)$ on $G + H$ by

$$f(x) = \begin{cases} 2, & \text{if } x = v; \\ 0, & \text{if } x \in V(H) \cup N_G(v); \\ 1, & \text{else.} \end{cases}$$

Since f satisfies condition $(i)(b)$ of Proposition 2.7, $f = (V_0, V_1, V_2) \in PRD(G+H)$ with $V_2 = \{v\}$ and $V_1 = V(G) \setminus N_G[v]$. Thus,

$$\begin{aligned} \gamma_R^P(G+H) \leq \omega_{G+H}(f) &= |V(G) \setminus N_G[v]| + 2 \\ &= m + 1 - \Delta(G). \end{aligned}$$

Similarly, $\gamma_R^P(G+H) \leq n + 1 - \Delta(H)$.

Now, pick $u \in V(G)$ and $v \in V(H)$ such that $deg_G(u) = \delta(G)$ and $deg_H(v) = \delta(H)$, and define $f = (V_0, V_1, V_2)$ on $G + H$ by

$$f(x) = \begin{cases} 2, & \text{if } x = u, v; \\ 1, & \text{if } x \in N_G(u) \cup N_H(v); \\ 0, & \text{else.} \end{cases}$$

Since f satisfies Proposition 2.7 (iii), $f \in PRD(G+H)$. Since $V_2 = \{u, v\}$ and $V_1 = N_G(u) \cup N_H(v)$,

$$\begin{aligned} \gamma_R^P(G+H) \leq \omega_{G+H}(f) &= |N_G(u) \cup N_H(v)| + 4 \\ &= 4 + \delta(G) + \delta(H). \end{aligned}$$

All of the above show that $\gamma_R^P(G+H) \leq \alpha$.

Now, let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of $G + H$. By Corollary 2.3(ii), since $m + n \geq 4$, $V_2 \neq \varnothing$. Assume $A_1 = V_2 \cap V(G) \neq \varnothing$. We consider two cases:

Case 1: Suppose that $A_2 = V_2 \cap V(H) = \varnothing$. If Proposition 2.7$(i)(a)$ holds for f, then

$$\omega_{G+H}(f) \geq n + \gamma_R^P(G) > n \geq n + 1 - \Delta(H) \geq \alpha.$$

On the other hand, if Proposition 2.7$(i)(b)$ holds for f, then

$$\omega_{G+H}(f) \geq 2 + |V(G) \setminus N_G[v]| \geq m + 1 - \Delta(G) \geq \alpha.$$

Case 2: Suppose that $A_2 = V_2 \cap V(H) \neq \varnothing$. If $|A_1| \geq 2$ and $|A_2| \geq 2$, then $V_0 = \varnothing$ and $\gamma_R^P(G + H) > m + n$, which is impossible. Assume that $|A_2| = 1$. We consider two subcases. First, suppose that $|A_1| \geq 2$. Then $V_0 \cap V(H) = \varnothing$, and since f is a γ_R^P-function of $G + H$, $V(G) \setminus N_G[A_1] \subseteq V_0$ (by Proposition 2.1) and $N_G(A_1) \setminus A_1 \subseteq V_1$. This means that $|V_1| \geq |V(H) \setminus V_2| + |N_G(A_1) \setminus A_1|$ so that

$$\omega_{G+H}(f) = (n-1) + |N_G(A_1) \setminus A_1| + 2|V_2| \geq n + 5 > n + 1 - \Delta(H).$$

Finally, suppose that $|A_1| = 1$. Let $A_1 = \{u\}$ and $A_2 = \{v\}$ for some $u \in V(G)$ and $v \in V(H)$. By Proposition 2.1, $f(x) = 0$ for all $x \in V(G + H) \setminus (N_G[u] \cup N_H[v])$. Thus,

$$\omega_{G+H}(f) \geq 2|A_1 \cup A_2| + |N_G(u) \cup N_H(v)| \geq 4 + \delta(G) + \delta(H) \geq \alpha.$$

All cases above imply that $\gamma_R^P(G + H) \geq \alpha$. ■

In particular, if $m \geq n$, then

$$\gamma_R^P(P_m + P_n) = \begin{cases} n-1, & \text{if } n \leq 6; \\ 6, & \text{if } n \geq 7. \end{cases} \quad \text{and} \quad \gamma_R^P(C_m + P_n) = \begin{cases} n-1, & \text{if } n \leq 7; \\ 7, & \text{if } n \geq 8. \end{cases}$$

2.2. On the corona of graphs

Let G and H be connected graphs. Adapting the notation used in [6], for each $v \in V(G)$, H^v denotes that copy of H which is joined with v in $G \circ H$. In case $H = \{x\}$, we write $V(H^v) = \{x^v\}$. Then $V(G + H) = \cup_{v \in V(G)} V(H^v + v)$, where $H^v + v = H^v + \langle v \rangle$.

It is worth noting that $K_1 \circ H = H + K_1$ for any graph H.

Theorem 2.9. *For nontrivial connected graphs G of order n,*

$$\gamma_R^P(G \circ K_1) = \min\{\omega_G(f) + n - |V_2| : f = (V_0, V_1, V_2) \in PRD(G)\}.$$

In particular, $\gamma_R^P(K_n \circ K_1) = n + 1$.

Proof: Write $H = \{x\}$, and put $\alpha = \min\{\omega_G(f) + n - |V_2| : f = (V_0, V_1, V_2) \in PRD(G)\}$. Let $f = (V_0, V_1, V_2) \in PRD(G)$. Define $f^* = (V_0^*, V_1^*, V_2^*)$ on $G \circ K_1$ by

$$f^*(z) = \begin{cases} f(z), & \text{if } z \in V(G); \\ 1, & \text{if } z = x^v \text{ for some } v \in V_0 \cup V_1; \\ 0, & \text{if } z = x^v \text{ for some } v \in V_2. \end{cases}$$

Then $f^* \in PRD(G \circ K_1)$ with $V_0^* = V_0 \cup \{x^v : v \in V_2\}$, $V_1^* = V_1 \cup \{x^v : v \in V_0 \cup V_1\}$ and $V_2^* = V_2$. Moreover,

$$\omega_{G\circ K_1}(f^*) = \omega_G(f) + n - |V_2|.$$

Thus, $\gamma_R^P(G \circ K_1) \leq \alpha$.

Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on $G \circ K_1$, and let A denote the set of all $u \in V_0 \cap V(G)$ for which $uv \notin E(G)$ for all $v \in V_2 \cap V(G)$. Then for each $u \in A$, $V_2 \cap N_{G\circ K_1}(u) = \{x^u\}$. Define $f^* = (V_0^*, V_1^*, V_2^*)$ on $G \circ K_1$ by

$$f^*(z) = \begin{cases} f(z), & \text{if } z \in V(G) \setminus A; \\ 1, & \text{if } z \in A \cup \{x^u : u \in (V_0 \cup V_1) \cap V(G)\}; \\ 0, & \text{if } z \in \{x^v : \ v \in V_2 \cap V(G)\}. \end{cases}$$

Then $f^* \in PRD(G \circ K_1)$ with $V_0^* = ((V_0 \cap V(G)) \setminus A) \cup \{x^u : u \in V_2 \cap V(G)\}$, $V_1^* = A \cup (V_1 \cap V(G)) \cup \{x^u : u \in (V_0 \cup V_1) \cap V(G)\}$ and $V_2^* = V_2 \cap V(G)$. Observe that $f(u) + f(x^u) = 2 = f^*(u) + f^*(x^u)$ for each $u \in A$, and $f(u) + f(x^u) \geq f^*(u) + f^*(x^u)$ for each $u \in V(G) \setminus A$. Thus,

$$\begin{aligned} \omega_{G\circ K_1}(f) &= \sum_{u\in A}(f(u) + f(x^u)) + \sum_{v\in V(G)\setminus A}(f(u) + f(x^u)) \\ &\geq \sum_{u\in A}(f^*(u) + f^*(x^u)) + \sum_{u\in V(G)\setminus A}(f^*(u) + f^*(x^u)) \\ &= \omega_{G\circ K_1}(f^*). \end{aligned}$$

Since f is a γ_R^P-function, $\omega_{G\circ K_1}(f) = \omega_{G\circ K_1}(f^*)$. Moreover, for each $u \in V_0^* \cap V(G)$, $u \in (V_0 \cap V(G)) \setminus A$ so that there exists a unique $v \in V_2 \cap V(G) = V_2^*$ such that $uv \in E(G)$. This means that the restriction $f^*|_G$ of f^* to G is a perfect Roman dominating function on G. Thus,

$$\begin{aligned} \gamma_R^P(G \circ K_1) = \omega_{G\circ K_1}(f^*) &= \omega_G(f^*|_G) + \sum_{v\in V(G)} f^*(x^v) \\ &= \omega_G(f^*|_G) + |\,(V_0 \cup V_1) \cap V(G)| \\ &= \omega_G(f^*|_G) + n - |V_2^* \cap V(G)| \\ &\geq \alpha. \end{aligned}$$

■

It follows from Theorem 2.9 that for all connected graphs G of order $n \geq 2$,

$$\gamma_R^P(G \circ K_1) \leq \gamma_R^P(G) + n - \lambda,$$

where $\lambda = \max\{|V_2| : (V_0, V_1, V_2) \text{ is a } \gamma_R^P\text{-function on } G\}$, and this bound is sharp. Verify that equality is attained if G is a cycle C_n ($n \geq 3$), a path P_n ($n \geq 2$), or any graph with $\gamma(G) = 1$.

Our desired result for more general graphs G and H will follow from the following characterization.

Theorem 2.10. *Let G and H be nontrivial graphs with G connected, and $f = (V_0, V_1, V_2)$. Then $f \in PRD(G \circ H)$ if and only if the following holds:*

- (i) *For all $v \in V_0 \cap V(G)$ either*
  - (a) *$V_2 \cap N_G(v) = \varnothing$ and $V_2 \cap V(H^v) = \{u\}$ with u satisfying $V_0 \cap V(H^v) \subseteq N_{H^v}(u)$; or*
  - (b) *$|V_2 \cap N_G(v)| = 1$ and $V(H^v) \subseteq V_1$;*
- (ii) *For all $v \in V_1 \cap V(G)$, the restriction $f|_{H^v}$ of f to H^v is a perfect Roman dominating function on H^v;*
- (iii) *For all $v \in V_2 \cap V(G)$ for which $V_0 \cap V(H^v) \neq \varnothing$, $V_0 \cap N_{H^v}(V_2 \cap V(H^v)) = \varnothing$.*

Proof: Assume that $f \in PRD(G \circ H)$. Let $v \in V_0 \cap V(G)$. Then there exists a unique $u \in V_2$ for which $u \in N_{G \circ H}(v) = V(H^v) \cup N_G(v)$. If $V_2 \cap N_G(v) = \varnothing$, then $V_2 \cap V(H^v) = \{u\}$ and $V_0 \cap V(H^v) \subseteq N_{H^v}(u)$. Suppose that $V_2 \cap N_G(v) \neq \varnothing$. Then $|V_2 \cap N_G(v)| = 1$ and $V_2 \cap V(H^v) = \varnothing$. Moreover, if $w \in V_0 \cap V(H^v)$, then there exists a unique $z \in V_2 \cap V(H^v)$ such that $wz \in E(H^v)$. Since $vz \in E(G \circ H)$, this is impossible. Thus, $V(H^v) \subseteq V_1$. This proves (i). Next, let $v \in V_1 \cap V(G)$, and let $w \in V_0 \cap V(H^v)$. Since f is a perfect Roman dominating function, there exists unique $u \in V_2$ for which $uw \in E(G \circ H)$. Since $v \in V_1$, $u \in V_2 \cap V(H^v)$ and $uw \in E(H^v)$. Thus, $f|_{H^v}$ is a perfect Roman dominating function on H^v, and (ii) holds. Statement (iii) is clear.

Conversely, suppose that conditions (i), (ii) and (iii) hold for f, and let $w \in V_0$. Then $w \in V(H^v + v)$ for some $v \in V(G)$. If $w = v$, then by condition (i), $V_2 \cap (V(H^v) \cup N_G(w)) = \{u\}$ for some $u \in V(G \circ H)$. This means that $V_2 \cap N_{G \circ H}(w) = \{u\}$. Suppose that $w \in V(H^v)$. We consider three cases:

Case 1: Suppose that $v \in V_0$. Since $w \in V_0 \cap V(H^v)$, $V(H^v) \nsubseteq V_1$. Thus, by condition (i) there exists $u \in V(H^v)$ for which $V_2 \cap V(H^v) = \{u\}$ and $V_0 \cap V(H^v) \subseteq N_{H^v}(u)$. This means that $V_2 \cap N_{G \circ H}(w) = \{u\}$.

Case 2: Suppose that $v \in V_1$. By condition (ii), there exists a unique $u \in V_2 \cap V(H^v)$ such that $uw \in E(H^v) \subseteq E(G \circ H)$. This implies that $V_2 \cap N_{G \circ H}(w) = \{u\}$.

Case 3: Suppose that $v \in V_2$. Since $w \in V_0 \cap V(H^v)$, condition (iii) implies that $w \notin N_{H^v}(V_2 \cap V(H^v)$. Thus, $V_2 \cap N_{G \circ H}(w) = \{v\}$.

Therefore, f is a perfect Roman dominating function on $V(G \circ H)$. ■

Corollary 2.11. *Let G and H be nontrivial graphs with G connected of order n. Then $\gamma_R^P(G \circ H) = 2n$.*

Proof: By Theorem 2.7, the function $f = (V_0, V_1, V_2)$ defined by $f(x) = 2$ for all $v \in V(G)$, and $f(x) = 0$ else, is a perfect Roman dominating function on $G \circ H$. Thus, $\gamma_R^P(G \circ H) \leq 2n$.

Now, let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on $V(G \circ H)$. Let $v \in V(G)$. Clearly, if $v \in V_2$, then $\sum_{x \in V(H^v + v)} f(x) \geq 2$. If $v \in V_0$, then by Proposition 2.10(i) and since

$|V(H^v| \geq 2$, $\sum_{x\in V(H^v+v)} f(x) \geq 2$. Finally, if $v \in V_1$, then by Proposition 2.10(ii), $\sum_{x\in V(H^v+v)} f(x) > 2$. Therefore,

$$\gamma_R^P(G \circ H) = \omega_{G\circ H}(f) = \sum_{v\in V(G)} \left(\sum_{x\in V(H^v+v)} f(x) \right) \geq 2n.$$

■

2.3. On the complementary prisms

Let $f = (V_0, V_1, V_2) \in PRD(G\overline{G})$. Suppose that for the restriction $f|_{\overline{G}} \notin PRD(\overline{G})$. Then there exists $v \in V(G)$ such that $\overline{v} \in V_0$ and $V_2 \cap N_{G\overline{G}}(\overline{v}) = \{v\}$. Let $u \in V_0 \cap V(G)$. There exists $w \in V(G\overline{G})$ such that $V_2 \cap N_{G\overline{G}}(u) = \{w\}$. If $w = \overline{u}$, then $\overline{uv} \notin E(\overline{G}$, and consequently, $uv \in E(G)$, a contradiction. Thus, $w \in V_2 \cap V(G)$. This proves the following lemma.

Lemma 2.12. *Let G be any graph. If $f \in PRD(G\overline{G})$, then $f|_G \in PRD(G)$ or $f|_{\overline{G}} \in PRD(\overline{G})$.*

Proposition 2.13. *Let G be a graph of order n. Then*

(i) $\gamma(G\overline{G}) < \gamma_R^P(G\overline{G})$;

(ii) $\gamma_R^P(G\overline{G}) = 2$ *if and only if* $n = 1$;

(iii) $\gamma_R^P(G\overline{G}) = 3$ *if and only if* $G \in \{K_2, \overline{K_2}\}$;

(iv) *If* $\gamma(G) = 1$, *then* $\gamma_R^P(G\overline{G}) \leq n + 1$ *and equality is attained if* $deg_G(v) \leq 3$ *for all* $v \notin Dom(G)$ *or* $\overline{G}$ *is the disjoint union of* $K_j \in \{K_1, K_2\}$.

Proof: Since $G\overline{G}$ is connected, (i) follows from Corollary 2.6.

If $n = 1$, then $G\overline{G} = K_2$ and $\gamma_R^P(G\overline{G}) = 2$. Suppose that $\gamma_R^P(G\overline{G}) = 2$, and let f be a γ_R^P-function of $G\overline{G}$. By Lemma 2.12, we may assume that $f|_G \in PRD(G)$. If $\omega_G(f|_G) = 1$, then $n = 1$. If $\omega_G(f|_G) = 2$, then $G = \{v\}$ with $f(v) = f|_G(v) = 2$ and $f(\overline{v}) = 0$.

If $G \in \{K_2, \overline{K_2}\}$, then $G\overline{G}$ is isomorphic to P_4. Thus, $\gamma_R^P(G\overline{G}) = 3$. Conversely, suppose that $\gamma_R^P(G\overline{G}) = 3$. By Proposition 2.3($iii$), $\Delta(G\overline{G}) = 2n - 2$. Let $v \in V(G\overline{G})$ be such that $deg_{G\overline{G}}(v) = 2n - 2$. Without loss of generality, assume that $v \in V(G)$. Since $N_{G\overline{G}}(v) \cap V(\overline{G}) = \{\overline{v}\}$, $deg_G(v) = 2n - 3 \leq n - 1$. Necessarily, $n \leq 2$. By (ii), $n = 2$ and $G = K_2$.

If $\gamma(G) = 1$, then by Proposition 2.2, $\gamma_R^P(G\overline{G}) \leq n+1$. First, suppose that $deg_G(v) \leq 3$ for all $v \notin Dom(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of $G\overline{G}$. Since $\omega_{G\overline{G}}(f) \leq n + 1$, $V_2 \neq \varnothing$. We consider two cases:

Case 1: Suppose that $V_2 \cap V(G) = \varnothing$. If $V(G) \subseteq V_1$, then $V(\overline{G}) \nsubseteq V_0$ so that $\omega_{G\overline{G}}(f) \geq n+1$. Suppose that $V(G) \cap V_0 \neq \varnothing$. Then

$$\begin{aligned}\omega_{G\overline{G}}(f) &= \sum_{w \in V_0 \cap V(G)} f(\overline{w}) + \sum_{w \in V_1 \cap V(G)} (f(w) + f(\overline{w})) \\ &\geq n+1.\end{aligned}$$

Case 2: Assume that $V_2 \cap V(G) \neq \varnothing$. We consider two subcases:

Subcase 2.1: Suppose that V_2 contains a dominating vertex v of G. Since f is a γ_R^P-function, $N_G(v) \cup \{\overline{v}\} \subseteq V_0$. Let $w \in V(G) \setminus \{v\}$. Suppose that $\overline{w} \in V_0$. There exists $u \in V(G)$ such that $N_{\overline{G}}(\overline{w}) \cap V_2 = \{\overline{u}\}$. Since $\overline{wv} \notin E(\overline{G})$, $u \neq v$. Thus, $u \in V_0$ and $v, \overline{u} \in N_{G\overline{G}}(u) \cap V_2$, a contradiction. This means that $f(\overline{w}) \geq 1$. Therefore, $\omega_{G\overline{G}}(f) = 2 + \sum_{w \in V(G) \setminus \{v\}} f(\overline{w}) \geq 2 + n - 1 = n + 1$.

Subcase 2.2: Suppose that $V_2 \cap Dom(G) = \varnothing$. Choos $v \in Dom(G)$. Put $A = \{w \in V(G) : f(w) = f(\overline{w}) = 0\}$. If $A = \varnothing$, then $f(w) + f(\overline{w}) \geq 1$ for all $w \in V(G)$ and since $V_2 \cap V(G) \neq \varnothing$, we have $\omega_{G\overline{G}}(f) \geq n+1$. Suppose that $A \neq \varnothing$. Here, we work on two subcases:

Subcase 2.2.1: Suppose that $v \in V_0$. If $f(\overline{v}) = 2$, then $V(G) \cap V_2 = \varnothing$ and so $f(\overline{u}) = 2$ for each $u \in V_0 \cap V(G)$. This implies that $\omega_{G\overline{G}}(f) \geq n+1$. Suppose that $f(\overline{v}) = 1$. Then there exists $u \in V(G)$ such that $V_2 \cap V(G) = \{u\}$. Moreover, for each $w \in A$, $wu \in E(G)$. Since $deg_G(u) \leq 3$ and $uv \in E(G)$, $|A| \leq 2$. Suppose that $A = \{w\}$. There exists $a \in V(G)$ such that $u \neq a$ and $N_{\overline{G}}(\overline{w}) \cap V_2 = \{\overline{a}\}$. Since $\alpha = (f(u) + f(\overline{u})) + (f(w) + f(\overline{w})) + (f(a) + f(\overline{a})) \geq 4$,

$$\omega_{G\overline{G}}(f) = \alpha + \sum_{x \in V(G) \setminus \{u,w,a\}} (f(x) + f(\overline{x})) \geq 4 + (n-4) + 1 = n+1.$$

Now, suppose that $A = \{w, z\}$. There exist $a, b \in V(G)$ such that $\overline{a}, \overline{b} \in V_2$, $\overline{wa}, \overline{zb} \in E(\overline{G})$ and $\overline{a}, \overline{b} \in N_{\overline{G}}(\overline{u})$. Thus, $f(\overline{u}) = f(a) = f(b) = 1$ and whether $a = b$ or $a \neq b$,

$$\alpha = (f(u) + f(\overline{u})) + (f(w) + f(\overline{w})) + (f(z) + f(\overline{z})) + (f(a) + f(\overline{a})) + \big(f(b) + f(\overline{b})\big) \geq 6.$$

Thus,

$$\omega_{G\overline{G}}(f) = \alpha + \sum_{x \in V(G) \setminus \{u,w,z,a,b\}} (f(x) + f(\overline{x})) \geq 6 + (n-6) + 1 = n+1.$$

Subcase 2.2.2: Suppose that $v, \overline{v} \in V_1$. For each $w \in A$, there exist distinct vertices $u, z \in V(G)$ such that $u, \overline{z} \in V_2$, $uw \in E(G)$ and $\overline{wz} \in E(\overline{G})$. Again, for each $u \in V_2 \cap V(G)$, since $deg_G(u) \leq 3$, there can only be at most two vertices $a, b \in A$ for which $ua, ub \in E(G)$. Using similar arguments, if $|A| \leq 2$, then $\omega_{G\overline{G}}(f) \geq n+1$. To proceed, we only have to consider the case where $3 \leq |A| \leq 4$. Other cases follow inductively.

Suppose that $A = \{x, y, w\}$. The only nontrivial scenario is the following: There exist $a, c \in V_2 \cap V(G)$ and $b \in V(G)$ such that $\overline{b} \in V_2$, $ac \notin E(G)$, $wc \in E(G)$, $\{x, y\} \subseteq N_G(a)$, and $\{\overline{x}, \overline{y}, \overline{w}\} \subseteq N_{\overline{G}}(\overline{b})$. Since $\overline{a}\overline{b} \in E(\overline{G})$, $f(\overline{a}) = 1$. Thus,

$$\begin{aligned}\omega_{G\overline{G}}(f) &= \sum_{u\in\{a,x,y,b,w,c\}} (f(u) + f(\overline{u})) + \sum_{u\in V(G)\setminus\{a,b,c,x,y,w\}} (f(u) + f(\overline{u}))\\ &\geq 7 + (n-7) + 2\\ &> n+1.\end{aligned}$$

Finally, suppose that $A = \{x, y, z, w\}$. It is enough to consider only the following nontrivial case: There exist $a, c \in V_2 \cap V(G)$ and $b \in V(G)$ such that $\overline{b} \in V_2$, $ac \notin E(G)$, $\{x, y\} \subseteq N_G(a)$, $\{w, z\} \subseteq N_G(c)$, and $\{\overline{x}, \overline{y}, \overline{z}, \overline{w}\} \subseteq N_{\overline{G}}(\overline{b})$. Since $\overline{a}\overline{b}, \overline{c}\overline{b} \in E(\overline{G})$, $f(\overline{a}) = f(\overline{c}) = 1$. Hence,

$$\begin{aligned}\omega_{G\overline{G}}(f) &= \sum_{u\in\{a,b,c,x,y,w,z\}} (f(u) + f(\overline{u})) + \sum_{u\in V(G)\setminus\{a,b,c,x,y,w,z\}} (f(u) + f(\overline{u}))\\ &\geq 8 + (n-8) + 2\\ &> n+1.\end{aligned}$$

All of the above cases show that $\gamma_R^P(G) = \omega_{G\overline{G}}(f) \geq n + 1$.

Next, suppose that $\overline{G}$ is the union of $K_j \in \{K_1, K_2\}$, and let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of $G\overline{G}$. As shown previously, we may assume that $V_2 \cap V(G) \neq \varnothing$, and if V_2 contains a dominating vertex of G, then $\omega_{G\overline{G}}(f) \geq n + 1$. Henceforth, we assume that $V_2 \cap Dom(G) = \varnothing$. Pick $v \in Dom(G)$. Then $\overline{v} \in Iso(\overline{G})$. Note that for all $\overline{x} \in Iso(\overline{G})$, $x \notin A = \{w \in V(G) : f(w) = f(\overline{w}) = 0\}$ so that $(f(x) + f(\overline{x})) \geq 1$. Also, for all $x, y \in V(G)$ for which $\overline{x}\overline{y} \in E(\overline{G})$, if $x \in A$, then $\overline{y} \in V_2$ and so $(f(x) + f(\overline{x})) + (f(y) + f(\overline{y})) \geq 2$. Thus, if $v \in V_0$ and $u \in V(G)$ such that $V_2 \cap V(G) = \{u\}$, then

$$\begin{aligned}\omega_{G\overline{G}}(f) &= (f(u) + f(\overline{u})) + \sum_{\overline{x}\in Iso(\overline{G})} (f(x) + f(\overline{x})) +\\ &\qquad \sum_{\overline{x}\overline{y}\in E(\overline{G})} ((f(x) + f(\overline{x})) + (f(y) + f(\overline{y})))\\ &\geq n+1.\end{aligned}$$

On the other hand, if $v \in V_1$, then $f(\overline{v}) = 1$ and

$$\begin{aligned}\omega_{G\overline{G}}(f) &= (f(v) + f(\overline{v})) + \sum_{\overline{x}\in Iso(\overline{G})\setminus\{\overline{v}\}} (f(x) + f(\overline{x})) +\\ &\qquad \sum_{\overline{x}\overline{y}\in E(\overline{G})} ((f(x) + f(\overline{x})) + (f(y) + f(\overline{y})))\\ &\geq n+1.\end{aligned}$$

Therefore, $\gamma_R^P(G\overline{G}) \geq n + 1$. ■

As shown by the graph G in Figure 1, strict inequality may be attained in Proposition 2.13(iv) if we remove the condition that $deg_G(v) \leq 3$ for all nondominating vertices v of G. For such G, $\gamma_R^P(G\overline{G}) = 6 < |V(G)| + 1$.

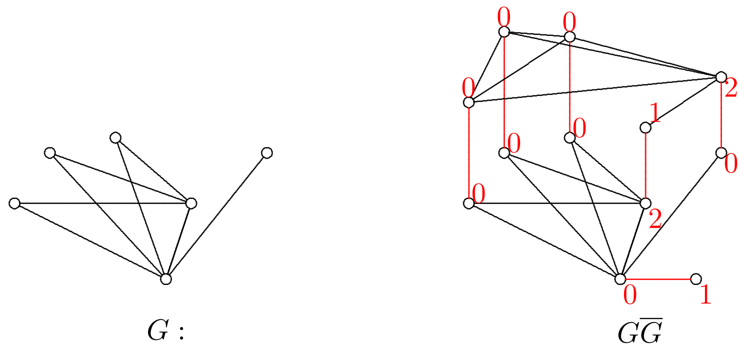

Figure 1: Graph G with $\gamma(G) = 1$ and $\gamma_R^P(G\overline{G}) < |V(G)| + 1$

Pick $G = K_n$. By Proposition 2.13(iv) and Corollary 2.5,

$$\gamma_R^P(G\overline{G}) = 1 + \max\{\gamma_R^P(G), \gamma_R^P(\overline{G})\}.$$

Observe also that if $v \in V(G)$, then $f = (V(G) \setminus \{v\}, \varnothing, \{v\}) \in PRD(G)$ and $\gamma_R^P(G\overline{G}) = \omega_G(f) + n - |V_2|$. The following result shows that these two expressions serve as sharp lower and upper bounds, respectively, of $\gamma_R^P(G\overline{G})$ for a general graph G.

Theorem 2.14. *For any graph G,*

$$1 + \max\{\gamma_R^P(G), \gamma_R^P(\overline{G})\} \leq \gamma_R^P(G\overline{G}) \leq \rho,$$

where $\rho = \min\{\omega_G(f) + n - |V_2| : f = (V_0, V_1, V_2) \in PRD(G) \cup PRD(\overline{G})\}$.

Proof: WLOG assume that for some $f = (V_0, V_1, V_2)$ on G, $\rho = \omega_G(f) + n - |V_2|$. Extend f to $G\overline{G}$ by defining $f(\overline{v}) = 0$ for all $v \in V_2$ and $f(\overline{v}) = 1$ for all $v \in V(G) \setminus V_2$. Then the extension $f \in PRD(G\overline{G})$ and $\gamma_R^P(G\overline{G}) \leq \omega_G(f) + n - |V_2|$. Thus, $\gamma_R^P(G\overline{G}) \leq \rho$.

In view of Proposition 2.13(iv), we assume that neither G nor $\overline{G}$ is a complete graph. WLOG, assume that $\gamma_R^P(G) \geq \gamma_R^P(\overline{G})$. Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on $G\overline{G}$. If $V(\overline{G}) \subseteq V_0$, then $V_2 = V(G)$ so that $\gamma_R^P(G\overline{G}) = 2|V_2| = |V(G\overline{G})|$. Since $G\overline{G}$ is connected, $n = 1$ by Corollary 2.5 and Corollary 2.3(ii). This is contradictory to our assumption. Thus, $V(\overline{G}) \cap (V_1 \cup V_2) \neq \varnothing$. If $V_2 \cap V(\overline{G}) = \varnothing$, then $g = (V_0 \cap V(G), V_1 \cap V(G), V_2) \in PRD(G)$. Since $V(\overline{G}) \cap V_1 \neq \varnothing$,

$$\gamma_R^P(G\overline{G}) = \omega_{G\overline{G}}(f) \geq \omega_G(g) + 1 \geq \gamma_R^P(G) + 1.$$

Suppose that $V_2 \cap V(\overline{G}) \neq \varnothing$, and let $A = \{v \in V_0 : V_2 \cap N_{G\overline{G}}(v) = \{\overline{v}\}\}$. Define $g = (V_0^*, V_1^*, V_2^*)$ on G by

$$g(x) = \begin{cases} f(x), & \text{if } x \in V(G) \setminus A; \\ 1, & \text{if } x \in A. \end{cases}$$

Then $g \in PRD(G)$ with $V_0^* = (V_0 \setminus A) \cap V(G)$, $V_1^* = A \cup (V_1 \cap V(G))$ and $V_2^* = V_2 \cap V(G)$. Since $\{\overline{v} : v \in A\} \subseteq V_2 \cap V(\overline{G})$,

$$\gamma_R^P(G\overline{G}) = \omega_G(g) + \sum_{x \in V(\overline{G})} f(x) - |A| \geq \omega_G(g) + 1 \geq \gamma_R^P(G) + 1.$$

■

If $G = C_5$, then G and $\overline{G}$ are isomorphic and $G\overline{G}$ is isomorphic to the Petersen graph. Observe that $\gamma_R^P(G\overline{G}) = 7$, $\gamma_R^P(G) = \gamma_R^P(\overline{G}) = 4$ and $\rho = 8$ so that

$$1 + \max\{\gamma_R^P(G), \gamma_R^P(\overline{G})\} < \gamma_R^P(G\overline{G}) < \rho.$$

This shows that strict inequality can be attained at each side of the inequalities in Theorem 2.14.

2.4. On the edge corona of graphs

Given graphs G and H, we write H^{uv} to denote that copy of H that is being joined with the endvertices of the edge $uv \in E(G)$ in the edge corona $G \diamond H$. If $H = \{x\}$, then we write $V(H^{uv}) = \{x^{uv}\}$.

For an $f \in PRD(G)$, we write for each $a, b \in \{0, 1, 2\}$,

$$E_{ab}(f; G) = \{uv \in E(G) : (f(u) = a \wedge f(v) = b) \vee (f(u) = b \wedge f(v) = a)\},$$

where "$\wedge$" and "$\vee$" denote "and" and "or", respectively.

Theorem 2.15. *Let G be a nontrivial connected graph and H any graph of order n. Then*

$$\gamma_R^P(G \diamond H) \leq \alpha,$$

where

$$\alpha = \min_{g \in PRD(G)} \left(\omega_G(g) + |E_{11}(g; G)|\gamma_R^P(H) + n\left(|E_{01}(g; G)| + |E_{22}(g; G)| + E_{00}(g; G)|\right)\right),$$

and this upper bound is sharp.

Proof: Let $g \in PRD(G)$. If no confusion arises, we write $E_{ab} = E_{ab}(g; G)$. Let $h \in PRD(H)$. For each $ab \in E(G)$, we define a copy h^{ab} of h on H^{ab}. Define the function $f = (V_0, V_1, V_2)$ on $G \diamond H$ by

$$f(x) = \begin{cases} g(x), & \text{if } x \in V(G); \\ h^{uv}(x), & \text{if } x \in V(H^{uv}), \text{ where } uv \in E_{11}; \\ 0, & \text{if } x \in V(H^{uv}), \text{ where } uv \in E_{02} \cup E_{12}; \\ 1, & \text{if } x \in V(H^{uv}), \text{where } uv \in E_{01} \cup E_{00} \cup E_{22}. \end{cases}$$

We claim that $f \in PRD(G \diamond H)$. First, note that $f|_G = g = (V_0 \cap V(G), V_1 \cap V(G), V_2 \cap V(G))$. Let $x \in V_0$. Suppose that $x \in V(G)$. Then $N_{G\diamond H}(x) = N_G(x) \cup \big(\cup_{u \in N_G(x)} V(H^{ux})\big)$. Since $g \in PRD(G)$, $|V_2 \cap N_G(x)| = 1$, say $V_2 \cap N_G(x) = \{z\}$. Let $u \in N_G(x)$, and let $y \in V(H^{xu})$. If $u \in V_0 \cup V_1$, then $y \in V_1$. On the other hand, if $u \in V_2$, then $y \in V_0$. Thus, $V_2 \cap V(H^{ux}) = \varnothing$. Since u is arbitrary, $V_2 \cap \big(\cup_{u \in N_G(x)} V(H^{ux})\big) = \varnothing$ and so $V_2 \cap N_{G\diamond H}(x) = \{z\}$. Suppose that $x \in V(H^{uv})$ for some $uv \in E(G)$. Then $N_{G\diamond H}(x) = \{u, v\} \cup N_{H^{uv}}(x)$. Since $f(x) = 0$, $uv \notin E_{00} \cup E_{22} \cup E_{01}$. If $uv \in E_{11}$, then $h^{uv}(x) = 0$ and there exists exactly one $y \in V(H^{uv})$ such that $xy \in E(H^{uv})$ and $f(y) = h^{uv}(y) = 2$. In this case, $V_2 \cap N_{G\diamond H}(x) = V_2 \cap N_{H^{uv}}(x) = \{y\}$. Suppose that $uv \in E_{02} \cup E_{12}$. Since $V(H^{uv}) \subseteq V_0$, either $V_2 \cap N_{G\diamond H}(x) = \{u\}$ or $V_2 \cap N_{G\diamond H}(x) = \{v\}$. Accordingly, $f \in PRD(G \diamond H)$. Therefore,

$$\begin{aligned}
\gamma_R^P(G \diamond H) &\le \omega_G(g) + |E_{11}|\omega_H(h) + \sum_{x \in \{V(H^{uv}) : uv \in E_{00} \cup E_{01} \cup E_{22}\}} f(x) \\
&= \omega_G(g) + |E_{11}|\omega_H(h) + n\left(|E_{01}| + |E_{22}| + E_{00}|\right).
\end{aligned}$$

Since h is arbitrary, the desired inequality holds.

Consider the graph $G \diamond P_3$ in Figure 2, where G is the caterpillar $ca(2, 0, 2)$ with the corresponding vertex labelling. The function g on $V(G)$ given by $g(x) = g(z) = 2$, $g(y) = 1$ and $g(x) = 0$ else is in $PRD(G)$. Since $E_{00} = E_{01} = E_{22} = E_{00} = \varnothing$, $\alpha \le \omega_G(g) = 5$ so that $\gamma_R^P(G \diamond P_3) \le 5$. Now, note that $\{x, z\}$ is the unique γ-set of $G \diamond P_3$. However, $\{x, z\}$

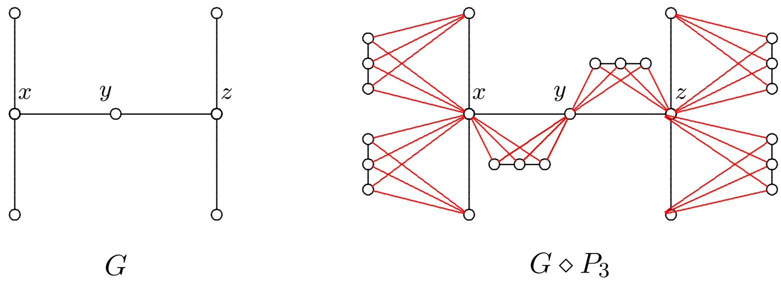

Figure 2: The edge corona $G \diamond P_3$ with $\gamma_R^P(G \diamond P_3) = 5$

does not form the $V_1 \cup V_2$ for any $f = (V_0, V_1, V_2) \in PRD(G \diamond P_3)$. Thus, $\gamma_R^P(G \diamond P_3) \ge 5$. ■

The value of α in Theorem 2.15 is not necessarily determined by a γ_R^P-function on G. Consider the two copies of the edge corona $P_5 \diamond C_4$ given in Figure 3 with the corresponding assignment of colours to the vertices. Here, we write $P_5 = \{x_1, x_2, x_3, x_4, x_5\}$. Observe that $f = (\{x_1, x_3, x_4\}, \varnothing, \{x_2, x_5\})$ is a γ_R^P-function on P_5 (see right-hand side figure), while $g = (\{x_1, x_5\}, \{x_3\}, \{x_2, x_4\}) \in PRD(P_5)$ but not a γ_R^P-function on P_5 (see left-hand side figure). Verify that $\gamma_R^P(P_5 \diamond C_4) = 5$ and is determined by the function g.

From Theorem 2.15 and as illustrated in the preceding example, the value of α in Theorem 2.15 is determined by the functions $g \in PRD(G)$ for which most of the sets

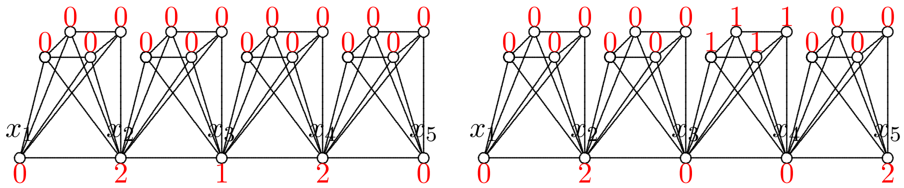

Figure 3: The edge corona $P_5 \diamond C_4$

$E_{00}(g;G)$, $E_{22}(g;G)$, $E_{11}(g;G)$ and $E_{01}(g;G)$ are empty. In view of such, the following observation can be easily verified.

Corollary 2.16. *Let H be any nontrivial graph of order m. then*

(*i*) *For the path P_n on $n \geq 2$ vertices, $\gamma_R^P(P_n \diamond H) = 3\lfloor \frac{n-2}{2} \rfloor + 2$.*

(*ii*) *If $m \geq 3$, then for the cycle C_n on $n \geq 3$ vertices,*

$$\gamma_R^P(C_n \diamond H) = \begin{cases} 3k, & \text{if } n = 2k; \\ 3k + 1 + \gamma_R^P(H), & \text{if } n = 2k+1. \end{cases}$$

(*iii*) *If $m \geq 3$, then for $2 \leq n \leq k$, $\gamma_R^P(K_{n,k} \diamond H) = 2n + k$.*

Theorem 2.17. *Let G be a nontrivial connected graph. Then*

$$\gamma_R^P(G \diamond K_1) = \min_{g \in PRD(G)} \left(\omega_G(G) + |E_{00}(g;G)| + |E_{01}(g;G)| + |E_{11}(g;G)| + |E_{22}(g;G)|\right).$$

Proof: Put

$$\alpha = \min\{\omega_G(G) + |E_{00}(g;G)| + |E_{01}(g;G)| + |E_{11}(g;G)| + |E_{22}(g;G)| : g \in PRD(G)\}.$$

By Theorem 2.15, $\gamma_R^P(G \diamond K_1) \leq \alpha$.

Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on $G \diamond K_1$. Suppose that the restriction $f|_G$ of f to G is not a perfect Roman dominating function on G. We will construct a γ_R^P-function g on $G \diamond K_1$ such that $\omega_{G \diamond K_1}(g) = \omega_{G \diamond K_1}(f)$ and its restriction $g|_G$ to G is a perfect Roman dominating function on G. There exists $u \in V_0 \cap V(G)$ such that $uv \notin E(G)$ for all $v \in V_2 \cap V(G)$. This means that there exists $v \in N_G(u)$ such that $V_2 \cap N_{G \diamond K_1}(u) = \{x^{uv}\}$.

Case 1: Suppose that $v \notin V_0$. Define $f^1 = (V_0^1, V_1^1, V_2^1)$ on $G \diamond K_1$ by $f^1(u) = f^1(x^{uv}) = 1$ and $f^1(x) = f(x)$ for all $x \in V(G \diamond K_1) \setminus \{u, x^{uv}\}$. Then $f^1 \in PRD(G \diamond K_1)$ with $\omega_{G \diamond K_1}(f^1) = \omega_{G \diamond K_1}(f)$.

Case 2: Suppose that $v \in V_0$. If $(N_G(v) \setminus \{u\}) \cap V_0 = \varnothing$, then take $f^1 = (V_0^1, V_1^1, V_2^1)$ on G given by $f^1(v) = 2$, $f^1(x^{uv}) = 0$ and $f^1(x) = f(x)$ for all $x \in V(G \diamond K_1) \setminus \{v, x^{uv}\}$. Then $f^1 \in PRD(G \diamond K_1)$ and $\omega_{G \diamond K_1}(f^1) = \omega_{G \diamond K_1}(f)$. Suppose that $B = (N_G(v) \setminus \{u\}) \cap V_0 \neq \varnothing$.

Necessarily, $x^{vw} \in V_1$ for each $w \in B$. In this case, take the function $f^1 = (V_0^1, V_1^1, V_2^1)$ on $G \diamond K_1$ given by

$$f^1(x) = \begin{cases} 2, & \text{if } x = v; \\ 0, & \text{if } x \in \{x^{uv}, x^{vw} : w \in B\}; \\ 1, & \text{if } x \in B; \\ f(x), & \text{if } x \in V(G \diamond K_1) \setminus (B \cup \{x^{vw} : w \in B\}). \end{cases}$$

Then $f^1 \in PRD(G \diamond K_1)$ with $V_0^1 = (V_0 \setminus \{v\}) \cup \{x^{uv}, x^{vw} : w \in B\}$, $V_1^1 = (V_1 \setminus \{x^{vw} : w \in B\}) \cup B$ and $V_2^1 = (V_2 \setminus \{x^{uv}\}) \cup \{v\}$. It is easy to verify that $f^1 \in PRD(G \diamond K_1)$ and $\omega_{G \diamond K_1}(f^1) = \omega_{G \diamond K_1}(f)$.

If $f^1|_G \notin PRD(G)$, then we follow the same process and obtain $f^2 \in PRD(G \diamond K_1)$ with $\omega_{G \diamond K_1}(f^2) = \omega_{G \diamond K_1}(f^1) = \omega_{G \diamond K_1}(f)$. If necessary, we do a finitely many repetitions of the process until we obtain a function $g = f^k \in PRD(G \diamond K_1)$ for which $\omega_{G \diamond K_1}(g) = \omega_{G \diamond K_1}(f)$ and $g|_G \in PRD(G)$. By the definition of α, $\gamma_R^P(G \diamond K_1) = \omega_{G \diamond K_1}(g) \geq \alpha$. ■

The value of $\gamma_R^P(G \diamond K_1)$ in Theorem 2.17 is determined by the functions $g \in PRD(G)$ for which the sets E_{22} and E_{11} are empty. With this observation, it can readily be verified that for $n \geq 1$ and $m \geq 3$,

$$\gamma_R^P(P_n \diamond K_1) = \lfloor \frac{n-1}{3} \rfloor + \gamma_R^P(P_n) \text{ and } \gamma_R^P(C_m \diamond K_1) = \lceil \frac{n}{3} \rceil + \gamma_R^P(C_m).$$

2.5. On the composition of graphs

Given $S \subseteq V(G[H])$, we write $S_G = \{x \in V(G) : (x, y) \in S \text{ for some } y \in V(H)\}$, which is called the *projection of* G on $G[H]$.

Proposition 2.18. *Let G and H be connected graphs, G noncomplete and H of order n with $\gamma(H) = 1$. Then*

$$\gamma_R^P(G[H]) \leq \alpha,$$

where $\alpha = \min\{(n-1)(|V_1| + |V_2 \cap N_G(V_2)|) + \omega_G(f) : f = (V_0, V_1, V_2) \in PRD(G)\}$.

Proof: Let $v \in V(H)$ for which $N_H[v] = V(H)$. Let $f = (V_0, V_1, V_2) \in PRD(G)$ such that $V_2 \neq \varnothing$. Define $g = (V_0^*, V_1^*, V_2^*)$ on $G[H]$ by

$$g((x, y)) = \begin{cases} 0, & \text{if } (x \in V_2 \setminus N_G(V_2) \ \wedge \ y \neq v) \ \vee \ (x \in V_0); \\ 1, & \text{if } (x \in V_2 \cap N_G(V_2) \ \wedge \ y \neq v) \ \vee (x \in V_1); \\ 2, & \text{if } x \in V_2 \text{ and } y = v. \end{cases}$$

with $V_0^* = ((V_2 \setminus N_G(V_2)) \times (V(H) \setminus \{v\})) \cup (V_0 \times V(H))$, $V_2^* = V_2 \times \{v\}$ and $V_1^* = (V_1 \cup V(H)) \cup ((V_2 \cap N_G(V_2)) \times (V(H) \setminus \{v\}))$. Let $(x, y) \in V_0^*$. If $x \in V_2$, then $x \notin N_G(V_2)$ so that $N_{G[H]}((x, y)) \cap V_2^* = \{(x, v)\}$. If $x \in V_0$, then there exists $u \in V_2$ such that $N_G(x) \cap V_2 = \{u\}$, which implies that $N_{G[H]}((x, y)) \cap V_2^* = \{(u, v)\}$. Thus, $g \in PRD(G[H])$. Therefore, $\gamma_R^P(G[H]) \leq |V_1^*| + 2|V_2^*| = (n-1)(|V_1| + |V_2 \cap N_G(V_2)|) + \omega_G(f)$. Since f is arbitrary, the desired inequality is established. ■

Proposition 2.19. *Let G be a nontrivial connected graph and $p \geq 2$. Then*

$$\gamma_R^P(G[K_p]) = \alpha,$$

where $\alpha = \min\{(n-1)(|V_1| + |V_2 \cap N_G(V_2)|) + \omega_G(f) : f = (V_0, V_1, V_2) \in PRD(G)\}$.

Proof: Let $f = (V_0, V_1, V_2)$ be a γ_P^R-function on $V(G[H])$. Then $V_2 \neq \varnothing$ and $V_0 \neq \varnothing$. First, we claim that $(V_0)_G \cap (V_1)_G = \varnothing$. Suppose not, and let $(x,y) \in V_1$ be such that $(x,z) \in V_0$ for some $z \neq y$. There exists unique $(u,v) \in V_2$ for which $(x,z)(u,v) \in E(G[K_p]$. If $u = x$, then since $y \neq v$, $(x,y)(u,v) \in E(G[K_p])$. Thus, whether $u = x$ or $x \neq u$, $(x,y)(u,v) \in E(G[K_p])$. By Proposition 2.1, there exists $(a,b) \in V_2 \setminus \{(u,v)\}$ such that $(x,y)(a,b) \in E(G[K_p])$. Using the same argument, whether $x = a$ or $x \neq b$, $(x,z)(a,b) \in E(G[K_p])$. This is a contradiction since $(x,z) \in V_0$.

Fix $v \in V(K_p)$. Define $A = \{(x,v) : x \in (V_0)_G \cap (V_2)_G\}$, $B = \{(x,y) \in V_2 : x \notin (V_0)_G\}$ and $C = \{(x,y) \in V_2 : x \in (V_0)_G, y \neq v\}$. Put

$$V_0^* = (V_0 \setminus A) \cup C, \;\; V_1^* = V_1, \;\text{ and }\; V_2^* = A \cup B.$$

Then $\{V_0^*, V_1^*, V_2^*\}$ forms a partition of $V(G[K_p])$. Note here that, in particular, since $(V_0)_G \cap (V_1)_G = \varnothing$ and $V_1 \cap V_2 = \varnothing$. Now, let $(x,y) \in V_0^*$.

Case 1: Suppose that $(x,y) \in V_0 \setminus A$. There exists $(u,w) \in V_2$ such that $N_{G[K_p]}((x,y)) \cap V_2 = \{(u,w)\}$. If $u \notin (V_0)_G$, then $(u,w) \in B$ and $N_{G[K_p]}((x,y)) \cap V_2^* = \{(u,w)\}$. On the other hand, if $u \in (V_0)_G$, then $(u,v) \in A$ and $N_{G[K_p]}((x,y)) \cap V_2^* = \{(u,v)\}$.

Case 2: Suppose that $(x,y) \in C$ and let $z \in V(K_p) \setminus \{y\}$ for which $(x,z) \in V_0$. Since $(x,y)(x,z) \in E(G[K_p])$ and $(x,y) \in V_2$, $N_{G[K_p]}((x,z)) \cap V_2 = \{(x,y)\}$. This means that $(x,w) \notin V_2$ for all $w \in V(K_p)\setminus\{y\}$ and $(u,w) \notin V_2$ for all $u \in N_G(x)$ and for all $w \in V(K_p)$. Thus, $N_{G[K_p]}((x,y)) \cap V_2^* = N_{G[K_p]}((x,y)) \cap A = \{(x,v)\}$.

Accordingly, the function $g = (V_0^*, V_1^*, V_2^*) \in PRD(G[K_p])$. Since $V_1^* = V_1$ and $|V_2^*| \leq |V_2|$, $\omega_{G[K_p]}(f) \geq \omega_{G[K_p]}(g)$. Because f is a γ_R^P-function of $G[K_p]$, $\omega_{G[K_p]}(f) = \omega_{G[K_p]}(g)$ and g is a γ_R^P-function of $G[K_p]$.

Define the function $h = (V_0^h, V_1^h, V_2^h)$ on G by

$$h(x) = \begin{cases} 2, & \text{if } x \in (V_2^*)_G\,; \\ 1, & \text{if } x \in (V_1^*)_G \setminus (V_2^*)_G\,; \\ 0, & \text{else.} \end{cases}$$

Let $x \in V_0^h$. Then $(x,y) \in V_0^*$ for all $y \in V(K_p)$. Pick $y \in V(K_p)$. There exists a unique $(u,v) \in V_2^*$ for which $(x,y)(u,v) \in E(G[K_p])$. It follows that $u \in V_2^h$ and $ux \in E(G)$. Moreover, u is unique in this sense as (u,v) is for (x,y). Thus, $h \in PRD(G)$.

Finally, let $x, u \in V_2^h$ for which $xu \in E(G)$. Let $y, v \in V(K_p)$ such that $(x,y), (u,v) \in V_2^*$. Since g is a γ_R^P-function of $G[K_p]$, $(x,a), (u,b) \in V_1^*$ for all $a \in V(K_p) \setminus \{y\}$ and for all

$b \in V(K_p) \setminus \{v\}$. On the other hand, by the definition of h, for each $x \in V_1^h$, $(x,y) \in V_1^*$ for all $y \in V(K_p)$. Thus, $|V_1^*| \geq p|V_1^h| + (p-1)|V_2^h \cap N_G(V_2^h)|$. Therefore,

$$\begin{aligned} \gamma_R^P(G[K_p]) = \omega_{G[K_p]}(g) &= |V_1^*| + 2|V_2^*| \\ &\geq p|V_1^h| + (p-1)|V_2^h \cap N_G(V_2^h)| + 2|V_2^h| \\ &= (p-1)\left(|V_1^h| + |V_2^h \cap N_G(V_2^h)|\right) + \omega_G(h) \\ &\geq \alpha. \end{aligned}$$

The desired equality is completed by Proposition 2.18 ■

Equality in Proposition 2.18 is possible even if H is not complete. Consider the graph $G[P_3]$ in Figure 4, with G being the caterpillar graph $ca(0,2,0,2,0)$. Observe that $\alpha = 7$.

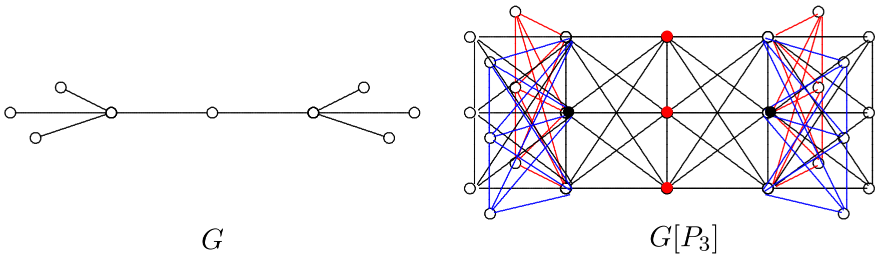

Figure 4: Graph G with $\gamma_R^P(G[P_3]) = 7$

On the other hand, $\gamma_R^P(G[P_3]) = 7$, which is determined by $(V_0, V_1, V_2) \in PRD(G[P_3])$, where V_1 and V_2 are the sets of all red and all black vertices, respectively, in $G[P_3]$ and $V_0 = V(G[P_3]) \setminus (V_1 \cup V_2)$.

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References

[1] H. Abdollhzadeh Ahangar. M. Chellali and S.M. Sheikholeslami, Outer independent double Roman domination. *Appl. Math. Comput.*, 364(124617), 2020.

[2] A. Alhashim. W. Desormeaux and T. Haynes. Roman domination in complementary prisms. *Australian Journal of Combinatorics*, 68(2):218–228, 2017.

[3] J. Arquilla and H. Fredricksen. "Graphing" an optimal grand strategy. *Military Operations Research*, 1:3–19, 1995.

[4] C. Berge. *The theory of Graphs and its Applications* . Wiley, New York, 1962.

[5] F. Buckley and F. Harary. *Distance in Graphs .* Addison-Wesley, Redwood City, CA, 1990.

[6] S. Canoy. R. Mollejon and J.G. Canoy. Hop dominating sets in graphs under binary operations. *European Journal of Pure and Applied Mathematics*, 12(4):1455–1463, 2019.

[7] B. Chaluvaraju and V. Chaitra. Roman domination in complimentary prism. *International J.Math. Combi.*, 2:24–31, 2012.

[8] E. Cockayne. P. Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi. Roman domination in graphs. *Discrete Mathematics*, 278:11–22, 2004.

[9] E. Cockayne and S. Hedetniemi. Towards a theory of domination in graphs. *Networks*, 7(3):1977, 1977.

[10] P. Dankelmann, D. Day, D. Erwin, S. Mukwembi, and H. Swart. Domination with exponential decay. *Discrete Mathematics*, 309:5877–5883, 2009.

[11] W. Desormeaux. T.W. Haynes and M.A. Henning. An extremal problem for total domination stable graphs upon edge removal. *Discrete Applied Mathematics*, 159:1048–1052, 2011.

[12] O. Favaron. , H. Karami and R. Khoeilar and S.M. Sheikholeslami. On the Roman domination number of a graph. *Discrete Mathematics*, 309:3447–3451, 2009.

[13] T. Haynes, S.T. Hedetniemi, and P.J. Slater. *Fundamentals of Domination in Graphs .* Marcel Dekker, Inc., New York, 1998.

[14] M. Henning. W. Klostermeyer and G. MacGillivray. Perfect Roman domination in trees. *Discrete Applied Mathematics*, 236:235–245, 2018.

[15] M. Henning and W. Klostermeyer. Perfect Roman domination in regular grahs. *Applicable Analysis and Discrete Mathematics*, 12(1):143–152, 2018.

[16] Y. Kwon and J. Lee. Perfect dominating sets in Cayley graphs. *Discrete Applied Mathematics*, 162:259–263, 2014.

[17] C.S. Revelle and K.E. Rosing. Defendens imperium romanum: a classical problem in military strategy. *Amer. Math. Monthly*, 107(7):585–594, 2000.

[18] I. Stewart. Defend the Roman Empire. *Sci. Amer.*, 281(6):136–139, 1999.

[19] J. Yue and J. Song. Note on the perfect Roman domination number of graphs. *Applied Mathematics and Computation*, 364:1–5, 2020.

[20] J. Yue, M. Wei, M. Li, and G. Liu. On the double roman domination of graphs. *Applied Mathematics and Computation*, 338:669–675, 2018.