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On γ-Sets in Rings

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Abstract. Let R be a ring with identity 1_R. A subset J of R is called a γ-set if for every $a \in R\backslash J$, there exist $b, c \in J$ such that $a + b = 0$ and $ac = 1_R = ca$. A γ-set of minimum cardinality is called a minimum γ-set.
In this study, we identified some elements of R that are necessarily in a γ-sets, and we presented a method of constructing a new γ-set.
Moreover, we gave: necessary and sufficient conditions for rings to have a unique γ-set; an upper bound for the total number of minimum γ-sets in a division ring; a lower bound for the total number of minimum γ-sets in a division ring; necessary and sufficient conditions for $T(x)$ and T to be equal; necessary and sufficient conditions for a ring to have a trivial γ-set; necessary and sufficient conditions for an image of a γ-set to be a γ-set also; necessary and sufficient conditions for a ring to have a trivial γ-set; and, necessary and sufficient conditions for the families of γ-sets of two division rings to be isomorphic.

2020 Mathematics Subject Classifications: 16

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1. Introduction

Let G be a group with identity e. A subset D of G is called a $\mathscr{D}$-set of G if for every x in $G\backslash D$, there exists $y \in D$ such that $xy = e = yx$. In other words, a subset of a group G is a $\mathscr{D}$-set only if every element not in D has its inverse in D. A smallest $\mathscr{D}$-set of G is called a *minimum* $\mathscr{D}$-set of G. The number of minimum $\mathscr{D}$-set of G is called the *index minimum.* If G is a finite group and $S = \{s \in G : s^2 = e\}$ (the elements of S will be called *involutions*), then the c-number of G is given by $|(G\backslash S)|/2$.

Let R be a ring with identity 1_R. A subset J of R is called a γ-set of R if for every $a \in R\backslash J$, there exist $b, c \in J$ such that $a + b = 0$ and $ac = 1_R = ca$. For example, consider the field $\mathbb{Z}_5$. Then the γ-sets of $\mathbb{Z}_5$ are $\{0, 1, 4, 2\}$, $\{0, 1, 4, 3\}$, and $\mathbb{Z}_5$. A γ-set of a finite

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ring having minimum cardinality is called a *minimum* γ-set. For example, $\{0, 1, 4, 2\}$ and $\{0, 1, 4, 3\}$ are minimum γ-sets $\mathbb{Z}_5$.

Here after please refer to [4], [5], [6], [7], [8], [9], [10] for the other concepts.

Motivated by the concept dominating sets in graphs, Buloron *et al.* [3] introduced the concept $\mathscr{D}$-set in a group. The concept $\mathscr{D}$-set uses the idea of dominating sets in some sense. For example, a $\mathscr{D}$-set E in a group requires that every element not in E must have its inverse in E in the same way that a dominating set D in a graph requires every element not in D must be a neighbor of some element in D.

Buloron *et al.* [3] gave some fundamental properties of $\mathscr{D}$-sets and some characterizations. Ontolan *et al.* [12] gave the number of minimum $\mathscr{D}$-sets in a group. Corcino *et al.* [2] presented some isomorphism results for some families of $\mathscr{D}$-sets.

Rosero and Baldado [1] continued the study of $\mathscr{D}$-sets by investigating the $\mathscr{D}$-sets that are generated by a set. Moreover, they introduced and investigated a parallel concept for rings, called γ-sets [11].

In this study, we continued the investigation of γ-sets.

2. Preliminary Results

This section presents some elementary properties of a γ-set. We denote by T_R the set of all γ-sets of R. Note that $T_R \neq \varnothing$ since R is a γ-set.

The next theorem, Theorem 1, is taken from [3]. It shows that the set of all γ-sets in a ring is a semi-group under the set operation union, and the set $T_R^C = \{J^C : J \text{ is a } \gamma\text{-set}\}$ is a semi-group under the set operation intersection.

Theorem 1. *[3] Let R be a ring with identity 1_R. Let T_R be the set of all γ-sets of R and $T_R^C = \{J^C : J \text{ is a } \gamma\text{-set}\}$. Then*

a.) The set T_R is a semi-group under the set operation union;

b.) The set T_R^C is a semi-group under the set operation intersection.

Remark 1 (b) is found in [4], while (c) is an exercise in [6] (Prob 24E, Chapter 5.1). Remark 1 (d) is a contrapositive of (c).

Remark 1. *Let R be a ring with identity 1_R and $a \in R$.*

a.) If a is a unit, then $-(a^{-1}) = (-a)^{-1}$.

b.) If a is a unit, then so is $-a$.

c.) If a is a unit, then a is not a zero divisor.

d.) If a is a zero divisor, then a is not a unit.

Remark 2 is clear, and sometimes are given in the exercises of some books.

Remark 2. *Let R be a ring with identity $1_R \neq 0$ and a be a unit of R.*

a.) $-a = a^{-1}$ *if and only if* $a = (-a)^{-1}$.

b.) $-a \neq a^{-1}$ *if and only if* $a \neq (-a)^{-1}$

c.) $a^2 = 1_R$ *if and only if* $-a = (-a)^{-1}$.

d.) $a^2 \neq 1_R$ *if and only if* $-a \neq (-a)^{-1}$.

e.) $2a = 0$ *if and only if* $(a)^{-1} = (-a)^{-1}$.

f.) $2a \neq 0$ *if and only if* $(a)^{-1} \neq (-a)^{-1}$.

Theorem 2, identified the elements of a ring that are necessarily in a γ-sets.

Theorem 2. *Let R be a ring with identity $1_R \neq 0$ and J be a γ-set of R.*

a.) If $2a = 0$, *then* $a \in J$.

b.) If $a^2 = 1_R$, *then* $a \in J$.

c.) If a is not a unit, then $a \in J$.

d.) If a is a zero-divisor, then $a \in J$.

Proof. Let R be a ring with identity $1_R \neq 0$ and J be a γ-set of R. (a) Assume that $2a = 0$ and $a \notin J$. Since J is a γ-set, there exists $b \in J$ such that $a + b = 0$. Hence, $a = a + 0 = a + (a + b) = (a + a) + b = 2a + b = 0 + b = b$, that is $a = b$. This is a contradiction.

(b) Assume that $a^2 = 1_R$ and $a \notin J$. Since J is a γ-set, there exists $c \in J$ such that $ac = 1_R = ca$. Hence, $a = a1_R = a(ac) = (aa)c = 1_Rc = c$, that is $a = c$. This is a contradiction.

(c) If a is not a unit, then a has no multiplicative inverse. Clearly, a is necessarily in J.

(d) If a is a zero-divisor, then by Remark 2 (b), a is not a unit. Hence, by (c) a must be in J. □

3. Constructing a γ-Set

In this section, we presented a method of constructing a γ-set from a γ-set.

The next theorem, Theorem 3, says that a unit a with $a^2 \neq 1_R$ and $2a \neq 0$ determines a γ-set.

Theorem 3. *Let R be a ring with identity $1_R \neq 0$, and J is a γ-set of R. If a is a unit with* $a^2 \neq 1_R$ *and* $2a \neq 0$, *then* $\big(J\backslash\{a, (-a)^{-1}\}\big) \cup \{a^{-1}, -a\}$ *and* $\big(J\backslash\{a^{-1}, -a\}\big) \cup \{a, (-a)^{-1}\}$ *are γ-sets of R.*

Proof. Let R be a ring with identity $1_R \neq 0$, and J is a γ-set of R. Let a be a unit of R with $a^2 \neq 1_R$ and $2a \neq 0$. Then by Remark 2 (d) and Remark 2 (f), $-a \neq (-a)^{-1}$ and $a^{-1} \neq (-a)^{-1}$. Consider $J_1 = (J\backslash\{a, (-a)^{-1}\}) \cup \{a^{-1}, -a\}$ and $J_2 = (J\backslash\{a^{-1}, -a\}) \cup \{a, (-a)^{-1}\}$.

Claim 1. $J_1 = (J\backslash\{a, (-a)^{-1}\}) \cup \{a^{-1}, -a\}$ is a γ-set

To show Claim 1 consider the following cases:

Case 1. $a \notin J$

If $a \notin J$, then $J = J_1$. Hence J_1 is a γ-set.

Case 2. $a \in J$

If $a \in J$, then let $b \in R\backslash J_1$ and consider the following subcases:

Subcase 1. $b \neq a$ and $b \neq (-a)^{-1}$

If $b \neq a$ and $b \neq (-a)^{-1}$, then $b \in R\backslash J \cup \{a^{-1}, -a\}$. Since J is a γ-set, there exist $c, d \in J_1$ such that $b + c = 0 = c + b$ and $bd = 1_R = db$.

Subcase 2. $b = a$

If $b = a$, then $a + (-a) = 0 = (-a) + a$ and $aa^{-1} = 1_R = a^{-1}a$.

Subcase 3. $b = (-a)^{-1}$

If $b = (-a)^{-1}$, then by Remark 1 (a) $(-a)^{-1} + a^{-1} = -a^{-1} + a^{-1} = 0 = a^{-1} + -a^{-1} = a^{-1} + (-a)^{-1}$ and $(-a)^{-1}(-a) = 1_R = (-a)(-a)^{-1}$.

This shows the claim.

Claim 2. $J_2 = (J\backslash\{a^{-1}, -a\}) \cup \{a, (-a)^{-1}\}$ is a γ-set

Proved similarly. □

Let R be a ring with identity $1_R \neq 0$, and J is a γ-set of R. A unit a with $a^2 \neq 1_R$ and $2a \neq 0$ is called a *super-couple*. Theorem 5 suggests that every super-couple determines a minimum γ-set, in the same way as in [3] that every non-involution determines a $\mathscr{D}$-set.

Theorem 4 give some of the conditions wherein a ring R has a unique γ-set, that is, $|T_R| = 1$.

Theorem 4. *Let R be a ring with identity $1_R \neq 0$ and J be a γ-set of R. Then $|T_R| > 1$ if and only if there exists a unit $u \in R$ such that $u^2 \neq 1_R$ and $2u \neq 0$.*

Proof. Let R be a ring with identity $1_R \neq 0$ and J be a γ-set of R. Suppose that $|T_R| > 1$. Then there exists a γ-set J in R with $J \neq R$. Let $x \in R\backslash J$. Since J is a γ-set, there exists $y, z \in J$ such that $x + y = 0 = y + x$ and $xz = 1_R = zx$. Thus, x is a unit. Moreover, since $y, z \in J$ and $x \in R\backslash J$, $x \neq y$ and $x \neq z$. Hence, by Remark 2 (d) and Remark 2 (f), $x^2 \neq 1_R$ and $2x \neq 0$, respectively.

Conversely, assume that there exists a unit $x \in R$ such that $x^2 \neq 1_R$ and $2x \neq 0$. Then by Theorem 5, $\big(R\backslash\{x^{-1}, -x\}\big) \cup \{x, (-x)^{-1}\}$ is a nontrivial γ-sets of R. Therefore, $|T_R| > 1$. □

Theorem 5. *Let R be a ring with identity $1_R \neq 0$ and J be a γ-set of R. Then $|T_R| = 1$ if and only if for all $a \in R$ either $a^2 = 1_R$ or $2a = 0$ or a is a zero-divisor.*

Proof. Proved similarly. □

4. An Equivalence Relation in $R\backslash S$

In this section, we presented an equivalence relation in $R\backslash S$ which will be useful in the next section.

Lemma 1. *Let R be a division ring and $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. The relation $\sim$ on $R\backslash S$ given by $x \sim y$ if and only if $x = y$ or $x = y^{-1}$ or $x = y$ or $x = (-y)^{-1}$ is an equivalence relation.*

Proof. Let R be a division ring and $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. Define a relation $\sim$ on $R\backslash S$ as follows: $x \sim y$ if and only if $x = y$ or $x = y^{-1}$ or $x = (-y)^{-1}$. Since $x = x$ for all $x \in R$, we have $x \sim x$ for all $x \in R\backslash S$. Hence, $\sim$ is reflexive. It can easily be shown that $\sim$ is symmetric and transitive. Thus, $\sim$ is an equivalence relation. □

Remark 3. *Let R be a division ring, and let $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. The equivalence relation $\sim$ in $R\backslash S$ of Lemma 1 partitions $R\backslash S$ into equivalence classes $[a] = \{x \in R\backslash S : x \sim a\} = \{x \in R\backslash S : x = a,$ or $x = a^{-1},$ or $x = -a,$ or $x = (-a)^{-1}\}$.*

5. Some Bounds on the Number of Minimum γ-Set

In this section, we established a sharp upperbound and a sharp lowerbound for the number of minimum γ-set in a finite division ring.

If R is a finite division ring, then we denote the partition of $R\backslash S$ in Remark 3 by $\mathscr{C} = \{[a_1], [a_2], \ldots, [a_c]\}$. In this case, we call c the c-number of R.

Lemma 2. *Let R be a finite division ring, and let $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. If $\mathscr{C} = \{[a_1], [a_2], \ldots, [a_c]\}$ is the partition of $R\backslash S$ in the sense of Remark 3, then $2 \leq |[a_i]| \leq 4$.*

Proof. Let R be a finite division ring and let $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. If $\mathscr{C} = \{[a_1], [a_2], \ldots, [a_c]\}$ is the partition of $R\backslash S$ in the sense of Remark 3, then $[a_i] = \{a_i, -a_i, a_i^{-1}, -a_i^{-1},\}$ for all $i = 1, 2, \ldots, c$. Since $a_i^2 \neq 1_R$ and $2a_i \neq 0$ for all i, Remark 2 implies that $\{a_i, -a_i^{-1}\} \cap \{-a_i, a_i^{-1}\} = \varnothing$ for all i. If $a_i = -a_i^{-1}$, then by Remark 2, $-a_i = a_i^{-1}$. Hence, in this case $|[a_i]| = 2$. On the hand, if $a_i \neq -a_i^{-1}$, then by Remark 2, $-a_i \neq a_i^{-1}$. Hence, in this case $|[a_i]| = 4$. Accordingly, $2 \leq |[a_i]| \leq 4$. □

By Theorem 3, each equivalence class $[a_i]$ determines two minimum γ-sets.

Lemma 3. *Let R be a finite division ring, and let $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. Then, $c \geq (|R| - |S|)/4$.*

Proof. Let R be a finite division ring and let $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. By Lemma 2, $|[a_i]| \leq 4$. Therefore, $c \geq (|R| - |S|)/4$. □

Lemma 4. *Let R be a finite division ring, and let $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. Then, $c \leq (|R| - |S|)/2$.*

Proof. Let R be a finite division ring and let $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. By Lemma 2, $2 \leq |[a_i]|$. Therefore, $c \leq (|R| - |S|)/2$. □

Theorem 6 give a necessary and sufficient condition for a γ-set to be minimum.

Theorem 6. *Let R be a finite division ring. Then, E is a minimum γ-set of R if and only if $E = S \cup \{-x_1, x_1^{-1}, -x_2, x_2^{-1}, \ldots, -x_c, x_c^{-1}\}$ where $x_i \in [a_i]$ for $i = 1, 2, \ldots, c$, and $\{[a_1], [a_2], \ldots, [a_c]\}$ is the partition of $R \backslash S$ in the sense of Remark 3.*

Proof. Suppose that E is a minimum γ-set of R and E is not of the form $S \cup \{-x_1, x_1^{-1}, -x_2, x_2^{-1}, \ldots, -x_c, x_c^{-1}\}$. If E is not of the form $S \cup \{-x_1, x_1^{-1}, -x_2, x_2^{-1}, \ldots, -x_c, x_c^{-1}\}$, then there exists $i \in \{1, 2, \ldots, j\}$ such that $-x_i, x_i^{-1}, (-x_i)^{-1} \in E$ or $x_i, -x_i, x_i^{-1}, (-x_i)^{-1} \in E$. Thus, $S \cup \{-x_1, x_1^{-1}, -x_2, x_2^{-1}, \ldots, -x_c, x_c^{-1}\}$ is a γ-set smaller than E. This is a contradiction.

Conversely, suppose that E is of the form $S \cup \{-x_1, x_1^{-1}, -x_2, x_2^{-1}, \ldots, -x_c, x_c^{-1}\}$ and E is not a minimum γ-set of R. If E is not a minimum γ-set of R, then then there exists $i \in \{1, 2, \ldots, j\}$ such that $x_i, -x_i, x_i^{-1} \notin E$. Since E is a γ-set and x_i^{-1}, $x_i = (x_i^{-1})^{-1} \in E$. This is a contradiction. □

The *index minimum* of a finite ring R is the number of minimum γ-sets of R and is denoted by $ind(R)$. Corollary 1 gives an upper bound on the number of minimum γ-set of a finite division ring, while Corollary 2 gives a lower bound on the number of minimum γ-set of a finite division ring.

Corollary 1. *Let R be a finite division ring. Then $ind(R) \leq 2^{(|R|-|S|)/2}$.*

Proof. Let R be a finite division ring and let $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. Then by Lemma 4, $c \leq (|R|-|S|)/2$. Moreover, by Theorem 3, each equivalence class $[a_i]$ determines two minimum γ-sets. Therefore, by the *multiplication principle*, $ind(R) \leq 2^{(|R|-|S|)/2}$. □

The bound in Corollary 1 is sharp. Equality holds for some fields. For example, if R is $\mathbb{Z}_5$, then the equality holds. To see this, we note that the minimum γ-sets of $\mathbb{Z}_5$ are $J_1 = \{0, 1, 4, 3\}$ and $J_1 = \{0, 1, 4, 3\}$. Hence, $ind(\mathbb{Z}_5) = 2$. This is equal to $2^{(|R|-|S|)/2} = 2^{(|\mathbb{Z}_5|-|\{0,1,4\}|)/2} = 2^{(5-3)/2} = 2$.

Corollary 2. *Let R be a finite division ring. Then $ind(R) \geq 2^{(|R|-|S|)/4}$.*

Proof. Let R be a finite division ring and let $S = \{x \in R : x^2 = 1_R$ or $2x = 0\}$. Then by Lemma 3, $c \geq (|R|-|S|)/4$. Moreover, by Theorem 3, each equivalence class $[a_i]$ determines two minimum γ-sets. Therefore, by *multiplication principle*, $ind(R) \leq 2^{(|R|-|S|)/2}$. □

The bound in Corollary 2 is also sharp. Equality holds for some fields. For example, if R is $\mathbb{Z}_7$, then the equality holds. To see this, we note that the minimum γ-sets of $\mathbb{Z}_7$ are $J_1 = \{0, 1, 2, 3, 6\}$ and $J_2 = \{0, 1, 4, 5, 6\}$. Hence, $ind(\mathbb{Z}_7) = 2$. This is equal to $2^{(|R|-|S|)/2} = 2^{(|\mathbb{Z}_7|-|\{0,1,6\}|)/2} = 2^{(7-3)/4} = 2$.

6. γ-Sets and Homomorphism of Rings

In this section, we gave some properties of γ-sets in relation to its homomorphic image.

We say that a set A precedes a set B if there exists an injective map from A to B. In this case, we write $A \preceq B$.

Theorem 7. *Let J be a γ-set of a ring. Then $R \backslash J \preceq J$.*

Proof. Let $f : R\backslash J \rightarrow J$ be a given by $f(x) = x^{-1}$, and let $x, y \in J$ with $x = y$. Since J is a γ-set and each unit of a ring has a unique multiplicative inverse, $x = y$ implies that $f^{-1}(x) = f^{-1}(y)$. This means that f is injective. Hence, $R\backslash J \preceq J$. □

Theorem 7 says that in a finite ring, a γ-set has more elements than its complement.

Lemma 5. *Let R be a ring, and $x \in R$. Then $T(x) = \{J \subseteq R : J$ is a γ-set and $x \in J\}$ is a semigroup under the operation union.*

Proof. It suffices to show that $T(x)$ is closed under the operation union. Let J_1 and J_2 be element of $T(x)$. Then by Theorem 1, $J_1 \cup J_2$ is a γ-set. Since clearly $x \in J_1 \cup J_2$, $J_1 \cup J_2 \in T(x)$. □

An element x of a ring R is called an *involution* if $x^2 = 1_R$, or $2x = 0$, or $x \neq -x^{-1}$.

Theorem 8. *Let x be a non-identity element of a ring. Then x is an involution if and only if $T(x) = T$.*

Proof. Let R be a ring, and suppose that x is an involution of R. If x is an involution, then by Theorem 2 x is contained in every γ-set of R. Thus, if J is a γ-set, then $J \in T(x)$, that is $T \subseteq T(x)$. Since clearly $T \supseteq T(x)$, we must have $T = T(x)$.

Conversely, assume that $T(x) = T$ and x is not an involution. If x is not an involution and $x \neq 1_R$, then $x \neq x^{-1}$. By Theorem 3, $J' = \big(J\backslash\{x, (-x)^{-1}\}\big) \cup \{x^{-1}, -x\}$ is also a γ-set. Note that $J' \in T$, but $J' \notin T(x)$, that is $T(x) \neq T$. This is a contradiction. □

Theorem 9. *Let R be a ring with identity and x be a non-zero element of R with $2x = 0$. Then every γ-set of R contains a non-trivial subring.*

Proof. Let J be a γ-set of a ring R and x be a non-zero element of R with $2x = 0$. Theorem 2 implies that 0 and x are elements of J. Thus, $\{0, x\}$ is a subring of R contained in J. □

Theorem 10. *Let R be a ring with identity. R has a trivial γ-set if and only if R is trivial.*

Proof. Let $R = \{0, 1\}$. Then clearly $\{0, 1\}$ is a γ-set of R.

Conversely, suppose that $J = \{0, 1\}$ is a γ-set of R and R is non-trivial. Let $x \in R$ with $x \neq 0$ and $x \neq 1_R$. Since J is a γ-set of R, there exists $y, z \in J$ such that $x + y = 0$ and $xz = 1_R$. Since the elements of J are 0 and 1 only, this implies that $x + 1 = 0$, that is $x = -1$. Hence, $x^2 = (-1)^2 = 1_R$. By Theorem 2, $x \in J$. This is a contradiction. □

Theorem 11. *Let T be the set of all γ-sets of a division ring R, and $S = \{x \in R : x^2 = 1_R\} \cup \{0\}$. Then $|T| = 1$ if and only if $R = S$.*

Proof. Assume that $|T| = 1$, and $R \neq S$. If $R \neq S$, then there exists $x \in R\backslash S$ such that $x^2 = 1_R$. Let $J \in T$ and consider the following cases:
Case 1. $x \notin J$
If $x \in J$, then by Theorem 3, $J' = \big(J\backslash\{x^{-1}, -x\}\big) \cup \{x, (-x)^{-1}\}$ is another γ-set. This is a contradiction.
Case 2. $x \in J$
If $x \in J$, then by Theorem 3, $J' = \big(J\backslash\{x, (-x)^{-1}\}\big) \cup \{x^{-1}, -x\}$ is another γ-set. This is a contradiction.

Conversely, suppose that $R = S$. Since R is a division ring, every non-zero element is an involution. Hence, by Theorem 10 if J is a γ-set of R, we must have $J = R$, that is R is the only γ-set R. Thus, $|T| = 1$. □

Theorem 12. *Let Q be a subring of R, and J be a γ-set of R. Then J is a γ-set of Q if and only if $Q = R$.*

Proof. Assume that J is a γ-set of Q, and $Q \neq R$. If $Q \neq R$, then there exists $x \in R\backslash Q$. Since $x \notin Q$, $x \notin J$. Since J, $x^{-1} \in J$. Since J is also a γ-set of Q, $x = xx^{-1} \in T$. This is a contradiction. The converse is clear. □

Theorem 13. *Let R_1 and R_2 be rings, and $\phi : R_1 \to R_2$ be an epimorphism of rings. If J is a γ-set of R_1, then $\phi(J)$ is a γ-set of R_2.*

Proof. Let J be a γ-set of R_1, and $y \in R_2\backslash\phi(J)$. If $y \in R_2\backslash\phi(J)$ and ϕ is an epimorphism, then there exists $x \in R_1$ such that $\phi(x) = y$. Note that $x \in R_1\backslash J$, otherwise $y = \phi(x) \in \phi(J)$. Since J is a γ-set, there exists $u, v \in J$ such that $xu = 1_{R_1}$ and $x + v = 0_{R_1}$. Clearly, $\phi(u), \phi(v) \in \phi(J)$. Since ϕ is a homomorphism, $\phi(x)\phi(u) = \phi(xu) = \phi(1_{R_1}) = 1_{R_2}$ and $\phi(x) + \phi(v) = \phi(x + v) = \phi(0_{R_1}) = 0_{R_2}$. Hence, there exists $\phi(u), \phi(v) \in \phi(J)$ such that $\phi(x)\phi(u) = 1_{R_2}$ and $\phi(x) + \phi(v) = 0_{R_2}$. This shows that $\phi(J)$ is a γ-set of R_2. □

Theorem 14. *Let R_1 and R_2 be rings, and $\phi : R_1 \to R_2$ be an isomorphism of rings. Then, J is a γ-set of R_1 if and only if $\phi(J)$ is a γ-set of R_2.*

Proof. Let J be a γ-set of R_1. Then by Theorem 7, $\phi(J)$ is a γ-set of R_2.

Conversely, let J be a subset of R_1, and suppose that $\phi(J)$ is a γ-set of R_2. Let $x \in R_1\backslash J$. Then $\phi(x) \in R_2\backslash\phi(J)$, otherwise $x = \phi^{-1}\phi(x) \in J$. Since $\phi(J)$ is a γ-set of R_2, there exists $u, v \in \phi(J)$ such that $\phi(x)u = 1_{R_2}$ and $\phi(x) + v = 0_{R_2}$. Note that $\phi^{-1}(u), \phi^{-1}(v) \in J$, otherwise $u = \phi(\phi^{-1}(u)) \in R_2\backslash\phi(J)$ and $v = \phi(\phi^{-1}(v)) \in R_2\backslash\phi(J)$. Since ϕ is an isomorphism, $x\phi^{-1}(u) = \phi^{-1}(\phi(x))\phi^{-1}(u) = \phi^{-1}(\phi(x)u) = \phi^{-1}(1_{R_2}) = 1_{R_1}$ and $x + \phi^{-1}(v) = \phi^{-1}(\phi(x)) + \phi^{-1}(v) = \phi^{-1}(\phi(x) + v) = \phi^{-1}(0_{R_2}) = 0_{R_1}$. Hence, there exists $\phi^{-1}(u), \phi^{-1}(v) \in J$ such that $x\phi^{-1}(u) = 1_{R_1}$ and $x + \phi^{-1}(v) = 0_{R_1}$. This shows that J is a γ-set of R_1. □

Theorem 15. *Let R be a ring, $T = \{J \subseteq R : J$ is a γ-set of $R\}$, and $T' = \{J' : J \in T\}$ where J' is the complement of J. Then, T is isomorphic to T'.*

Proof. Let R be a ring, $T = \{J \subseteq R : J$ is a γ-set of R$\}$, and $T' = \{J' : J \in T\}$. Define $\phi : T \to T'$ by $J \mapsto J'$ where J' is the complement of J. Then clearly ϕ is bijective. Now, let $J_1, J_2 \in T$. Then

$$\begin{aligned} \phi(J_1 \cup J_2) &= (J_1 \cup J_2)' \\ &= J_1' \cap J_2' \\ &= \phi(J_1) \cap \phi(J_2). \end{aligned}$$

This shows that ϕ is an isomorphism, that is T is isomorphic to T' as a semigroup. □

7. Separating γ-Sets

Our objective in this section is to show the statement: Let R and S be rings. Then T_R is isomorphic to T_S if and only if $|G\backslash S_R| = |H\backslash S_S|$.

We borrowed here some ideas presented by Joris N. Buloron in [2] to show the results.

We denote the set of all involutions of a division ring D by S_D, that is, $S_D = \{x \in R : x^2 = 1_D,$ or $2x = 0,$ or $a \neq -a^{-1}\}$.

Let J be a γ-set of a division ring D. Then J is called a *separating* γ-set of D if for every $x \in J\backslash S_D$, $x^{-1} \notin D$. Note that for a finite division ring D, the separating γ-sets are just the minimum γ-sets. Also note that if J is not a *separating* γ-set, then there exists $x \in D\backslash S_D$ such that $x, x^{-1} \in J$.

Lemma 6. *Let D be a division ring and J be a γ-set. If J is not a separating γ-set, then J can be expressed as a union of two distinct separating γ-sets.*

Proof. Let J be a γ-set that is not separating. Define a relation $\sim$ on $J\backslash S_D$ as follows: $x \sim y$ if and only if $x = y$ or $y = x^{-1}$. Then $\sim$ is an equivalence relation, that is, $\sim$ partitions $J\backslash S_D$ into equivalence classes. For each $x \in J\backslash S_D$, the equivalence class containing x is $\bar{x} = \{x, x^{-1}\}$. By the *Axiom of Choice*, there exists a set Δ such that $\Delta \cap \bar{x}$ is a singleton set for all $x \in J\backslash S_D$. It is easy to see that $\Delta \cup S_D$ and $J\backslash\Delta$ is a separating γ-set, and $J = (\Delta \cup S_D) \cup (J\backslash\Delta)$. □

A careful observation would suggest that a separating γ-set cannot be expressed as a union of two distinct γ-sets. The next lemma is anchored on this idea.

Lemma 7. *Let D be a division ring and J be a γ-set of D. J is not a separating $\mathscr{D}$-set if and only if it is a union of two or more distinct γ-sets.*

Proof. Let J be a γ-set of D and assume that J is not a separating γ-set. Then by Lemma 6, $J = E \cup F$ for some separating γ-sets E and F. Note that E and F must be distinct, otherwise, $J = E \cup F = E$ (which is a contradiction since E is a separating γ-set while J is not).

Conversely, assume that $J = E \cup F$ for some γ-sets E and F, with $E \neq F$. If one of E and F is not a separating γ-set, then clearly, $J = E \cup F$ is not a separating γ-set. So we assume that E and F are both separating γ-sets. Since $E \neq F$, $E \backslash F \neq \varnothing$. Let $x \in E \backslash F$. Since F is a γ-set, $x^{-1} \in F$. Hence, $x, x^{-1} \in D$. This implies that D is not a separating γ-set. □

At this point, we will now state some consequence of the above lemma.

The following definitions are helpful in the succeeding statements.

Let x be an element of a division ring D. We denote by $T_D(x)$ the family of all γ-sets containing x, that is, $T_D(x) = \{D \in T_D : x \in D\}$. Similarly, we denote by $T_{sep(D)}(x)$ the family of all separating γ-sets containing x, that is, $T_{sep(D)}(x) = \{D \in T_{sep(D)} : x \in D\}$.

Lemma 8. *Let D be a non-trivial division ring and x be an element of D. Then the following statements are equivalent.*

(i) *x is an involution.*

(ii) $T_D(x) = T_D$.

(iii) $T_{sep(D)}(x) = T_{sep(D)}$.

Proof. (1) ⇒ (2) Suppose that x is an involution and $T_D(x) \neq T_D$. If $T_D(x) \neq T_D$, then there exists a γ-set B such that $x \notin B$. Since x is an involution and B is a γ-set, $x = x^{-1} \in B$. This is a contradiction. Hence, $T_D(x) = T_D$.
(2) ⇒ (1) Suppose that $T_D(x) = T_D$ and x is not an involution. Let J be a γ-set of D and x be a non-involution. Consider the following cases:
Case 1. $x^2 \neq 1_D$
If $x^2 \neq 1_D$, that is, $x \neq x^{-1}$, then we note that $J \backslash \{x\}$ is a γ-set that do not contain x. This is a contradiction since $T_D(x) = T_D$. Therefore, x must be an involution.
Case 2. $2x \neq 0$
If $2x \neq 0$, that is, $x \neq -x$, then we note that $J \backslash \{x\}$ is a γ-set that do not contain x. This is a contradiction since $T_D(x) = T_D$. Therefore, x must be an involution.
Case 3. $x = -x^{-1}$
If $x = -x^{-1}$, then we note that $J \backslash \{x\}$ is a γ-set that do not contain x. This is a contradiction since $T_D(x) = T_D$. Therefore, x must be an involution.
(1) ⇒ (3) Suppose that x is an involution and $T_{sep(D)}(x) \neq T_{sep(D)}$. If $T_{sep(D)}(x) \neq T_{sep(D)}$, then there exists a separating γ-set B such that $x \notin B$. Since x is an involution and B is a γ-set, $x = x^{-1} \in B$. This is a contradiction. Hence, $T_{sep(D)}(x) = T_{sep(D)}$.
(3) ⇒ (1) Suppose that $T_{sep(D)}(x) = T_{sep(D)}$ and x is not an involution. If x is not an involution, then consider the following cases:
Case 1. $x^2 \neq 1_D$

If $x^2 \neq 1_D$, then $x \neq x^{-1}$. Let H be a separating γ-set. Then $(H\backslash\{x\}) \cup \{x^{-1}\}$ is a separating γ-set that do not contain x. This is a contradiction since $T_{sep(D)}(x) = T_{sep(D)}$. Therefore, x must be an involution.

Case 2. $2x \neq 0$

If $2x \neq 0$, then $x \neq -x$. Let H be a separating γ-set. Then $(H\backslash\{x\}) \cup \{-x\}$ is a separating γ-set that do not contain x. This is a contradiction since $T_{sep(D)}(x) = T_{sep(D)}$. Therefore, x must be an involution.

Case 3. $x = -x^{-1}$

If $x = -x^{-1}$, then $x = (-x)^{-1}$. Let H be a separating γ-set. Then $(H\backslash\{x\}) \cup \{-x\}$ is a separating γ-set that do not contain x. This is a contradiction since $T_{sep(D)}(x) = T_{sep(D)}$. Therefore, x must be an involution. □

The next proposition shows that an isomorphism preserves the state of being *separating* in the same way as it preserves other properties.

Lemma 9. *Let D_1 and D_2 be division rings, and $\varphi : T_{D_1} \to T_{D_2}$ be an isomorphism. Then J is a separating γ-set of D_1 if and only if $\varphi(J)$ is a separating γ-set of D_2.*

Proof. Let D_1 and D_2 be division rings, and $\varphi : T_{D_1} \to T_{D_2}$ be an isomorphism. Suppose that J is a separating γ-set of D_1 and $\varphi(J)$ is not a separating γ-set of D_2. If $\varphi(J)$ is not a separating γ-set of D_2, then by Lemma 6, $\varphi(J) = E \cup F$ for some distinct separating γ-sets E and F in D_2. It is easy to see that there exist distinct γ-sets E' and F' such that $\varphi(E') = E$ and $\varphi(F') = F$. Thus, $\varphi(J) = E \cup F = \varphi(E') \cup \varphi(F') = \varphi(E' \cup F')$. Since φ is injective, we have $J = E' \cup F'$. This is a contradiction (by Lemma 6). Therefore, $\varphi(J)$ must be a separating γ-set of D_2.

Conversely, assume that $\varphi(J)$ is a separating γ-set of H and J is not a separating γ-set of D_1. If J is not a separating γ-set of D_1, then by Lemma 6, $J = E \cup F$ for some γ-sets E and F with $E \neq F$. Thus, $\varphi(J) = \varphi(E \cup F) = \varphi(E) \cup \varphi(F)$. Since φ is injective, $\varphi(E) \neq \varphi(F)$. This is a contradiction (by Lemma 6). Therefore, J must be a separating γ-set of D_1. □

Lemma 10. *Let D_1 and D_2 be division rings and $\varphi : T_{D_1} \to T_{D_2}$ be an isomorphism. Let J be a separating γ-set of D_1 and $x \in D_1\backslash J$. Then there exists a unique $y \in D_2\backslash\varphi(J)$ such that $\varphi(J \cup \{x\}) = \varphi(J) \cup \{y\}$.*

Proof. Let D_1 and D_2 be division rings and $\varphi : T_{D_1} \to T_{D_2}$ be an isomorphism. Let J be a separating γ-set of D_1 and $x \in D_1\backslash J$. If $x \in D_1\backslash J$, then $x \notin J$. Note that $\varphi(J) \cup \{x\} \neq \varphi(J)$ since J is a separating γ-set and $\varphi(J) \cup \{x\}$ is not (by Lemma 9). Hence, $(\varphi(J) \cup \{x\})\backslash\varphi(J) \neq \varnothing$. Now, we claim that $(\varphi(J) \cup \{x\})\backslash\varphi(J)$ is singleton. Suppose it is not. Without loss of generality, assume that $\{u, v\} = (\varphi(J) \cup \{x\})\backslash\varphi(J)$. If $\{u, v\} = (\varphi(J) \cup \{x\})\backslash\varphi(J)$, then $u, v \notin \varphi(J)$. Since $\varphi(J)$ is a γ-set $u^{-1}, u^{-1} \in \varphi(J)$. Thus, $A = \varphi(J)$, $B = (\varphi(J)\backslash\{u^{-1}\}) \cup \{u\}$, and $C = (\varphi(J)\backslash\{v^{-1}\}) \cup \{v\}$ are three distinct separating γ-sets. Note that $\varphi(J \cup \{x\}) = A \cup B \cup C$. Hence, $J \cup \{x\} = J \cup \varphi^{-1}(B) \cup \varphi^{-1}(C)$ where J, $\varphi^{-1}(B)$, $\varphi^{-1}(C)$ are three distinct separating γ-sets. This is a contradiction.

Therefore, $(\varphi(J) \cup \{x\}) \backslash \varphi(J)$ must be singleton. Let $y \in (\varphi(J) \cup \{x\}) \backslash \varphi(J)$. Then there exists $y \in D_2 \backslash \varphi(J)$ such that $\varphi(J \cup \{x\}) = \varphi(J) \cup \{y\}$. □

The next result give necessary and sufficient conditions for two division rings to have isomorphic families of γ-set.

Theorem 16. *Let D_1 and D_2 be division rings. Then T_{D_1} is isomorphic to T_{D_2} if and only if there exists a bijection $\sigma : D_1 \backslash S_{D_1} \to D_2 \backslash S_{D_2}$.*

Proof. Let $\varphi : T_{D_1} \to T_{D_2}$ be an isomorphism. Define $\sigma : D_1 \backslash S_{D_1} \to D_2 \backslash S_{D_2}$ as follows. Let J be a separating γ-set of D_1 and $x \in D_1 \backslash S_{D_1}$. Without loss of generality, choose $x \notin J$. If $x \notin J$, then $x \in D_1 \backslash J$. By Lemma 10, there exists $y \in D_2 \backslash \varphi(J)$ with $\varphi(J \cup \{x\}) = \varphi(J) \cup \{y\}$. Now, we define $\sigma(x) = y$ and $\sigma(x^{-1}) = y^{-1}$.

We first show that σ is injective. Let $a, b \in D_1 \backslash S_{D_1}$ with $a \neq b$. Let $J_j = (J \backslash \{a\}) \cup \{a^{-1}\}$ and $J_k = (J \backslash \{b\}) \cup \{b^{-1}\}$. Then $a \in D_1 \backslash J_j$ and $b \in D_1 \backslash J_k$. By Lemma 10, there exist $u \in D_2 \backslash \varphi(J_j)$, and $v \in D_2 \backslash \varphi(J_k)$ such that $\varphi(J_j \cup \{a\}) = \varphi(J_j) \cup \{u\}$ and $\varphi(J_k \cup \{b\}) = \varphi(J_k) \cup \{v\}$. Without loss of generality, assume that $a \notin J$ and $b \notin J$. If $a \notin J$ and $b \notin J$, then $\sigma(a) = u$ and $\sigma(b) = v$. In the sense of the proof of Lemma 10, $\varphi(J) \backslash \varphi(J_j)$ and $\varphi(J) \backslash \varphi(J_j)$ are singleton sets. Thus, if $u = v$, then $\varphi(J \cup \{a\}) = \varphi(J \cup \{b\})$. Since φ is an isomorphism, $J \cup \{a\} = J \cup \{b\}$. Thus, if $a, b \notin J$, then $a = b$. This is a contradiction.This shows that σ is injective.

Next, we show that σ is surjective. Let $y \in D_2 \backslash S_{D_2}$ and J be a separating γ-set of D_1. Without loss of generality, assume that $y \notin J$. If $y \notin J$, then $y \in D_2 \backslash \varphi(J)$. Since φ^{-1} is also an isomorphism, by Lemma 10, there exists $x \in D_1 \backslash J$ such that $\varphi^{-1}(\varphi(J) \cup \{y\}) = J \cup \{x\}$, that is $\varphi(J) \cup \{y\} = \varphi(J \cup \{x\})$. This implies that there exists $x \in D_1 \backslash S_{D_1}$ such that $\sigma(x) = y$. This shows that σ is surjective.

Accordingly, σ is bijective.

For the converse, consider the bijective function $\sigma : D_1 \backslash S_{D_1} \to D_2 \backslash S_{D_2}$ given by $\sigma(x) = y$ and $\sigma(x^{-1}) = y^{-1}$ where $x \notin J$, and y and J are in the same sense as in the above arguments. Define $\varphi : T_{D_1} \to T_{D_2}$ as follows. Let J be in T_{D_1}, then $J = S_{D_1} \cup A$ for some subset A of $D_1 \backslash S_{D_1}$. Let $\varphi(J) = S_{D_2} \cup \sigma(A)$. Then it is easy to show that φ is an isomorphism. □

Corollary 3. *Let D_1 and D_2 be division rings. Then, T_{D_1} is isomorphic to T_{D_2} if and only if $|D_1 \backslash S_{D_1}| = |D_2 \backslash S_{D_2}|$.*

Proof. The given statement follows from Theorem 16. □

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