EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 2, 2021, 601-607
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Using area method in secondary school geometry

Samed J.Aliyev[1]*, Shahin M.Aghazade[1], Goncha Z.Abdullayeva[1]

[1] *Department of Methods of Mathematics and its Teaching, Faculty of Mechanics and Mathematics, Baku State University, Baku, Z.Khalilov str.23,, AZ1148, Azerbaijan*

Abstract. Areas are actively used in the solution of many geometrical problems. In this work, smart and laconic solutions are found for various problems by means of the area method. The well-known trigonometric addition formula is also proved. In the area method, the given formulas are divided into the parts, whose areas are then calculated using problem data.

2020 Mathematics Subject Classifications: 97G10, 97G30, 97G40

Key Words and Phrases: Area method, additive property of area, excircles of a triangle, addition formula.

Overcoming the difficulties faced by the schoolchildren in solving geometrical problems is still actual today. Including geometrical methods of problem solving in secondary school programs is one of the way to solve this problem within the concept of humanitarian school education. In this work, we consider the importance and the possibility of including the area method in secondary school programs, which has a special and very important role in geometrical methods allowing to solve a broad scope of problems.

Rarely mentioned in scientific-methodical literature, the area method is nonetheless often used to solve problems in mathematical olympiads and other competition events.

The works [2-7] describe the ways to solve some problems by the area method. Also lets mention [1], co-authored by the first author, where, using the area as an auxiliary element, the solution is found in a quite simple and smart way.

The area method involves various properties of areas to establish the relations between the problem data and the unknowns. The most often used ones are the additive property and the property of the ratio of areas which help to reduce the problem to the solution of some equation or to direct calculation.

Note that the area method is used to solve the problems which involve areas, and it is especially important for those problems which do not involve areas. In the latter ones, the area is introduced to the problem as an auxiliary element. Despite looking obvious, the area method is unfamiliar to majority of teachers and students due to the current

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v14i2.3942

Email addresses: samed59@bk.ru (S. J.Aliyev),
shahinaghazade@bsu.edu.az (S. M.Aghazade),a.q.z.41@mail.ru (G. Z.Abdullayeva)

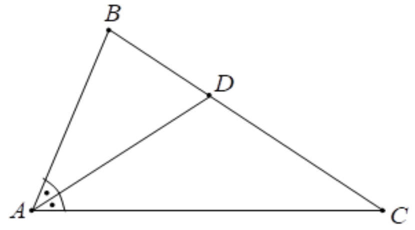

Figure 1: Triangle ABC with bisector AD.

situation with teaching geometry and the methods of solving geometrical problems in secondary schools.

How do we use the area method? We consider the area of the given figure as a sum of the areas of its parts. We calculate the areas of each part in a suitable way to obtain the equation which either gives the solution or significantly facilitates its search. To illustrate the method, we will consider simplest problems where the area can be calculated in two ways. For example, let an isosceles triangle be given, and let its base and the height drawn to its base be known; find the height drawn to its side. First, the area of this triangle is half of the product of its base and the height drawn to the base. On the other hand, it equals to the half of the product of the side and the corresponding height, with the side of triangle easily calculated by the Pythagorean theorem.

Lets give some examples which illustrate the use of area method.

Find the bisector AD of the triangle ABC , where $AB = c$, $AC = b$, $\angle BAC = \alpha$.

Lets calculate the area of the triangle ABC (Fig. 1):

$$S_{ABC} = S_{ABD} + S_{ADC}, \tag{1}$$

where

$$S_{ABC} = \frac{1}{2} \cdot AB \cdot AC \cdot \sin\alpha = \frac{1}{2} \cdot c \cdot b \cdot \sin\alpha,$$

$$S_{ABD} = \frac{1}{2} \cdot AB \cdot AD \cdot \sin\frac{\alpha}{2} = \frac{1}{2} \cdot c \cdot AD \cdot \sin\frac{\alpha}{2},$$

$$S_{ADC} = \frac{1}{2} \cdot AD \cdot AC \cdot \sin\frac{\alpha}{2} = \frac{1}{2} \cdot b \cdot AD \cdot \sin\frac{\alpha}{2}.$$

Then, substituting the obtained expressions into (1), we get the equation

$$\frac{1}{2} \cdot c \cdot b \cdot \sin\alpha = \frac{1}{2} \cdot c \cdot AD \cdot \sin\frac{\alpha}{2} + \frac{1}{2} \cdot b \cdot AD \cdot \sin\frac{\alpha}{2}.$$

After some transformations we obtain

$$AD = \frac{2 \cdot c \cdot b \cdot \cos\frac{\alpha}{2}}{b + c}.$$

The conditions of this problem do not involve areas, but note that the bisector divides the triangle into the parts whose areas can be easily calculated using the conditions of the

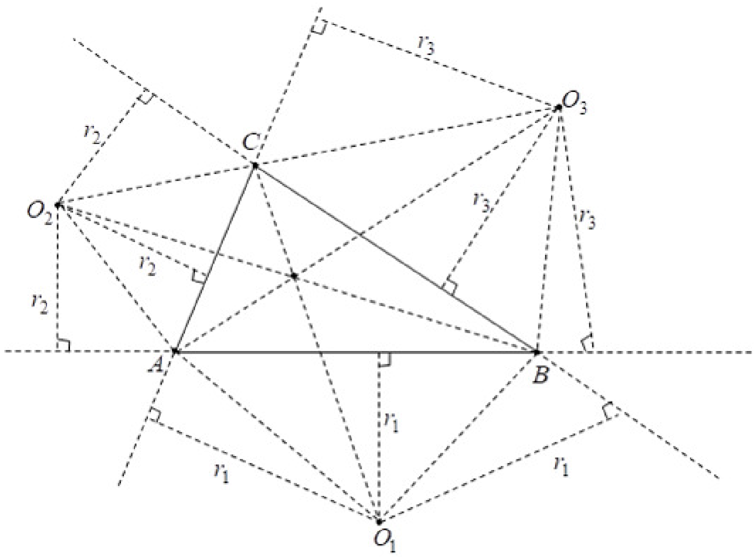

Figure 2: Points O_1, O_2, O_3 are the centers of the excircles of the triangle ABC.

problem. The area of the triangle itself can also be easily calculated. Therefore the area method is useful here. Having introduced the area as an auxiliary element, we solved the problem in a simple way.

Consider another problem, where the additive property of area and the formulas for the calculation of areas help to find the lengths of intervals.

Find the radii of the incircle and excircles of the triangle ABC , where $AB = 15, BC = 14, AC = 13$.

As the semiperimeter p of the triangle ABC is equal to 21, the area of this triangle can be found by Herons formula:

$$S_{ABC} = \sqrt{p\,(p-a)\,(p-b)\,(p-c)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84.$$

Consequently, the radius of the incircle of ABC is equal to

$$r = \frac{S_{ABC}}{p} = \frac{84}{21} = 4.$$

Let r_1, r_2, r_3 be the radii of the excircles of the triangle ABC (Fig. 2). Lets first find r_1. Note that this radius is a height in the triangles O_1BC, O_1AC, O_1AB. Then

$$S_{ABC} = S_{O_1BC} + S_{O_1AC} - S_{O_1AB} = \frac{1}{2}BC \cdot r_1 + \frac{1}{2}AC \cdot r_1 - \frac{1}{2}AB \cdot r_1.$$

Hence we have $2 \cdot 84 = r_1\,(14 + 13 - 15)$ or $r_1 = 14$.

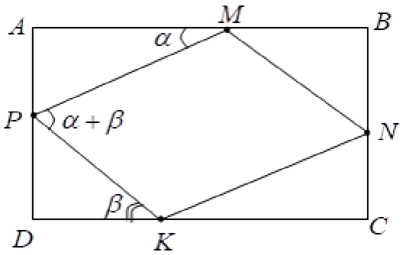

Figure 3: Rectangle $ABCD$ and the inscribed rhombus $MNKP$.

Similarly we find r_2 and r_3 :

$$S_{ABC} = S_{O_2BC} + S_{O_2AB} - S_{O_2AC} = \frac{1}{2}BC \cdot r_2 + \frac{1}{2}AB \cdot r_2 - \frac{1}{2}AC \cdot r_2,$$

$2 \cdot 84 = r_2\,(14 + 15 - 13)$ or $r_2 = \frac{21}{2}$;

$$S_{ABC} = S_{O_3AB} + S_{O_3AC} - S_{O_3BC} = \frac{1}{2}AB \cdot r_3 + \frac{1}{2}AC \cdot r_3 - \frac{1}{2}BC \cdot r_3,$$

$2 \cdot 84 = r_3\,(15 + 13 - 14)$ or $r_3 = 12$.

Note that after having found the radii r ,r_1 and r_2, we could also find the radius r_3 by the well-known formula

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

The area method obviously makes the solution of this problem easier as the problem becomes less complex with no need for some logical considerations. Moreover, a synthetic and technical problem becomes a nice one.

The area method is one of those which helps to find nice and laconic solutions for different problems. Being internal with respect to geometry, it is general for problem solving.

Didactic importance of the area method is that it is a study subject and a tool for teaching the next lesson (for example, the lesson entitled Similarity) simultaneously. Within the framework of humanitarization of the education, this method is of great importance from cultural and aesthetic aspects. The use of area method provides the possibility to develop interdisciplinary relationships: some trigonometric formulas are easily proved by this method.

Lets give a nice example which illustrates the possibility of proving some trigonometric formulas by the area method.

Consider the rectangle $ABCD$ and the rhombus $MNKP$ inscribed in $ABCD$ with the sides equal to 1 and the angles $\angle AMP = \alpha$,$\angle PKD = \beta$ (Fig. 3).

The angle at the vertex P of the rhombus is equal to $\alpha+\beta$, and the area of the rhombus is $S_{MNPK} = \sin{(\alpha + \beta)}$.

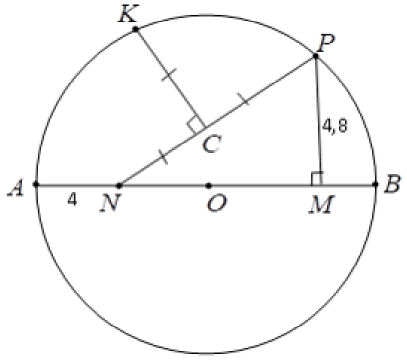

Figure 4: Circle centered at O with diameter AB.

On the other hand, the sides of the rhombus divide the rectangle into the pairs of equal triangles AMP and CKN, BMN and DKP, with catheti $\cos\alpha$, $\sin\alpha$ and $\cos\beta$, $\sin\beta$, respectively. Then from the obvious equality

$$S_{MNKP} = S_{ABCD} - 2S_{AMP} - 2S_{BMN}$$

after simplification we have

$$S_{MNKP} = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

Hence, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.

The question arises when solving problems: given the problem which does not involve the areas, how to determine whether it will be useful to apply the area method in this case? There is no general recommendations for this case, but the following tips based on the analysis of large number of problems solvable by the area method may be very useful.

Using additive property of area will most likely help to solve your problem by the area method if the problem involves

1) incircles and excircles of a triangle, circles tangent to two sides of a triangle, radius, angle, etc;

2) points lying on the sides of a polygon;

3) point inside a triangle which divides it into several triangles, projections of this point onto the sides of a triangle;

4) proof of assertion that three points lie on the same straight line.

Now lets consider a problem which seems, at first glance, not related to areas. Applying the area method to this problem significantly facilitates its solution.

Let the point O be the center of the given circle,$NC = KC = CP$, $KC \perp NP$,$PM \perp AB$, $AN = 4$,$PM = 4,8$ (Fig. 4). Find the radius of this circle.

Denote by R the radius of this circle. As $NC = KC = CP$,the triangle PKN is rectangular and $\angle PKN = 90°$. From $KC \perp NP$ it follows that this rectangular triangle PKN is even isosceles: $PK = KN$. Let KE be a chord passing through the point N. As $\angle PKN = 90°$, the interval PE connecting the points P and E passes through the point O, i.e. PE is a diameter of the circle (Fig. 5).

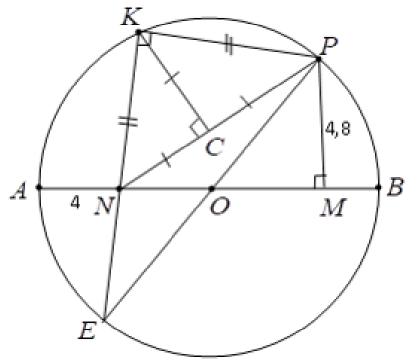

Figure 5: Circumscribed circle of rectangular triangle PKE.

By intersecting chords theorem we have

$$KN \cdot NE = AN \cdot NB$$

or $KN \cdot NE = 4\,(2R - 4)$.

On the other hand, as

$$S_{NEP} = \frac{NE \cdot KP}{2} = \frac{KN \cdot NE}{2} = 2\,(2R-4)\,,$$

$$S_{NOP} = \frac{NO \cdot 4,8}{2} = 2,4\,(R-4)\,,$$

by

$$S_{NEP} = 2S_{NOP}$$

we have $2\,(2R-4) = 4,8\,(R-4)$ or $R = 14$.

Taking into consideration the results of experimental research and recommendations by the leading experts and teachers, we arrive at the conclusion that some changes to secondary school geometry course, in other words, the inclusion of the area method in secondary school geometry course as one of problem solving methods, would not only enrich students knowledge, but would also contribute to further development of geometry.

References

[1] S.J.Aliyev, S.N.Efendi. Using area method in geometric problems. The Way of Science, International Scientific Journal, (2016), No4, (26), p.15-17. (in Russian)

[2] I.A.Kushnir. Auxiliary element method. Kvant, (1974), No2, p.46-51. (in Russian)

[3] I.D.Novikov. Area method. Kvant, (1971), No12, p.41-46. (in Russian)

[4] Y.P.Ponarin. Elementary geometry. Vol. 1, MSNMO, (2008), 312 p. (in Russian)

[5] V.V.Prasolov. Using the area. Kvant, (1986), No5, p.16-19. (in Russian)

[6] E.G.Gotman. Planimetry problems and methods to solve them. Prosvesheniye, (1996), 240 p. (in Russian)

[7] I.F.Sharigin. Learning to solve geometric problems. Matematika v shkole, (1989), No2, p.87-101. (in Russian)