



On the E-infinity algebras

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Abstract. In this paper we study an elementary use of E -infinity modules and E -infinity algebras as together they have a use in terms of describing triangulated categories. Also, we show an interpretation of E -infinity algebras where the modules are fibrant objects within the categories of differential graded co-algebras and co-modules.

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1. Introduction

An operad homology theory appeared in Mays investigation of iterated loop spaces in [11]. The operad model has some properties of the operations there in, for example, commutativity and associativity encoded in every operad as realized by the algebra involved. The classification of the DG-modules over ring k is defined by dg-mod. Examples of algebra over the operads in dg-mod include A_∞ -algebras and E_∞ -algebras generalizing the concept of associativity and commutativity. An E_∞ -algebra is a DG-module with multiplication present which is the associativity and commutativity up to all higher homotopies. An example of E_∞ -algebras corresponding to $n = \infty$ has been given in [1]. We present the essential explanations and meanings of E_∞ -algebras, E_∞ -modules and their fundamental properties in this study. This is in addition to the presentation of a derived category, finishing up with the depiction of triangulated classes using E_∞ -algebras.

We present an understanding of E_∞ -algebras and E_∞ -modules as the fibrant objects in the category of the model of certain DG-co-algebras (co-modules) following [9]. From the idea of [9], we provide conceptual construction of the E_∞ -functor categories and use it to construct the canonical bi-algebra structure of the cobar-construction of the simplicial complex of an associative algebra.

In the second section, we will present the definition of E_∞ -algebra with some examples and study some theories and properties of it.

In the third part: we explain the idea of co-algebras and the bar-cobar construction. This

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is followed by some relationships and examples.

The fourth section is concerned with the study of morphisms and relationships in E_∞ -algebra.

In the fifth section, we will discuss some important theories in E_∞ -algebra with its proof, and we will also present examples as an application.

2. Mathematical background

We start by briefly recalling the fundamental definitions of E_∞ -algebras and E_∞ -modules. This is to build up a picture of the different relations between them. We'll provide and concentrate on the interpretation of the fibrant objects of the E_∞ -algebras as in the model category of the differential graded co-algebras. For a gentler introduction, see [3], [2] and [10].

For the associated absolute field, we have used F . Since W is denoted as the graded vector space, i.e. $W = \bigotimes_{p \in Z} \mathcal{W}^p$, we define $\mathcal{S}W$ or $W[1]$ as the graded space with $(\mathcal{S}W)^p = \mathcal{W}^{p+1}$ for each $p \in Z$. $\mathcal{S}W$ is the shift of W .

Definition 1. [12]

An operad \mathfrak{S} is comprised of a symmetric monoidal category as part of a collection $\mathfrak{S}(j)_{j \geq 0}$. Each $\mathfrak{S}(j)$ is enriched by the activity of the symmetric group, Σ_j and that of the morphisms.

$$\gamma_{s,r_1,r_2,\dots,r_s} : \mathfrak{S}(s) \otimes \mathfrak{S}(r_1) \otimes \dots \otimes \mathfrak{S}(r_s) \longrightarrow \mathfrak{S}(r_1 + \dots + r_s) \tag{1}$$

for every decision of the components; $s, r_1, \dots, r_s \geq 0$, to the associativity and unit axioms, are fulfilled.

An E_∞ \mathcal{F} -algebra \mathcal{A} is a graded space with a map, $\theta_j : \mathfrak{S}(j) \otimes (SA)^{(j)} \longrightarrow (SA)$ and unit, $\eta : x \longrightarrow A$ to such an extent that the clear associativity, commutativity and the unit diagrams are commutative.

Example 1. [2]

A graded space $\mathcal{A} = \mathcal{M}[\varepsilon]/(\varepsilon^2)$ with the trivial \mathcal{A} -infinity structure given by map m_2 by multiplication of \mathcal{M} , since the maps $m_n = 0$ for each value $n \neq 2$, where \mathcal{M} is the ordinary algebra for $N \geq 1$ and ε be uncertain of degree $(2 - N)$. We characterize the linear map $f : \mathcal{M}^{\otimes N} \longrightarrow \mathcal{M}$ and the deformed multiplication,

$$m'_n = \left\{ \begin{array}{ll} m_n & n \neq N \\ m_N + \varepsilon f & n = N \end{array} \right\}$$

\mathcal{A} endowed with m'_n is \mathcal{A} -infinity algebra if and only if f is Hochschild cocycle for \mathcal{M} .

Definition 2. A weak E_∞ - \mathcal{F} -algebra \mathcal{A} is the graded space with map, $\theta_0 : \mathfrak{S}(j) \otimes x \longrightarrow (SA)$, $\theta_j, j \geq 0$. The morphisms, $f : \mathcal{A} \longrightarrow \mathcal{B}$ of E_∞ -algebras are the maps $\theta_n : \mathfrak{S}(j) \otimes (SA)^{(n)} \longrightarrow (SA)$ homogeneous of the degree zero with the ultimate objective that, $\forall n \geq 1$ we get;

$$\sum_{i+j+l=n} f_{i+1+l} \circ (1^{\otimes i} \otimes b_j \otimes 1^{\otimes l}) = \sum_{i_1+\dots+i_s=n} b_s \circ (f_{i_1} \otimes \dots \otimes f_{i_s}) \tag{2}$$

For any two morphisms (f, g) as $(f \circ g)$ is given as,

$$(f \circ g)_j = \sum_{i_1 + \dots + i_s = j} f_s \circ (g_{i_1} \otimes \dots \otimes g_{i_s}) \tag{3}$$

Proposition 1. [10]

For each E -infinity algebra \mathcal{A} , there is the universal E -infinity algebra morphism $\phi : U(\mathcal{A}) \rightarrow \mathcal{A}$ to a differential graded algebra $U(\mathcal{A})$. Moreover, the morphism ϕ is an E -infinity quasi isomorphism.

Proposition 2. If \mathcal{A} be E -infinity algebra and $f_1 : \mathcal{A} \rightarrow \mathcal{V}$ is a quasi-isomorphism (semi-isomorphism) of complexes since \mathcal{V} is the complex. Then the complex \mathcal{V} admits a structure of E -infinity algebra s.h. f_1 extends to an E -infinity quasi-isomorphism $f_1 : \mathcal{A} \rightarrow \mathcal{V}$.

Theorem 1. If \mathcal{A} is an E -infinity algebra, then $\mathcal{H}^*(\mathcal{A})$ admits an E -infinity algebra structure since:

- (1) $b_1 = 0$, b_2 is induced from $b_2^{\mathcal{A}}$,
- (2) The identity element in the homology is induced by the E -infinity quasi-isomorphism $\mathcal{A} \rightarrow \mathcal{H}^*(\mathcal{A})$.

Note that E_∞ -quasi-isomorphism is trivial. If, $b_1 = 0$ then E_∞ -algebra is minimal. The minimal model of an E -infinity algebra \mathcal{A} is the space $\mathcal{H}^*(\mathcal{A})$ endowed by the structure provided by the theorem.

Definition 3. The Yoneda product is characterized between Ext -groups over general rings. However, for algebras over fields, the presentation can be simplified using canonical resolutions. For any associative algebra \mathcal{B} with a unital, there is a projective resolution $P \rightarrow \mathcal{M}$ and a right \mathcal{B} -module \mathcal{M} . Let the DG-endomorphism algebra $\mathcal{A} = Hom_{\mathcal{B}}(P, P)$ of P with the n^{th} part of \mathcal{A} comprise of the graded object morphisms of degree n where its differential is the super commutator with a differential of P . Thus, \mathcal{A} is specifically an E -infinity algebra with a minimal model. The homology $\mathcal{H}_*^{E_n}(\mathcal{A})$ is isomorphic for m_2 to $Ext_{\mathcal{B}}^*(\mathcal{M}, \mathcal{M})$, which is the Yoneda algebra.

Definition 4. Let \mathcal{A} be the strict unit for an E -infinity algebra. It is an element, $1 \in E^0$, which is the unit of m_2 and such that, for $n \neq 2$, the map b_n takes the value 0 when one of its contentions rises to 1. $\mathcal{H}_*^{E_n}(\mathcal{A})$ is the homological unit for associative algebra \mathcal{A} with the multiplication induced by m_2 . Consequently, the Homological unitality is saved under E -infinity quasi-isomorphism.

Proposition 3. Each homologically unital E -infinity algebra is E -infinity quasi-isomorphic to a strictly unital E -infinity algebra. An E -infinity module over \mathcal{A} is a spectrum \mathcal{M} with maps,

$$\lambda_j : \mathfrak{S}(j) \times (\mathcal{A}^{j-1} \wedge \mathcal{M}) \rightarrow \mathcal{M} \tag{4}$$

that are λ_j , namely suitably, unital, associative, and equivalent.

An E -infinity module \mathcal{M} is an \mathcal{F} -module with the unital, equivariant, and associative systems within the action maps

$$\lambda_j : \mathfrak{S}(j) \otimes \mathcal{A}^{j-1} \otimes \mathcal{S}\mathcal{M} \rightarrow \mathcal{S}\mathcal{M} \tag{5}$$

That homogeneous of degree 1 is such that the identity of the definition 1 holds for $j \geq 1$. We define an ∞ -algebra \mathcal{A} as a module over itself.

The map of \mathcal{A} -modules is semi-isomorphic if there is an actuating isomorphism on the homology.

Definition 5. A derived category is characterized as the stable homotopic category of spectra and signified by $\bar{h}\mathfrak{S}$. If the map of spectra induces an isomorphism in the homotopy groups, then it is weak equivalence and $\bar{h}\mathfrak{S}$ is constructed from a homotopy category of the spectra by formally altering the weak equivalences.

The derived category of \mathcal{A} -modules $D_{\mathcal{A}} = \bar{h}\mathcal{M}_{\mathcal{A}}$ is constructed from the homotopy category of \mathcal{A} -modules by formally altering a semi-isomorphisms.

The exact triangle sequence, $\mathcal{M} \xrightarrow{f} N \rightarrow C f \rightarrow \sum M$ prompts a triangulation of the derived category, for a map $f : \mathcal{M} \rightarrow N$.

Definition 6. [9]

The derived category $D_{\infty}(\mathcal{A})$ is the localization of the category E -infinity modules with degree 0 morphisms regarding a class of a quasi-isomorphisms.

Note that, the objects of the derived category $D_{\infty}(\mathcal{A})$ are E -infinity modules, and its morphisms are obtained from the morphisms of E -infinity modules by formally rearranging every single semi isomorphisms and, $D(\text{Mod } \mathcal{A}) \rightarrow D_{\infty}\mathcal{A}$.

Theorem 2. [11] The category of \mathcal{F} -linear algebraically triangulated T with the split idempotent and the generator G . Then for $m_1 = 0$, the structure of E -infinity algebra is as follows:

$$\mathcal{A} = \bigotimes_{n \in \mathbb{Z}} \text{Hom}_T(G, G[n]) \tag{6}$$

m_2 is given as composition and that the functor;

$$T \rightarrow \text{Grmod}(\mathcal{A}, m_2), \quad U \mapsto \bigotimes_{n \in \mathbb{Z}} \text{Hom}_T(G, G[n])$$

which lifts to the triangle equivalence, $T \rightarrow \text{per}(\mathcal{A})$.

Definition 7. [8]

A cyclic fibration (co-fibration) is the map with fibration (co-fibration) and weak equivalence.

A cofibrant object is a one of a kind morphism $(\phi \rightarrow \mathcal{X})$ from the underlying item that is a co-fibration. The fibrant object is the special morphism $(\mathcal{X} \rightarrow *)$ concerning the terminal object which is a fibration.

Definition 8. The classes are supposed to satisfy the Quillen axioms as following:

- (1) C Have limits and co-limits which is finite.
- (2) If μ and γ are composable in C , and for any two of μ, γ and μ, γ are weak equivalences, at that point also is the third.
- (3) \mathcal{A} The draw in the morphism's category of C of a weak equivalence, fibration, or

cofibration is individually a weak equivalence, fibration, or cofibration.

(4) The following commuting arrow diagram

$$\begin{array}{ccc}
 \mathcal{A} & \longrightarrow & \mathcal{X} \\
 i \downarrow & l \nearrow & \downarrow p \\
 \mathcal{B} & \longrightarrow & \mathcal{Y}
 \end{array} \tag{7}$$

with a cofibration i and a fibration p , if i or p is weak equivalence, then the lifting l exists making both triangles commute.

Every morphism can be considered as (1) a cyclic cofibration taken after by a fibration, and as (2) a cofibration took after by a cyclic fibration.

Definition 9. [12]

The morphisms $(\mathcal{W} \cap \text{fib})$ of \mathcal{W} are called trivial fibrations. The morphisms in $(\mathcal{W} \cap \mathcal{C})$ of \mathcal{W} called trivial cofibrations. Fibration involves the morphisms with privilege lifting property for any trivial cofibrations and complex C of the morphisms with the left lifting property concerning all trivial fibration. A left (right) legitimate model class is a one where the weak equivalences are steady under push forward along cofibrations.

Example 2. [11]

The class on the form $C = C^+(\text{Mod } \mathcal{A})$ of the left bounded complex

$$\dots \longrightarrow 0 \longrightarrow \dots \longrightarrow X^p \longrightarrow X^{p+1} \longrightarrow \dots$$

of right modules over the ring \mathcal{A} .

For an arbitrary \mathcal{W} be a class of a semi-isomorphisms C , the set of morphism $i : \mathcal{X} \longrightarrow \mathcal{Y}$, $\forall i^n, n \in \mathbb{Z}$, is injective and fib is the set of morphisms $p : \mathcal{X} \longrightarrow \mathcal{Y}$ with morphism $p^n, n \in \mathbb{Z}$, is surjective. From [10], we get C is a model classification. Since C is the underlying and the terminal protest, consequently the morphism $0 \longrightarrow \mathcal{X}$ is dependably a cofibration, since the morphism $\mathcal{X} \longrightarrow 0$ is fibration iff the all components $\mathcal{X}^n, n \in \mathbb{Z}$ are injective

$$\begin{array}{ccc}
 X & \longrightarrow & 0 \\
 \sim \searrow & & \nearrow \\
 & I &
 \end{array} \tag{8}$$

For a self-assertive class C , an object \mathcal{X} is fibrant if the morphism $\mathcal{X} \longrightarrow *$ is a fibration. Correspondingly, if the morphism $\phi \longrightarrow Y$ is a cofibration, then the object \mathcal{Y} is cofibrant. All complexes \mathcal{X} are cofibrant and a complex \mathcal{Y} is fibrant if and only if it has injective parts.

In the following section, we study the co-algebras With examples illustrated.

3. Co-algebras and the cobar construction

In this part, we recall the idea of co-algebras and the bar-cobar construction. This is followed by some relations and examples. The fundamental references are [13],[5] and [14].

Definition 10. An algebraic operad comprises a gathering of the chain complexes, $O(n), n \geq 0$, an accumulation of a chain maps

$$\gamma : O(k) \otimes O(j_1) \otimes \cdots O(j_k) \longrightarrow O(j_1 + \cdots + j_k) \tag{9}$$

An O -coalgebra is a chain complex C together with chain maps

$$\theta : O(j) \otimes C \longrightarrow C^j$$

Where, O is an operad, fulfilling the conditions;

(i) *Associativity:* For $\sum_{s=1}^k j_s = j$, then the diagram;

$$\begin{array}{ccc} O(k) \otimes O(j_1) \otimes \cdots O(j_k) \otimes C & \xrightarrow{\gamma \otimes id} & O(j) \otimes C \\ id \otimes \theta \downarrow & & \downarrow \theta \\ & & C^j \\ & & \uparrow \theta^k \\ O(j_1) \otimes \cdots O(j_k) \otimes C^k & \xrightarrow{shuffe} & O(j_1) \otimes \cdots O(j_k) \otimes C \end{array} \tag{10}$$

is commutes.

(ii) *Unity:* The accompanying diagram commutes:

$$\begin{array}{ccc} R \otimes C & \xrightarrow{\cong} & C \\ \gamma \otimes id \downarrow & \nearrow \theta & \\ O(1) \otimes C & & \end{array} \tag{11}$$

(iii) *Equivariance:* For a discretionary component $\sigma \in \Sigma_j$, the accompanying graph commutes:

$$\begin{array}{ccc} O(j) \otimes C & \xrightarrow{\sigma \otimes id} & O(j) \otimes C \\ \theta \downarrow & & \downarrow \theta \\ C^j & \xrightarrow{\sigma} & C^j \end{array} \tag{12}$$

The morphism in O -coalgebras are the map commuting strictly with the above structure. The class O -coalgebras will be referred to by $CoAlg_O$.

We characterize W as the class semi-isomorphisms and Fib as the arrangement of surjective morphisms. Consider $(C \circ f)$ as the set of morphisms i to such an extent that it is present in each commutative square of strong bolts in Alg .

Definition 11. [7]

A graded coalgebra over K is graded K -module C with a comultiplication of degree 0, to such an extent that the accompanying diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \Delta \\ C \otimes C & \xrightarrow{1 \otimes \Delta} & C \otimes C \otimes C \end{array}$$

this called co-associativity A coderivation on co-algebra is the map $G : C \longrightarrow C$ satisfying co-Leibniz rule, that is, the accompanying diagram commutes:

$$\begin{array}{ccc}
 C & \xrightarrow{G} & C \\
 \Delta \downarrow & & \downarrow \Delta \\
 C \otimes C & \xrightarrow{G \otimes + 1 \otimes G} & C \otimes C
 \end{array}$$

Definition 12. [4]

A DG-coalgebra is graded coalgebra with co-derivation $P : C \rightarrow C$ of degree (-1) such that, $P^2 = 0$.

Example 3. The fundamental cause of an evaluated graded co-algebra is co-tensor co-algebra of graded K -module:

$$T(V) = \sum_{n=0}^{\infty} V^{\otimes n} \tag{13}$$

The co-multiplication is formed as;

$$\Delta(v_1, \dots, v_n) = \sum_{i=0}^n \otimes(v_{i+1}, \dots, v_n) \tag{14}$$

since (v_1, \dots, v_n) stands for $v_1 \otimes \dots \otimes v_n$.

So, for every graded co-algebra C and the linear map $C \rightarrow V$, there is one of a kind expansion to co-algebra map $C \rightarrow T(V)$ with the end goal that the diagram;

$$\begin{array}{ccc}
 C & \longrightarrow & T(V) \\
 \downarrow & & \downarrow \\
 V & = & V
 \end{array}$$

is commutes.

Definition 13. [12]

A co-algebra C is a co-complete if the union of the compositions of the canonical projection as the kernel's maps

$$C \rightarrow C^{\otimes n} \rightarrow \left(\frac{C}{K}\right)^{\otimes n}, \quad n \geq 2$$

with the iterated co-multiplication.

Definition 14. [11]

For the n^{th} complex $Hom_k^{\bullet}(C, \mathcal{A})$ is n degree space of homogeneous K -linear maps $f : C \rightarrow \mathcal{A}$ and differential maps f to $(d \circ f - (-1)^n f \circ d)$. This complex turns into a differential graded algebra for the convolution characterized by;

$$f * g = \mu \circ (f \otimes g) \circ \Delta$$

The maps $\tau : C \rightarrow \mathcal{A}$, which is homogeneous k -linear of degree 1, is twisting cochain if it is homogeneous and fulfills

$$d(\tau) + \tau * \tau = 0, \quad \varepsilon \circ \tau \circ \varepsilon = 0 \tag{15}$$

Proposition 4. [14]

Define $T_w(C, \mathcal{A})$ a class of the twisting cochains. Then for $\mathcal{A} \in Alg$, the functor

$$C \circ g \longrightarrow Sets, \quad C \longmapsto T_w(C, \mathcal{A})$$

is representable.

In the following section, we study the relation and morphisms in E_∞ -algebra and we introduce the definition of Massy sequence and Massy product.

4. Basic statement on E -infinity algebras

In the current part, we consider the fundamental relations in the E_∞ -modules. The augmented E_∞ -algebra is equipped with morphism $\varepsilon : \mathcal{A} \longrightarrow k$.

Definition 15. [13]

The complex $H_k^\bullet(C, A)$ becomes an augmented E_∞ -algebra for the convolution operation;

$$b_n(g_1, \dots, g_n) = b_n^A \circ (g_1 \otimes \dots \otimes g_n) \circ \Delta^{(n)} \tag{16}$$

Where, $\Delta^{(n)}$ is iterate of taking values in \otimes^n for a coalgebra $\in \circ g$ and an augmented ∞ -algebra A .

Let $\infty(\cdot, A)$ be the arrangement of all arrangement of the "Maurer-Cartan equation",

$$\sum_{n \geq 1} b_n(\tau, \dots, \tau) = 0$$

Proposition 5. [2]

$B_\infty A$ is $T^c(SA)$ enriched with the one of a kind co-derivation whose composition with a basic projection $BA \downarrow SA$ has the segments

$$b_n : (SA)^{\otimes n} \longrightarrow SA, \quad n \geq 1 \tag{17}$$

Example 4. Let $A = TV$, where $V = k$ is concentrated with degree 1. Endow A with the novel differential whose confinement to $V \subset TV$ is

$$V = k \xrightarrow{\sim} k \otimes k = V^{\otimes 2} \subset TV \tag{18}$$

Then A is the semi isomorphic to its sub-algebra k , which is fibrant-cofibrant.

If A was fibrant-cofibrant, then the inclusion $k \longrightarrow TV$ should admit a left inverse up to homotopy in the feeling of C . Since there are non-zero maps $h : A \longrightarrow k$ with degree -1 such that $\varepsilon \circ h = 0$, then A cannot be fibrant or cofibrant, since A is the cobra construction on a non-complete dg-coalgebra.

Definition 16. A C -comodule is a chain complex D together with chains maps, $\lambda : O(j) \otimes D \longrightarrow D \otimes C^{j-1}$ for O is an operad and C is an O -coalgebra, fulfilling the conditions:

(i) Associativity: $\sum_{s=1}^k j_s = j$, and

$$\begin{array}{ccc}
 O(k) \otimes O(j_1) \otimes \cdots O(j_K) \otimes D & \xrightarrow{\gamma \otimes id} & O(j) \otimes C \\
 & & \downarrow \theta \\
 & & D \otimes C^{j-1} \\
 id \otimes \lambda \downarrow & & \uparrow \lambda \otimes \theta^{k-1} \\
 O(j_1) \otimes \cdots O(j_K) \otimes D \otimes C^{k-1} & \xrightarrow{shuffle} & O(j_1) \otimes D \otimes \cdots O(j_K) \otimes C
 \end{array}$$

is commutes.

(ii) *Unity: the accompanying outline commutes:*

$$\begin{array}{ccc}
 R \otimes D & \xrightarrow{\cong} & D \\
 \gamma \otimes id \downarrow & \nearrow \theta & \\
 O(1) \otimes D & &
 \end{array}$$

(iii) *Equivariance: Let $\sigma \in \Sigma_{j-1} \subset \Sigma_j$, then the accompanying outline is a commute:*

$$\begin{array}{ccc}
 O(j) \otimes D & \xrightarrow{\sigma \otimes id} & O(j) \otimes D \\
 \theta \downarrow & & \downarrow \theta \\
 D \otimes C^{j-1} & \xrightarrow{id \otimes \sigma} & D \otimes C^{j-1}
 \end{array}$$

A morphism in C -comodules is a homeomorphism of abelian groups commuting with the above structure, see [10].

Definition 17. For all classes of all right dg - A -modules and category $dg - C$ -comodules M . Then M is co-complete, $M \rightarrow M \otimes \bar{C}^{\otimes n}, n \geq 2$. Then the match

$$\begin{array}{ccc}
 Mod A & & \\
 ! \otimes_{\tau} A & \uparrow \downarrow & ! \otimes_{\tau} C \\
 Comc C & &
 \end{array}$$

is a couple of adjoint functors.

Theorem 3. .

(i) The category $Comc C$ concedes to a special structure of model category whose weak equivalences are the morphisms f such that $f \otimes_{\tau} A$ be a semi isomorphism and whose cofibrations are injective morphisms.

(ii) The functors $! \otimes_{\tau} C$ and $! \otimes_{\tau} A$ induce semi-inverse equivalences

$$D(A) \xrightarrow{\sim} D(C)$$

since $D(C)$ the localization of $Comc C$ is regarding classes of weak equivalences.

By comparing the adjunction morphism, $B_{\infty} A = C \rightarrow B\Omega C = B_{\infty}(\Omega C)$ to canonical E -infinity morphism $A \rightarrow \Omega B_{\infty} A$, which is a semi-isomorphism which is universal among the E -infinity morphisms from A to a dg -algebra, since A be an augmented E_{∞} -algebra and $B_{\infty} A = C$.

Definition 18. If we let $U(A) = \Omega B_{\infty} A$, at that point there is canonical cyclic twisting cochain $\tau : B_{\infty}(A) \rightarrow U(A)$. From theorem 3 we have an equivalence

$$D(U(A)) \xrightarrow{\sim} D(B_\infty(A))$$

Definition 19. [6]

Assume that $Mod_\infty A$ is the grouping of E_∞ -modules over $Ab_n^M, n \geq 2$ which are vanish when one of the contentions is 1. The morphisms $g_n, n \geq 2$ are strictly unital (vanish if the arguments is 1). This category is isomorphic to the order of all E -infinity modules and all E -infinity morphisms (to more see [11]).

Suppose that M is in $(Mod_\infty A)$. The datum of the arbitrary E -infinity module structure over \bar{A} and the datum of its strictly unital E -infinity module structure over A are equivalent each to other. it is also equivalent to a co-module differential in the induced co-module $(M \otimes B_\infty A)$. $(B_\infty M)$ is the induced co-module supplied with the differential corresponding to a given E -infinity module structure on M . The functor,

$$Mod_\infty A \longrightarrow ComcB_\infty(A), M \longrightarrow B_\infty M$$

exists.

Proposition 6. [1]

The functor $M \longrightarrow B_\infty M$ induces the following equivalence:

- (i) The equivalence onto subcategory fibrant (cofibrant) objects $(ComcB_\infty(A))_{cf}$ of $ComcB_\infty(A)$.
- (ii) $(Mod_\infty A)/\text{homotopy} \xrightarrow{\sim} D(C)$.

Definition 20. [2]

The derived classification for a non-augmented E -infinity algebra $D_\infty A$, is the kernel of the functor $D_\infty(A^+) \longrightarrow D_\infty(k)$, since $A^+ = A \oplus k$ be the increased E -infinity algebra obtained by adjoining k and the augmentation $A^+ \longrightarrow k$ yields a functor

$$Mod_\infty A^+ \longrightarrow Mod_\infty k$$

Proposition 7. The cohomology $H^*(M)$ is unital $H^*(A)$ -module i.e. M includes a place with the kernel iff M is homologically unital since A is homologically unital. The category $D_\infty(A)$ is compactly created a triangulated category and has the free A -module of the rank one as a generator of the compact.

Definition 21. [9]

Let B be a differential polynormal algebra with,

$$B = B^1 \supset B^2 \supset \dots \supset B^n \supset \dots \tag{19}$$

if a natural map $f : B \longrightarrow \hat{B}$ is an isomorphism, then B is perfect algebra. Since \hat{B} is polynormal algebra and given by; $\hat{B} = \lim B/B^n$.

Definition 22. Let the E_∞ -algebra A . Outline the Massy sequence (a^2, \dots, a^n) of the elements $a_i \in SA^{\otimes i}$ such that,

$$(\pi(2) \otimes \dots \otimes 1 + 1 \otimes \dots \otimes \pi(2))(a^n) = 0$$

$$(\pi(3) \otimes \dots \otimes 1 + 1 \otimes \dots \otimes \pi(3))(a^n) + (\pi(2) \otimes \dots \otimes 1 + 1 \otimes \dots \otimes \pi(2))(a^{n-1}) = 0$$

.....

$$(\pi(n-1) \otimes 1 + 1 \otimes \pi(n-1))(a^n) + (\pi(n-2) \otimes 1 + 1 \otimes (\pi(n-2)))(a^{n-1}) + \dots + (\pi(2) \otimes 1 + 1 \otimes \pi(2))(a^3) = 0$$

and the Massy product is given by;

$$\mu(a^2, \dots, a^n) = \pi(2)(a^2) + \dots + \pi(2)(a^n)$$

all parts in A are decomposable if they're images of Massy product. And therefore, the module of the indecomposable elements JA is that factor A concerning the decomposable elements.

Definition 23. Let B be the graded E_∞ -algebra and $\hat{F}B$ is that the B -construction. From the short exact sequence; $0 \rightarrow \hat{F}^1 B^i \rightarrow \hat{F} B^p \rightarrow 0$ we've got the long exact sequence;

$$\dots \rightarrow H_n(\hat{F}B)^{p_n} \rightarrow H_n(\hat{F}^1 B) \rightarrow \dots \tag{20}$$

with the projections; $i : \hat{F}^1 B \rightarrow \hat{F} B, p : \hat{F} B \rightarrow B$. And for all $x \in B$, if $x \in \ker v_n : B \rightarrow H_n(\hat{F}^1 B)$. Then x is a primitive element.

Now we can present the results that we studied to clarify important relationships of morphisms in the homology and cohomology theory of E_∞ -algebra.

5. Main Result

Through our study of E_∞ -algebra and providing some definitions of perfect algebra and primitive and indecomposable elements, we will study and prove the relationships between them in the (co)homology theory through the following theories:

Theorem 4. Let A and NA are the perfect algebra and N -construction, respectively. For the homology of NA , the space of the primitive elements $PH_n(NA)$ is isomorphic to indecomposable elements space; $PH_n(NA) \cong JA$.

Proof. Since $PH_n(NA)$ is primitive space then;

$$PH_n(NA) = Im\{H_n(\hat{F}H_n(NA)) \mapsto H_n(NA)\}$$

since $H_n(\hat{F}NA) \cong A$, then $PH_n(NA) \cong ImA \rightarrow H_n(NA) \cong JA$.

Theorem 5. Consider $JH_n(FU)$ be the space of indecomposable components in $H_n(FU)$, wherever U is perfect algebra and PU be the primitive space. Then $JH_n(FU) \cong PU$.

Proof. For the indecomposable elements of E_∞ -algebra we tend to get;

$$JH_n(FU) = Im\{H_n(FU) \rightarrow H_n(BH_n(FU))\}$$

since $H_n(BH_n(FU)) = U$, we get

$$JH_n(FU) \cong Im\{H_n(FU) \rightarrow U\} \cong PA$$

Theorem 6. For the ideal algebra U , and therefore the n^{th} -homology $U_n = H_n(BA)$ is the graded space with the approximation property. Then we get of $U^n = H^n(BA)$ in E_∞ -algebra as,

$$\sum_{n \geq 0} \pi_n(q \otimes \cdots \otimes q) \bar{\pi}(n+2)(x) = 0, \quad x \in \bar{K} \tag{21}$$

Now we present some examples as an application to what we got as results from previous theories.

Example 5. Consider perfect algebra $U = \mathfrak{S}_1^{(m)}$. For the cohomology $H^n(U)$, we discover the generator $a_1 \in H_1(U)$ corresponding $e_1 \in \mathfrak{S}_1^m$ and satisfy that,

$$\pi_{n-2}(a_1 \otimes \cdots \otimes a_2) = 0, \quad 2 \leq n \leq m$$

and we get the even-dimensional of cohomologies have the generators $a_2^n = a_2 \cdots a_2 \in H^{2n}(U)$ and isomorphic to C . However, the odd-dimensional cohomology has the generators $a_2^n \cdot pa_1 \in H^{2n+1}(A)$ and isomorphic to C .

Example 6. For the ideal algebra $U = \mathfrak{S}_1^{(m)}$ and also the short exact sequence $0 \rightarrow \mathfrak{S}_1^{(m)} \rightarrow \mathfrak{S}_1 \rightarrow \mathfrak{S}_1^m \rightarrow 0$, we discover that the cohomology $H^1(U)$ has generators a_{m+1}, \dots, a_{2m+1} that like $e_{m+1}, \dots, e_{2m+1} \in U$ and satisfy that;

$$\begin{aligned} a_{m+1}a_{m+1} &= 0, \\ a_{m+1}a_{m+2} + a_{m+2}a_{m+1} &= 0, \\ \dots & \\ a_{m+1}a_{2m+1} + \dots + a_{2m+1}a_{m+1} &= 0, \\ \pi_1(a_{m+1} \otimes a_{m+1} \otimes a_{m+1}) + a_{m+2}a_{2m+1} + \dots + a_{2m+1}a_{m+2} &= 0 \\ \dots & \end{aligned}$$

6. Conclusion

We have studied the basic statements previously made on E -infinity modules and E -infinity algebras. We also demonstrate an interpretation of the E -infinity algebras and modules as fibrant objects within the category differential graded co-algebras and co-modules. We have also presented new properties and examples to explain the idea.

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