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On the Diophantine Equation $M_p^x + (M_q + 1)^y = z^2$

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Abstract. In this paper, we study and solve the exponential Diophantine equation of the form $M_p^x + (M_q + 1)^y = z^2$ for Mersenne primes M_p and M_q and non-negative integers x, y, and z. We use elementary methods, such as the factoring method and the modular arithmetic method, to prove our research results. Several illustrations are presented, as well as cases where solutions to the Diophantine equation do not exist.

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Key Words and Phrases: Diophantine equation, Exponential Diophantine equation, Mersenne primes

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1. Introduction

A number of researchers have been studying the exponential Diophantine equations of the form $a^x + b^y = z^2$. This includes Aggarwal, Burshtein, Kumar, Sroysang, Rabago, among others (cf. [1], [2], [3], [5], [6], [8], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [24], [27]). Some of them have studied these equations in relation to Mersenne primes. They focused on the case where one of the bases a and b is a Mersenne prime. In particular, some considered $M_2 = 3$, $M_3 = 7$ and $M_5 = 31$, which are actually the first three Mersenne prime numbers. Records show that Sroysang [25] proved that the solutions of $3^x + 2^y = z^2$ are $(0, 1, 2), (3, 0, 3)$, and $(2, 4, 5)$. Asthana and Singh [4] proved that $3^x + 13^y = z^2$ has exactly four non-negative integer solutions, and these are $(1, 0, 2)$, $(1, 1, 4), (3, 2, 14)$ and $(5, 1, 6)$. Rabago [16] proved that the triples $(4, 1, 10)$ and $(1, 0, 2)$ are the only solutions to the Diophantine equation $3^x + 19^y = z^2$, and that $(2, 1, 10)$ and $(1, 0, 2)$ are the only two solutions to $3^x + 91^y = z^2$. Sroysang [26] also showed that the $7^x + 8^y = z^2$ has the only solution $(x, y, z) = (0, 1, 3)$. Another work of Sroysang [23] shows

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that the equation $31^x + 32^y = z^2$ has no non-negative integer solution. Chotchaisthit [9] aimed to study $p^x + (p+1)^y = z^2$ in the set of nonnegative integers and where p is a Mersenne prime.

These works motivate the researchers to study the Diophantine equations of the form $M_p^x + (M_q + 1)^y = z^2$, where M_p and M_q are Mersenne primes. Factoring and modular arithmetic methods are the elementary methods used in the study.

Using the factoring method, an equation, say $f(x_1, x_2, ..., x_n) = 0$, will be written as

$$f_1(x) \cdot f_2(x) \cdot ... \cdot f_k(x) = c,$$

where $f_1, f_2, ...f_k \in \mathbb{Z}[x]$, $x = (x_1, x_2, ..., x_n)$, $c \in \mathbb{Z}$. Given the prime factorization of c, we obatin finitely many decompositions into k factorizations $c_1, c_2, ...c_k$. Every factorization yields a system of equations

$$\begin{cases} f_1(x_1, x_2, ..., x_n) = c_1 \\ f_2(x_1, x_2, ..., x_n) = c_2 \\ \qquad\qquad \vdots \\ f_k(x_1, x_2, ..., x_n) = c_k. \end{cases}$$

The complete set of solutions is obtained by solving all such systems.

The modular arithmetic method, on the other hand, is widely used in proving non-solvability of a given equation or at least reducing the set of integers where we can find possible solutions. Properties of modular arithmetic are utilized in deriving the results.

For interesting examples using the above methods, the reader is referred to the book by Andreescu et al. [28].

2. Main Results

The following definition and lemmas are needed in this study.

Definition 1. *A Mersenne prime is a prime number of the form* $2^p - 1$*, where* p *is also a prime number.*

Lemma 1. *All Mersenne primes are congruent to* $3 \pmod 4$*.*

Proof. Since a Mersenne prime is of the form $2^p - 1$, it follows that $p \geq 2$. Thus, $2^p \equiv 0 \pmod 4$, yielding $2^p - 1 \equiv -1 \pmod 4$ or $3 \pmod 4$.

Lemma 2 (Mihailescu's Theorem [12])**.** *The quadruple* $(3, 2, 2, 3)$ *is the unique solution for the Diophantine equation* $a^x - b^y = 1$*, where* a, b, x *and* y *are integers with* $min\{a, b, x, y\} > 1$*.*

The main result for this study is stated as follows.

Theorem 1. *Every nonnegative integer solution (M_p, M_q, x, y, z) of the Diophantine equation $M_p^x + (M_q+1)^y = z^2$, where M_p and M_q are Mersenne primes, takes any of the following forms:*

i. $(M_p, 7, 0, 1, 3)$

ii. $(3, M_q, 1, 0, 2)$

iii. $\left(M_p, M_q, 2, \dfrac{p+2}{q}, 2^p+1\right)$

Proof. Let us consider first the case when one of the exponents x and y is zero.

If $x = 0$, then regardless of any Mersenne prime M_p, we have the equation

$$1 + (M_q + 1)^y = z^2.$$

If $y = 0$, then $z^2 = 2$, not a perfect square. If $y = 1$, then $z^2 - 2^q = 1$. By Mihailescu's theorem, $z = 3$ and $q = 3$. Then, $M_q = 7$, a Mersenne prime and thus $(M_p, M_q, x, y, z) = (M_p, 7, 0, 1, 3)$ for any Mersenne prime M_p is a solution.

If $y > 1$, then by Mihailescu's Theorem, $2^q = 2$ giving us $q = 1$ which is not possible because q must be a prime number.

If $y = 0$, then regardless of any Mersenne prime M_q, we have the equation

$$M_p^x + 1 = z^2.$$

Substituting $M_p = 2^p - 1$ to the equation above will lead to the equation

$$(2^p - 1)^x + 1 = z^2.$$

If $x = 0$, then $z^2 = 2$, not a perfect square. If $x = 1$, then $z^2 = 2^p$. Let $z = 2^a$. Then, $2^{2a} = 2^p$, which implies that $2a = p$. Using the primality of p, we get $a = 1$ and $p = 2$. This results to $z = 2$ and $M_p = 2^p - 1 = 3$, a Mersenne prime. Hence, $(M_p, M_q, x, z) = (3, M_q, 1, 0, 2)$, for any Mersenne prime M_q, is a solution. If $x > 1$, then by Mihailescu's Theorem, $z = 3, x = 2$ and $2^p - 1 = 2$, a contradiction.

We are now left with the case where $\min\{x, y\} \geq 1$. We note that all Mersenne primes are congruent to $3 \pmod 4$. Hence, $M_p \equiv 3 \pmod 4$ and
$M_q + 1 \equiv 0 \pmod 4$. Thus, for any positive integer y,

$$M_p^x + (M_q+1)^y \equiv \begin{cases} 3 \pmod 4 & \text{for odd } x \\ 1 \pmod 4 & \text{for even } x. \end{cases}$$

Because $z^2 \equiv 1 \pmod 4$, we can say that x is even. Thus, there exists a positive integer k such that $x = 2k$. So, $M_p^{2k} + (M_q+1)^y = z^2$. By substituting $M_q = 2^q - 1$ for some prime q, we get the equation $(M_p)^{2k} + 2^{qy} = z^2$. It can be expressed as $z^2 - (M_p)^{2k} = 2^{qy}$. Factoring the left side of the equation leads to

$$(z + M_p^k)(z - M_p^k) = 2^{qy}.$$

There exist nonnegative integers α and β with $\alpha > \beta$ and $\alpha + \beta = 2qy$ such that $(z + M_p^k)(z - M_p^k) = 2^{\alpha+\beta}$. This implies that $(z + M_p^k) = 2^\alpha$ and $(z - M_p^k) = 2^\beta$, which gives $2M_p^k = 2^\beta(2^{\alpha-\beta} - 1)$. Equating the odd parts and the even parts leads to the system

$$\begin{cases} 2^\beta = 2 \\ 2^{\alpha-\beta} - 1 = M_p^k. \end{cases}$$

The first equation implies that $\beta = 1$. Then, the second equation becomes

$$2^{\alpha-1} - M_p^k = 1.$$

If $k > 1$ and $\alpha > 2$, there is no solution by Mihailescu's Theorem. If $\alpha = 2$, then $M_p^k = 1$. This gives the value $k = 0$, a contradiction to k being positive. If $k = 1$, then $x = 2$, $z = 2^p+1$ and $2^{\alpha-1} - M_p = 1$ or in equivalent form $2^{\alpha-1} = 2^p$. This implies that $\alpha = p+1$. Since $\alpha + \beta = qy$ and $\beta = 1$, it follows that $p + 2 = qy$ or $y = \dfrac{p+2}{q}$. If $q|p + 2$, then we have the set of solutions

$$\{(M_p, M_q, x, y, z)\} = \left\{ \left(M_p, M_q, 2, \frac{p+2}{q}, 2^p + 1 \right) \right\}.$$

□

By Theorem 1, the positive integer solutions of $M_p^x + (M_q + 1)^y = z^2$ are given by $(M_p, M_q, x, y, z) = \left(M_p, M_q, 2, \dfrac{p+2}{q}, 2^p + 1 \right)$. Given the Mersenne prime M_p, the solutions can be found by finding all primes q that divide $p + 2$. It should be checked also if the corresponding Mersenne number $2^q - 1$ is a Mersenne prime. The number of solutions depends on how many primes q that divide $p + 2$ such that M_q is a Mersenne prime. Let us take take the case of $M_3 = 7$ and $M_{13} = 8191$.

Example 1. *Find the positive integer solution of* $7^x + (M_q + 1)^y = z^2$, *where* M_q *is a Mersenne prime.*

Solution. Theorem 1 asserts that $(M_q, x, y, z) = \left(M_q, 2, \dfrac{4}{q}, 9 \right)$ if $q|4$. The only prime q that divides 4 is 2, and it happens that $M_2 = 3$ is a Mersenne prime. In conclusion, $(3, 2, 2, 8)$ is the unique positive integer solution.

Example 2. *Find the positive integer solution of* $8191^x + (M_q + 1)^y = z^2$, *where* M_q *is a Mersenne prime.*

Solution. Theorem 1 guarantees that $(M_q, x, y, z) = \left(M_q, 2, \dfrac{15}{q}, 8193 \right)$ if $q|15$. The primes that divide 15 are 3 and 5. If $q = 3$, then $y = 5$ and $M_q = 7$, a Mersenne prime. Hence, we have $(7, 2, 5, 8193)$ as a solution. If $q = 5$, then $y = 5$ and $M_q = 31$. Thus, $(31, 2, 3, 8193)$ is another solution.

Let us also solve some examples where M_p and M_q are given. Consider the equations $3^x + 8^y = z^2$ and $31^x + 128^y = z^2$.

Example 3. *Find the positive integer solution of* $3^x + 8^y = z^2$.

Solution. Here, $M_p = 3$, where $p = 2$ and $M_q = 7$, where $q = 3$. Theorem 1 guarantees that a positive integer solution exists if $q|p+2$. Since $3 \nmid 4$, it follows that there is no positive integer solution.

Example 4. *Find the positive integer solution of* $31^x + 128^y = z^2$.

Solution. Here, $M_p = 31$, where $p = 5$ and $M_q = 127$, where $q = 7$. Theorem 1 asserts that $(x, y, z) = \left(2, \frac{p+2}{q}, 9\right)$ is a solution if $q|p+2$. Hence,
$(x, y, z) = (2, 1, 9)$ is the unique solution.

3. Conclusion and Recommendation

In this work, using the factoring and modular arithmetic methods, the Mihailescu's theorem, and the fact that every Mersenne prime is of the form $4k+3$, we were able to show that the Diophantine equation $M_p^x + (M_q+1)^y = z^2$, where M_p and M_q are Mersenne primes, have the following nonnegative integer solutions (M_p, M_q, x, y, z), namely
$(M_p, 7, 0, 1, 3)$, $(3, M_q, 1, 0, 2)$ and $\left(M_p, M_q, 2, \frac{p+2}{q}, 2^p+1\right)$.

The following table presents some positive integer solutions of the Diophantine equation $M_p^x + (M_q+1)^y = z^2$ for the first five Mersenne primes M_p.

Table 1. Some Positive Integer Solutions of $M_p^x + (M_q+1)^y = z^2$

M_p	p	$p+2$	q	y	M_q	(M_p, M_q, x, y, z)
3	2	4	2	2	3	(3,3,2,2,5)
7	3	5	5	1	31	(7,31,2,1,9)
31	5	7	7	1	127	(31,127,2,1,33)
127	7	9	3	3	7	(127,7,2,3,129)
8191	13	15	3	5	7	(8191,7,2,5,8193)
8191	13	15	5	3	31	(8191,31,2,3,8193)

The next table presents some particular cases of the exponential Diophantine equation $M_p^x + (M_q+1)^y = z^2$, wherein no solutions can be obtained. The unsolvability of these equations is achieved because the prime q fails to divide $p+2$.

Table 2. List of Some Unsolvable Cases of $M_p^x + (M_q+1)^y = z^2$

M_p	p	M_q	q	$p+2$	$M_p^x + (M_q+1)^y = z^2$
3	2	127	5	4	$3^x + 128^y = z^2$
7	3	7	3	5	$7^x + 8^y = z^2$
31	5	7	3	7	$31^x + 8^y = z^2$
127	7	31	5	9	$127^x + 32^y = z^2$
8191	13	127	7	15	$8191^x + 128^y = z^2$

The results presented in this study contribute to the repository of knowlege in the theory of numbers, especially in solving exponential Diophantine equations.

For possible extensions, the reader may try to solve the following Diophantine equations in $\mathbb{N}_0$:

(i) $M_p^x + (M_q + k)^y = z^2$, where $k \geq 1$, and M_p and M_q are Mersenne primes;

(ii) $M_p^x + (M_q + 1)^y = z^n$,where $n \geq 1$, and M_p and M_q are Mersenne primes; and

(iii) $(M_q - 1)^x + M_p^y + (M_q + 1)^z = w^2$, where M_p and M_q are Mersenne primes.

Since the equation under consideration in this study is equivalent to $(2^p-1)^x+2^{qy} = z^2$, the reader may get additional results when compared to or combined with results on similar/related Diophantine equations, such as $x^2 - 2^r = p^n$ [29] and $x^2 - D = p^n$ (cf. [10], [11] [31], [30]). Lastly, to find other results for the equation under consideration and the suggested equations above, the reader might get interested in applying other methods such as the linear forms in logarithms, like what was done in the paper by Bugeaud [7].

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