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Chio's-like method for calculating the rectangular (non-square) determinants: Computer algorithm interpretation and comparison

Armend Salihu[1], Fahri Marevci[2,*]

[1] *South East European University, Ilindenska n.335, 1200 Tetovo, Macedonia*
[2] *Faculty of Education, University of Prishtina, Prishtine 10000, Kosovo*

Abstract. In this paper, we present an approach for the calculation of rectangular determinants, where in addition to the mathematical formula, we also provide a computer algorithm for their calculation. Firstly, we present a method similar to Sarrus method for calculating the rectangular determinant of the order 2×3. Secondly, we present an approach for calculating the rectangular determinants of order $m \times n$ by adding a row with all elements equal to one (1) in any row, as well as an application of Chio's rule for calculating the rectangular determinants. Thirdly, we find the time complexity and comparison of the computer execution time of calculation of the rectangular determinant based on the presented algorithms and comparing them with the algorithm based on the Laplace method.

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Key Words and Phrases: Determinants, Rectangular Determinants, Chio's-like Method, Computer Algorithm

1. Rectangular determinants definition

Let A be $m \times n$ a rectangular matrix:

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \tag{1}$$

its determinant, where $m \leq n$ is the sum (See: [2, 6]):

*Corresponding author.
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Email addresses: ar.salihu@gmail.com (A. Salihu), fahrimarevci@hotmail.com (F. Marevci)

$$det(A_{m\times n}) = |A_{m\times n}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

$$= \sum_{1<j_1<\cdots<j_m<n} (-1)^{r+s} \begin{vmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_n} \\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{mj_1} & a_{mj_2} & \cdots & a_{mj_m} \end{vmatrix}, \tag{2}$$

where $r = 1 + \cdots + m$, $s = j_1 + \cdots + j_m$. If $m > n$, then $det(A_{m\times n}) = det(A_{m\times n})^T$.

This determinant has many known standard characteristics, such as the validity of overall row expansion of Laplace method. It is a symmetric multi-linear function with respect to rows.

It is shown that this determinant has the extension of Laplace along rows that is valid for each $1 \leq i \leq m$ [7]:

$$det(A_{m\times n}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_j^i, \tag{3}$$

where A_j^i is the minor of the element a_{ij}.

In the following, the computer algorithm used to calculate rectangular determinants using Laplace expansion is presented. The algorithm is based on the Rezaifer's algorithm used to calculate square determinants using the Laplace method [8].

Algorithm 1.1: Recursive algorithm ***det_Laplace*** for Laplace method to calculate rectangular determinants

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Step 1: Insert the rectangular determinant A
Step 2: Determine the order of rectangular determinant $m \times n$
  $[m, n] = size(A)$;
Step 3: Calculate rectangular determinants using Laplace Method
  Initialize $d = 0$;
  Create Loop for i from 1 to n
    $d = d + (-1)\hat{}(1 + i) * A(1, i) * det\_Laplace(A(2 : m, [1 : i - 1 \quad i + 1 : n]))$;
  end
Step 4: Display the result of the determinant

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2. Main results

Proposition 2.1: For a rectangular determinant of order 2×3 the following Sarrus-like formula holds:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = \begin{matrix} + & + & + & \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} \\ a_{21} & a_{22} & a_{23} & a_{21} \end{vmatrix} \\ - & - & - & \end{matrix}$$

$$= a_{11}a_{22} + a_{12}a_{23} + a_{13}a_{21} - a_{12}a_{21} - a_{13}a_{22} - a_{11}a_{23}. \quad (4)$$

Proof: The proof follows immediately from the definition of rectangular determinant. ■

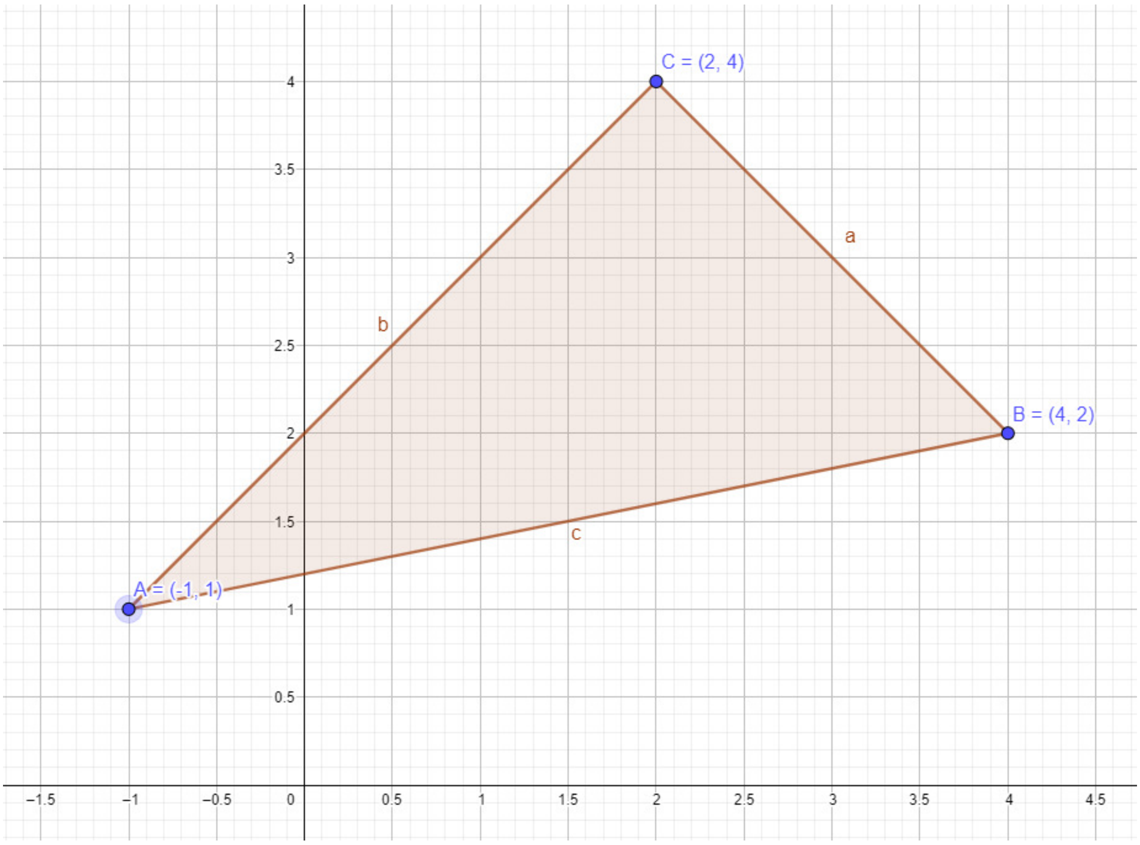
In the following is presented a computer algorithm to calculate rectangular determinants of order 2×3 based on Proposition 2.1.

Algorithm 2.1: Proposition 2.1 method to calculate rectangular determinants of order 2×3

Step 1: Insert the rectangular determinant of order 2×3
Step 2: Expand determinant horizontally add first column after the last column
  $A = [A \quad A(1 : n-1, 1 : 1)];$
Step 3: Exchange first row with the second row of expanded determinant
  B = flip(A);
Step 4: Calculate expanded determinants using Proposition 2.1
  Initialize: $c = 0$, $d = 0$;
  Create Loop for i from 0 to 2
    Initialize: a = 1, b = 1;
    Create Loop for j from 0 to 1
      $a = a * A(j+1, i+j+1);$
      $b = b * B(j+1, i+j+1);$
    end
    $c = c + a;$
    $d = d + b;$
  end
Step 5: Calculate the final result of rectangular determinant
  $e = c - d;$
Step 6: Display the result of the determinant

The following example shows an application of Proposition 2.1 for calculation of the area of a given triangle.

Example 2.1 Calculate the area of the triangle given in the following figure.



$$S=\frac{1}{2}\cdot\left|\begin{vmatrix}x_1&x_2&x_3\\y_1&y_2&y_3\end{vmatrix}\right|=\frac{1}{2}\cdot\left|\begin{vmatrix}-1&4&2\\1&2&4\end{vmatrix}\right|=\frac{1}{2}\cdot\left|\begin{vmatrix}\overset{+}{-1}&\overset{+}{4}&\overset{+}{2}&-1\\\underset{-}{1}&\underset{-}{2}&\underset{-}{4}&1\end{vmatrix}\right|$$

$$=\frac{1}{2}\cdot|(-1)\cdot 2+4\cdot 4+2\cdot 1-1\cdot 4-2\cdot 2-4\cdot(-1)|$$

$$=\frac{1}{2}\cdot|-2+16+2-4-4+4|=\frac{1}{2}\cdot|12|=6$$

The following Theorem concerns the calculation of the rectangular determinants of order $m\times n$, adding a row with all elements equal to one, the special case can be used to convert the rectangular determinant of order $(n-1)\times n$ to the square determinant of order $n\times n$.

Theorem 2.1: For a rectangular determinant of order $m\times n$ the following formula holds:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n} = (-1)^{(m+1)+i} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{(m+1)\times n}, \quad (5)$$

where $1 \le i \le m+1$, and $m+n$ is odd.

Proof: The proof follows immediately from formula 18 in [10] (or Ex. v, 32 in [2]), and property 5 in [6]. ■

In the following is presented a computer algorithm for calculating rectangular determinants of order $m \times n$ based on Theorem 2.1, adding one row of all elements equal to 1 in any row.

Algorithm 2.2: Theorem 2.1 method to calculate rectangular determinants of order $m \times n$

Step 1: Insert the rectangular determinant A
Step 2: Determine the order of rectangular determinant $m \times n$
$[m, n] = size(A)$;
Step 3: Create a row of order $1 \times n$ with all elements equal to 1
$X(1:n) = 1$
Step 4: Insert where to add the $1 \times n$ row with all elements equal to 1
Insert k;
Step 5: Create rectangular determinant of order $(m+1)\times n$ from given determinant and the row with all elements equal to 1
$B = [A(1:k-1, 1:n); X; A(k:m, 1:n)]$;
Step 6: Calculate the final result of created determinant
$d = (-1)\hat{}(m+1+k) * det\_Laplace(B)$; *//Rectangular or square determinant*
Step 7: Display the result of the determinant

In the following it is given an example, in which the row containing all elements equal to 1 is placed in the third row.

Example 2.2: Let be given a rectangular matrix of order 4×5. Its determinant can be calculated based on Theorem 2.1:

$$\begin{vmatrix} 3 & 5 & -6 & 1 & 4 \\ -2 & 4 & 9 & 2 & -4 \\ 5 & 7 & 1 & -3 & 6 \\ 6 & -9 & 5 & -4 & 1 \end{vmatrix} = (-1)^{(4+1)+3} \begin{vmatrix} 3 & 5 & -6 & 1 & 4 \\ -2 & 4 & 9 & 2 & -4 \\ 1 & 1 & 1 & 1 & 1 \\ 5 & 7 & 1 & -3 & 6 \\ 6 & -9 & 5 & -4 & 1 \end{vmatrix} = 3970$$

Theorem 2.2 (Chio's-like method for rectangular determinants): For a rectangular determinant of order $m \times n$, in cases for 2×3, 2×4 and 3×4, the following formula holds:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n} = \frac{|A_c|}{a_{11}^{m-2}} + (-1)^m \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}_{m\times(n-1)}, \tag{6}$$

where:

$$|A_c| = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{m1} & a_{m2} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{vmatrix} \end{vmatrix}_{(m-1)\times(n-1)} \tag{7}$$

and $a_{11} \neq 0$.

Proof: In the following we will prove the Theorem 2.2 for the rectangular determinant of order 2×3. First we multiply the elements of the second row by $a_{11} \neq 0$:

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}_{2\times 3} = \frac{1}{a_{11}} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \end{vmatrix}_{2\times 3}$$

$$= \frac{1}{a_{11}} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} \end{vmatrix}_{2\times 3} = \frac{1}{a_{11}} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \end{vmatrix}_{2\times 3}$$

$$= \frac{a_{11}}{a_{11}} \cdot \overbrace{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \end{vmatrix}_{1\times 2}}^{A_c} - \frac{a_{12}}{a_{11}} \cdot \begin{vmatrix} 0 & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \end{vmatrix}_{1\times 2} + \frac{a_{13}}{a_{11}} \cdot \begin{vmatrix} 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{vmatrix}_{1\times 2}$$

$$A_c + \frac{a_{12}}{a_{11}} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}_{2\times 2} - \frac{a_{13}}{a_{11}} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2\times 2}$$

$$= A_c + \frac{a_{12}}{a_{11}} \cdot (a_{11}a_{23} - a_{21}a_{13}) - \frac{a_{13}}{a_{11}} \cdot (a_{11}a_{22} - a_{21}a_{12})$$

$$= A_c + a_{12}a_{23} - Red\frac{a_{12}a_{13}a_{21}}{a_{11}} - a_{13}a_{22} + Red\frac{a_{12}a_{13}a_{21}}{a_{11}} = A_c + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}_{2\times 2}$$

$$= \frac{1}{a_{11}^{0}} \cdot A_c + (-1)^2 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}_{2\times 2}.$$

For order 2×4, we have as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{vmatrix}_{2\times 4} = \frac{1}{a_{11}} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} & a_{11}a_{24} \end{vmatrix}_{2\times 4}$$

$$= \frac{1}{a_{11}} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} & a_{11}a_{24} - a_{21}a_{14} \end{vmatrix}_{2\times 4}$$

$$= \frac{1}{a_{11}} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \end{vmatrix}_{2\times 4}$$

$$= \frac{a_{11}}{a_{11}} \cdot \overbrace{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \end{vmatrix}_{1\times 3}}^{A_c} - \frac{a_{12}}{a_{11}} \cdot \begin{vmatrix} 0 & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \end{vmatrix}_{1\times 3}$$

$$+ \frac{a_{13}}{a_{11}} \cdot \begin{vmatrix} 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \end{vmatrix}_{1\times 3} - \frac{a_{14}}{a_{11}} \cdot \begin{vmatrix} 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \end{vmatrix}_{1\times 3}$$

$$= A_c - \frac{a_{12}}{a_{11}} \left(0 - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \right) + \frac{a_{13}}{a_{11}} \left(0 - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \right)$$

$$- \frac{a_{14}}{a_{11}} \left(0 - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \right)$$

$$= A_c - \frac{a_{12}}{a_{11}} (0 - a_{11}a_{23} + a_{21}a_{13} + a_{11}a_{24} - a_{21}a_{14}) + \frac{a_{13}}{a_{11}} (0 - a_{11}a_{22} + a_{21}a_{12} + a_{11}a_{24} - a_{21}a_{14})$$

$$- \frac{a_{14}}{a_{11}} (0 - a_{11}a_{22} + a_{21}a_{12} + a_{11}a_{23} - a_{21}a_{13})$$

$$= A_c + a_{12}a_{23} - Red\frac{a_{12}a_{13}a_{21}}{a_{11}} - a_{12}a_{24} + Green\frac{a_{12}a_{14}a_{21}}{a_{11}} - a_{13}a_{22} + Red\frac{a_{12}a_{13}a_{21}}{a_{11}} + a_{13}a_{24} - Blue\frac{a_{13}a_{14}a_{21}}{a_{11}} + a_{14}a_{22}$$

$$- Green\frac{a_{12}a_{14}a_{21}}{a_{11}} - a_{14}a_{23} + Blue\frac{a_{13}a_{14}a_{21}}{a_{11}} = A_c + a_{12}a_{23} - a_{12}a_{24} - a_{13}a_{22} + a_{13}a_{24} + a_{14}a_{22} - a_{14}a_{23}$$

$$= A_c + (a_{12}a_{23} - a_{13}a_{22}) - (a_{12}a_{24} - a_{14}a_{22}) + (a_{13}a_{24} - a_{14}a_{23})$$

$$= A_c + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} = A_c + (-1)^2 \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \end{vmatrix}_{2\times 3}.$$

For order 3×4, we have as follows:

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix}_{3\times 4} = \frac{1}{a_{11}^2} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} & a_{11}a_{24} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} & a_{11}a_{34} \end{vmatrix}_{3\times 4}.$$

Based on the properties of determinants, we can multiply one row with one element and add/subtract the other row:

$$A = \frac{1}{a_{11}^2} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} & a_{11}a_{24} - a_{14}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} & a_{11}a_{34} - a_{14}a_{31} \end{vmatrix}_{3\times 4}$$

$$= \frac{1}{a_{11}^2} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} \end{vmatrix}_{3\times 4}.$$

Based on Laplace's method, we expand the last determinant according to the first row.

$$= \frac{a_{11}}{a_{11}^2} \cdot \overbrace{\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} \end{vmatrix}_{2\times 3}}^{A_c} - \frac{a_{12}}{a_{11}^2} \cdot \begin{vmatrix} 0 & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\ 0 & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} \end{vmatrix}_{2\times 3}$$

$$+ \frac{a_{13}}{a_{11}^2} \cdot \begin{vmatrix} 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} \end{vmatrix}_{2\times 3} - \frac{a_{14}}{a_{11}^2} \cdot \begin{vmatrix} 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}_{2\times 3}$$

$$= \frac{1}{a_{11}} A_c - \frac{a_{12}}{a_{11}^2} \cdot \begin{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} \end{vmatrix}_{2\times 2} + \frac{a_{13}}{a_{11}^2} \cdot \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} \end{vmatrix}_{2\times 2}$$

$$-\frac{a_{14}}{a_{11}^2}\cdot\begin{vmatrix}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}\\ \begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}\end{vmatrix}_{2\times 2}$$

$$=\frac{1}{a_{11}}A_c-a_{12}a_{23}a_{34}+\frac{a_{12}a_{14}a_{23}a_{31}}{a_{11}}+Red\frac{a_{12}a_{13}a_{21}a_{34}}{a_{11}}-Green\frac{a_{12}a_{13}a_{14}a_{21}a_{31}}{a_{11}^2}+a_{12}a_{24}a_{33}-Blue\frac{a_{12}a_{14}a_{21}a_{33}}{a_{11}}$$

$$-brown\frac{a_{12}a_{13}a_{24}a_{31}}{a_{11}}+Green\frac{a_{12}a_{13}a_{14}a_{21}a_{31}}{a_{11}^2}+a_{13}a_{22}a_{34}-Orange\frac{a_{13}a_{14}a_{22}a_{31}}{a_{11}}-Red\frac{a_{12}a_{13}a_{21}a_{34}}{a_{11}}+Green\frac{a_{12}a}{}$$

$$-a_{13}a_{24}a_{32}+Purple\frac{a_{13}a_{14}a_{21}a_{32}}{a_{11}}+brown\frac{a_{12}a_{13}a_{24}a_{31}}{a_{11}}-Green\frac{a_{12}a_{13}a_{14}a_{21}a_{31}}{a_{11}^2}-a_{14}a_{22}a_{33}+Orange\frac{a_{13}a_{14}a_{22}a_{3}}{a_{11}}$$

$$+Blue\frac{a_{12}a_{14}a_{21}a_{33}}{a_{11}}-Green\frac{a_{12}a_{13}a_{14}a_{21}a_{31}}{a_{11}^2}+a_{14}a_{23}a_{32}-Purple\frac{a_{13}a_{14}a_{21}a_{32}}{a_{11}}-\frac{a_{12}a_{14}a_{23}a_{31}}{a_{11}}+Green\frac{a_{12}a_{13}a_{14}a}{a_{11}^2}$$

$$=\frac{1}{a_{11}}A_c-(a_{12}a_{23}a_{34}+a_{13}a_{24}a_{32}+a_{14}a_{22}a_{33}-a_{14}a_{23}a_{34}-a_{13}a_{22}a_{34}-a_{12}a_{24}a_{32})$$

$$=\frac{1}{a_{11}}A_c+(-1)^3\begin{vmatrix}a_{12} & a_{13} & a_{14}\\ a_{22} & a_{23} & a_{24}\\ a_{32} & a_{33} & a_{34}\end{vmatrix}_{3\times 3}.$$

The proof is complete. ■

In the following is presented an algorithm based on Theorem 2.2. As we have computationally tested, we have seen that the algorithm holds also for any determinant of order $m\times n$.

Algorithm 2.3: Recursive algorithm ***det_Chio*** for Theorem 2.2 (Chio's-like) method to calculate rectangular determinants of order $m\times n$

Step 1: Insert the rectangular determinant A

Step 2: Determine the order of rectangular determinant $m\times n$

  $[m,n]=size(A);$

Step 3: Checking if A(1,1) is equal to 0

  if $A(1,1)=0$

    Exchange rows to find nonzero element

Step 4: Calculating sub matrices

Initialize $B = 0$;
Create Loop for i from 1 to m-1
Create Loop for j from 1 to n-1
$B(i,j) = A(1,1) * A(i+1,j+1) - A(1,j+1) * A(i+1,1)$
end
end
Step 5: Calculate the final result of rectangular determinant
$d = 1/A(1,1)\hat{}(m-2) * det\_Chio(B) + (-1)\hat{}m * det\_Chio(A(1:m,2:n))$;
Step 6: Display the result of the determinant

Remark 2.1: We have computationally tested Algorithm 2.3 for orders 3×4, 3×5, , 499×501 and 500×501, and compared with the Algorithm 1.1 (Laplace). For $m = n$ the Chio's Theorem holds (See: [1, 3]).

In the following, it is given an example of the calculation of a rectangular determinant based on Chio's-like formula and the result are compared with those obtained by Laplace method.

Example 2.3: Let us calculate the following determinant

$$\begin{vmatrix} 2 & -5 & 1 & 4 & 3 \\ 1 & 3 & -2 & 1 & 4 \\ -4 & 2 & 1 & 3 & 1 \end{vmatrix}_{3\times 5}$$

Solution:

1. Chio's-like formula:

$$\begin{vmatrix} 2 & -5 & 1 & 4 & 3 \\ 1 & 3 & -2 & 1 & 4 \\ -4 & 2 & 1 & 3 & 1 \end{vmatrix}_{3\times 5} = \frac{1}{2}\begin{vmatrix} \begin{vmatrix} 2 & -5 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & -5 \\ -4 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ -4 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -4 & 1 \end{vmatrix} \end{vmatrix}_{2\times 4}$$

$$+(-1)^3\begin{vmatrix} -5 & 1 & 4 & 3 \\ 3 & -2 & 1 & 4 \\ 2 & 1 & 3 & 1 \end{vmatrix}_{3\times 4} = \frac{1}{2}\begin{vmatrix} 11 & -5 & -2 & 5 \\ -16 & 6 & 22 & 14 \end{vmatrix}_{2\times 4} - \begin{vmatrix} -5 & 1 & 4 & 3 \\ 3 & -2 & 1 & 4 \\ 2 & 1 & 3 & 1 \end{vmatrix}_{3\times 4}$$

$$= -\frac{126}{2} - 104 = -167$$

2. Laplace method:

$$\begin{vmatrix} 2 & -5 & 1 & 4 & 3 \\ 1 & 3 & -2 & 1 & 4 \\ -4 & 2 & 1 & 3 & 1 \end{vmatrix}$$

$$= (-1)^{1+1}\cdot 2\cdot \begin{vmatrix} 3 & -2 & 1 & 4 \\ 2 & 1 & 3 & 1 \end{vmatrix} + (-1)^{1+2}\cdot(-5)\cdot\begin{vmatrix} 1 & -2 & 1 & 4 \\ -4 & 1 & 3 & 1 \end{vmatrix}$$

$$+(-1)^{1+3}\cdot 1\cdot\begin{vmatrix}1 & 3 & 1 & 4\\ -4 & 2 & 3 & 1\end{vmatrix}+(-1)^{1+4}\cdot 4\cdot\begin{vmatrix}1 & 3 & -2 & 4\\ -4 & 2 & 1 & 1\end{vmatrix}+(-1)^{1+5}\cdot 3\cdot\begin{vmatrix}1 & 3 & -2 & 1\\ -4 & 2 & 1 & 3\end{vmatrix}$$

$$=2\cdot(-17)+5\cdot(-9)+25-4\cdot 44+3\cdot 21=-34-45+25-176+63=-167$$

Corollary 2.1: For a rectangular determinant of order mn, in cases for 2×3, 2×4 and 3×4 the following formula holds:

$$\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{k1} & a_{k2} & \cdots & a_{kn}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{vmatrix}_{m\times n}=\frac{(-1)^{k-1}}{a_{k1}^{m-2}}\begin{vmatrix}\begin{vmatrix}a_{k1} & a_{k2}\\ a_{11} & a_{12}\end{vmatrix} & \cdots & \begin{vmatrix}a_{k1} & a_{kn}\\ a_{11} & a_{1n}\end{vmatrix}\\ \vdots & \ddots & \vdots\\ \begin{vmatrix}a_{k1} & a_{k2}\\ a_{k-1,1} & a_{k-1,2}\end{vmatrix} & \cdots & \begin{vmatrix}a_{k1} & a_{kn}\\ a_{k-1,1} & a_{k-1,n}\end{vmatrix}\\ \begin{vmatrix}a_{k1} & a_{k2}\\ a_{k+1,1} & a_{k+1,2}\end{vmatrix} & \cdots & \begin{vmatrix}a_{k1} & a_{kn}\\ a_{k+1,1} & a_{k+1,n}\end{vmatrix}\\ \vdots & \ddots & \vdots\\ \begin{vmatrix}a_{k1} & a_{k2}\\ a_{m1} & a_{m2}\end{vmatrix} & \cdots & \begin{vmatrix}a_{k1} & a_{kn}\\ a_{m1} & a_{mn}\end{vmatrix}\end{vmatrix}_{(m-1)\times(n-1)}$$

$$+(-1)^{m+k-1}\begin{vmatrix}a_{k2} & a_{k3} & \cdots & a_{kn}\\ a_{12} & a_{13} & \cdots & a_{1n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,n}\\ a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m2} & a_{m3} & \cdots & a_{mn}\end{vmatrix}_{m\times(n-1)}, \qquad (8)$$

where, $1\le k\le m$, and $a_{k1}\neq 0$.

Proof: Proof of Corollary 2.1 is derived from Theorem 2.2 and property of interchanging two rows, property 5 by interchanging the rows of the matrix in [6].

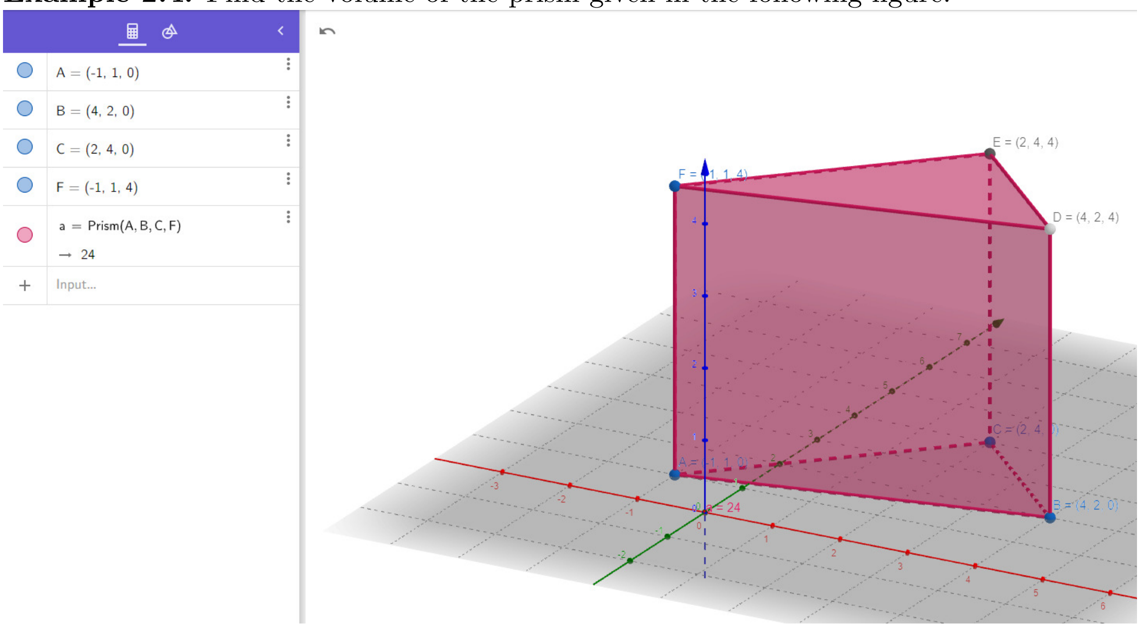
Note 2.1: In special cases when a_{11} is equal to zero, then Corollary 2.1 is applied, in the case when all elements of the first column are equal to 0, then the first column is eliminated and the sign before the determinant changes when m - (number of rows) is odd.

In the following is presented an algorithm based on Corollary 2.1.

Algorithm 2.4: Recursive algorithm ***det_Chio_Cor*** for Corollary 2.1 (Chio's-like) method to calculate rectangular determinants of order $m\times n$

Step 1: Insert the rectangular determinant A
Step 2: Determine the order of rectangular determinant $m \times n$
$[m, n] = size(A);$
Step 3: Select the row k of pivot element
Step 4: Checking if A(k,1) is equal to 0
if $A(k, 1) = 0$
Message: Select another k
Repeat Step 4
Step 5: Create modified matrix
if $k <= 0 \ || \ k > m + 1$
B=A
else
B=[A(k,:);A(1:k-1, 1:n);A(k+1:m, 1:n)]
Step 6: Calculating sub matrices
Initialize C=0;
Create Loop for i from 1 *to* $m - 1$
Create Loop for j from 1 *to* $n - 1$
$C(i, j) = B(1, 1) * B(i + 1, j + 1) - B(1, j + 1) * B(i + 1, 1)$
end
end
Step 7: Calculate the final result of rectangular determinant
$d = (-1)\hat{}(k-1)*(1/A(k, 1)\hat{}(m-2)*det\_Chio\_Cor(C)+(-1)\hat{}m*det\_Chio\_Cor(B(1 : m, 2 : n)))$
Step 8: Display the result of the determinant

Example 2.4: Find the volume of the prism given in the following figure.



$$V = \frac{1}{2} \cdot \left| \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} \right| = \frac{1}{2} \cdot \left| \begin{vmatrix} -1 & 4 & 2 & -1 \\ 1 & 2 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix} \right|$$

$$= \frac{1}{2} \cdot \left| \frac{(-1)^{2-1}}{1^{3-2}} \begin{vmatrix} \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} \end{vmatrix} + (-1)^{3+2-1} \begin{vmatrix} 2 & 4 & 1 \\ 4 & 2 & -1 \\ 0 & 0 & 4 \end{vmatrix} \right| = \frac{1}{2} \cdot \left| \begin{vmatrix} 6 & 6 & 0 \\ 0 & 0 & 4 \end{vmatrix} - 48 \right|$$

$$= \frac{1}{2} \cdot |24 - 24 - 48| = \frac{1}{2} \cdot |-48| = 24$$

3. Time complexity and execution time comparison of rectangular determinants calculation

For Algorithm 1.1, based on Laplace method, the time complexity is as follows [4, 9]:

- Loop of row expansion is $O(n)$;
- Creation of minor matrix is $O(n)$;
- Recursion is $O((n-1)!)$.

The asymptotic time complexity of the Algorithm 1.1 is $O((n-1)! \cdot max(n, n) = O(n!)$.

Time complexity of square determinant calculation of LU Decomposition Algorithm is as follows [9, 11]:

- Swapping rows is $O(n^2)$;
- First loop is $O(n^3)$, since:
  - + Loop i (Rows) is $O(n)$;
  - + Loop j (Columns) is $O(n)$; and
  - + Loop k (Elements subtraction) is $O(n)$;
- Second loop (pivot multiplication) is $O(n)$.

The asymptotic time complexity of LU Decomposition Algorithm is $O(max(n^2, n^3, n) = O(n^3)$.

Since the Algorithm 2.2 increases the number of rows for one, then time complexity of Algorithm 2.2 is same as Algorithm 1.1 using Laplace method. For special case of order $(n-1) \times n$ transforming to square determinant, and since time complexity of square determinants is $O(n^3)$ (LU decomposition), therefore for this case time complexity of Algorithm 2.2 is $O(n^3)$.

Regarding the time complexity of Algorithms 2.3 and 2.4 is as follows:

- Nested loop is $O(m \cdot n)$, since:
  - + Loop i (Rows) is $O(m)$; and
  - + Loop j (Columns) is $O(n)$;
- Creation of block matrices of order 2×2 is $O(m \cdot n)$;
- Recursion is $O(m-1)$.

The asymptotic time complexity of the Algorithms 2.3 and 2.4 is $O((m-1)\cdot max(m\cdot n,n)) = O(m^2\cdot n)$.

For additional algorithm complexity analysis see [5].

As can be seen from the time complexity of Algorithms 1.1, 2.2, 2.3 and 2.4, one can conclude that the algorithm based on Laplace-based method is slower than other algorithms. While, comparing the Algorithm 2.2 (for special case of order $(n-1)\times n$ with Algorithms 2.3 and 2.4, one can concluded that the execution time should be approximately the same as closely as m is to n.

Regarding the execution time of the rectangular determinant calculation, a computer with the following characteristics is used:

Table1: Computer characteristics used to simulate the calculation of determinants.

Name:	Dell
Model:	Vostro 15-3578
CPU:	Intel Core i7-8th gen 8550U 1.80 GHZ
RAM:	8 GB DDR4
GPU:	FULL HD Display 15.6” 1920x1080, AMD Radeon (TM) 520, 2048 MB GDDR5
HDD:	480 GB SSD

While software used for this simulation are presented in Table 2.

Table 2: Computer tools used for determinant calculation simulation:

OS	Windows 10 Pro 64-bit, Version 1803 (OS Build 17134.765
Software	MATLAB, Version 9.0.0321247 (R2016a), 64-bit (win64)

In this paper we have realized three comparisons of execution time of rectangular determinant calculation:

(i) First comparison is between Theorem 2.1 (adding a row with all elements equal to 1), Theorem 2.2 (Chio's like method) and Laplace method, for rectangular determinant of order $(n-1)\times n$, the results are presented in seconds in Table 3;

(ii) Second comparison is between Theorem 2.2 (Chio's like method) and Laplace method, for rectangular determinant of order $m\times n$, $m=10$, n from 11 to 25, the results are presented in seconds in Table 4;

(iii) Third comparison is between Theorem 2.2 (Chio's like method) and Laplace method, for rectangular determinant of order $m\times n$, m from 5 to 19, $n=20$, the results are presented in seconds in Table 5;

For the first comparison it is generated a random matrix of order $(n-1)\times n$ and tic toc is used to calculate the execution time, the MATLAB function is presented in the following:

Function 3.1: First comparison for order $(n-1) \times n$

---

n = Order of determinant
$A = rand(n-1, n)$;
disp(' ')
disp('Rectangular determinant of order $(n-1) \times n$ calculation adding one row with all elements equal to 1')
tic
$d1 = det\_1(A)$
toc
disp(' ')
disp('Rectangular determinant of order $(n-1) \times n$ calculation using Chios-like method')
tic
$d2 = det\_Chio(A)$
toc
disp(' ')
disp('Rectangular determinant of order $(n-1) \times n$ calculation using Laplace method')
tic
$d3 = det\_Laplace(A)$
toc

Results are presented in seconds in Table 3.

Table 3: Rectangular determinant calculation comparison between Theorem 2.1, Theorem 2.2 and Laplace method:

2*Det. Order	Theorem 2.1	Theorem 2.2	Laplace	Comparison		
	1	2	3	2-1	3-1	3-2
9×10	0.0004	0.0010	0.0020	0.0006	0.0016	0.0010
10×11	0.0007	0.0019	0.0021	0.0012	0.0014	0.0002
19×20	0.0003	0.0006	0.0041	0.0003	0.0038	0.0035
20×21	0.0010	0.0010	0.0059	0.0000	0.0049	0.0049
29×30	0.0003	0.0009	0.0141	0.0006	0.0138	0.0132
30×31	0.0006	0.0012	0.0151	0.0006	0.0145	0.0139
39×40	0.0007	0.0024	0.0279	0.0017	0.0272	0.0255
40×41	0.0006	0.0015	0.0317	0.0009	0.0311	0.0302
49×50	0.0004	0.0018	0.0518	0.0014	0.0514	0.0500
50×51	0.0006	0.0019	0.0593	0.0013	0.0587	0.0574
59×60	0.0007	0.0041	0.0997	0.0034	0.0990	0.0956
60×61	0.0007	0.0030	0.1074	0.0022	0.1067	0.1045
69×70	0.0024	0.0066	0.3043	0.0043	0.3019	0.2977
70×71	0.0008	0.0070	0.2967	0.0062	0.2959	0.2897
79×80	0.0010	0.0092	0.4361	0.0082	0.4351	0.4269
80×81	0.0009	0.0076	0.4657	0.0067	0.4648	0.4581
89×90	0.0010	0.0086	0.6169	0.0077	0.6160	0.6083
90×91	0.0011	0.0089	0.6351	0.0078	0.6340	0.6262
99×100	0.0008	0.0125	0.8656	0.0116	0.8648	0.8531
100×101	0.0009	0.0125	0.8861	0.0116	0.8852	0.8736
149×150	0.0025	0.0253	3.2421	0.0228	3.2396	3.2168
150×151	0.0010	0.0266	3.4131	0.0256	3.4121	3.3865
199×200	0.0007	0.0566	10.0962	0.0559	10.0955	10.0395
200×201	0.0026	0.0553	9.9452	0.0527	9.9426	9.8899
249×250	0.0018	0.0962	32.5930	0.0945	32.5912	32.4967
250×251	0.0030	0.0935	31.6585	0.0905	31.6556	31.5650
299×300	0.0036	0.1686	65.4817	0.1651	65.4781	65.3131
300×301	0.0035	0.1787	64.7036	0.1752	64.7001	64.5249

As can be seen in Table 3, there are some cases when the execution time of algorithms based on Theorem 2.1 and Theorem 2.2, shows that higher order of determinants are executed faster than lower order determinants. This is due to a very short period of execution time and the computer process priority and resource allocation (since execution was on windows environment).

For the second comparison we generated a random matrix of order $10 \times n$, for $11 \leq n \leq 25$ and tic toc is used to calculate execution time, the MATLAB function is presented in the following:

Function 3.2: Second comparison for order $10 \times n$, for $11 \leq n \leq 25$

---

m=10
n = Number of columns
A = rand(m,n);
disp(' ')
disp('Rectangular determinant of order $10 \times n$, for $11 \leq n \leq 25$ calculation using Chios-like method')
tic
$d1 = det\_Chio(A)$
toc
disp(' ')
disp('Rectangular determinant of order $10 \times n$, for $11 \leq n \leq 25$ calculation using Laplace method')
tic
$d2 = det\_Laplace(A)$
toc

Results are presented in seconds in Table 4.

Table 4: Rectangular determinant calculation comparison between Theorem 2.2 and Laplace method, m=10, n from 11 to 25:

2*Det. Order	Theorem 2.2	Laplace	Comparison
	1	2	2-1
10×11	0.0010	0.0032	0.0021
10×12	0.0009	0.0074	0.0064
10×13	0.0051	0.0221	0.0170
10×14	0.0082	0.0666	0.0584
10×15	0.0206	0.1834	0.1627
10×16	0.0494	0.4661	0.4166
10×17	0.1225	1.1035	0.9810
10×18	0.3933	3.6824	3.2891
10×19	0.8419	7.7522	6.9102
10×20	1.6791	15.5346	13.8555
10×21	3.1759	29.8277	26.6518
10×22	5.7377	60.5595	54.8218
10×23	10.1708	98.7320	88.5612
10×24	17.7038	196.3848	178.6810
10×25	29.4464	291.9469	262.5005

For the third comparison is used to generate a random matrix of order $m \times 20$, for $5 \leq m \leq 19$ and tic toc is used to calculate the execution time, the MATLAB function is presented in the following:

Function 3.3: Third comparison for order $m \times 20$, for $5 \leq m \leq 19$

```
m=10
n = Number of columns
A = rand(m,n);
disp(' ')
disp('Rectangular determinant of order m × 20, for 5 ≤ m ≤ 19 calculation using Chios-like method')
tic
    d1 = det_Chio(A)
toc
disp(' ')
disp('Rectangular determinant of order m × 20, for 5 ≤ m ≤ 19 calculation using Laplace method')
tic
    d2 = det_Laplace(A)
toc
```

Results are presented in seconds in Table 5.

Table 5: Rectangular determinant calculation comparison between Theorem 2.2 and Laplace method, m from 5 to 19, n=20:

2*Det. Order	Theorem 2.2	Laplace	Comparison
	1	2	2-1
5×20	0.1177	0.4406	0.3229
6×20	0.2762	1.4091	1.1329
7×20	0.5743	3.6031	3.0288
8×20	0.9602	7.2210	6.2608
9×20	1.4076	11.8139	10.4063
10×20	1.6498	15.7704	14.1206
11×20	1.6005	16.9290	15.3285
12×20	1.4516	21.1123	19.6607
13×20	0.8583	10.3589	9.5006
14×20	0.4422	5.9305	5.4883
15×20	0.1862	2.6365	2.4503
16×20	0.6209	0.9372	0.3163
17×20	0.0183	0.2529	0.2346
18×20	0.0052	0.0557	0.0505
19×20	0.0011	0.0048	0.0038

4. Conclusion

In this paper, we have presented a method for calculating 2×3 order of rectangular determinants, which is similar to the Sarus's method. We also presented the respective computer algorithm.

In addition, we have presented the possibility of adding a row with all elements equal to 1 in any row, based on the definition of Cullis/Radic, as well as the properties of determinants for interchanging two rows of rectangular determinant. As a special case we considered the case when the order of the determinant is $(n-1) \times n$ which can be transformed into square determinants. We also have presented the corresponding algorithm (See: Theorem 2.1 and Algorithm 2.2).

Regarding the Chio's-like method for calculating rectangular determinant, it is presented a Theorem 2.2, which is proven for orders 2×3, 2×4 and 3×4, while the computer algorithm is presented in Algorithm 2.3 and holds for determinants of order $m \times n$, considering cases when pivot element is equal to zero. Then we have obtained Corollary 2.1 where as a pivot element can be used any of the elements of first column. In cases when all elements of the first column are zero, if the number of rows is odd, the first column can be eliminated considering sign change.

Based on the time complexity of the presented algorithms, the algorithm based on Laplace method has time complexity of $O(n!)$, the algorithm based adding a row of all elements equal to 1 (transforming rectangular determinant of order $(n-1) \times n$, to the square determinant of order $n \times n$) has time complexity same as square determinant that is $O(n^3)$. The time complexity of algorithm based on Chio's-like method is $O(m^2 \cdot n)$. Based on the comparison of time complexity of these algorithms one can conclude that the slowest algorithm is based on Laplace method, and other algorithms are approximately the same, depending on how close is m to n.

In the third part of the paper, we have compared the computational speed of the rectangular determinants by different methods. The first comparison is made for the rectangular determinants of the order $(n-1) \times n$, between Theorem 2.1, by adding a row of all elements equal to 1 in any row (Algorithm 2.2), Theorem 2.2, based on Chio's-like method (Algorithm 2.3) and the Laplace method (Algorithm 1.1). From the analysis of the obtained results, we have noticed that Algorithm 2.2 is more effective than the other two algorithms. This algorithm has a very small advantage over Algorithm 2.3 since the time complexity of both Algorithms 2.2 and 2.3 is approximately the same, and there is seen significant advantage over Algorithm 1.1, the most obvious advantage is observed in determinants of higher orders, since the time complexity of Laplace method is $O(n!)$ and much higher than $O(n^3)$. From this comparison a significant advantage was observed between Algorithm 2.3 compared to Algorithm 1.1, which is obvious also based on time complexity. Similarly a significant difference was observed in the higher orders determinants.

The second comparison is made for the orders of rectangular determinants ranging from 10×11 to 10×25, between Algorithm 2.3 and Algorithm 1.1. Also in this comparison, an obvious advantage was observed regarding the computational speed of the calculation of the rectangular determinant of Algorithm 2.3 compared to Algorithm 1.1, especially the significant difference was observed in the higher orders of columns.

The last comparison in this paper is to compare the speed of calculation of rectangular determinants by Algorithm 2.3 and Algorithm 1.1, for orders from 5×20 up to 19×20.

In this regard, it has been observed an obvious advantage of the computational speed of rectangular determinants calculation with Algorithm 2.3 compared to Algorithm 1.1. Unlike the previous comparisons, here we have noticed that a higher difference is when the number of columns is approximate twice the number of rows. This occurs because the highest possible value of combinations n choose k is attained when k is half of the n.

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