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# On $k$-Fair Total Domination in Graphs

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**Abstract.** Let $G = (V(G), E(G))$ be a simple non-empty graph. For an integer $k \geq 1$, a *$k$-fair total dominating set* (*$kftd$-set*) is a total dominating set $S \subseteq V(G)$ such that $|N_G(u) \cap S| = k$ for every $u \in V(G) \backslash S$. The *$k$-fair total domination number* of $G$, denoted by $\gamma_{kftd}(G)$, is the minimum cardinality of a *$kftd$-set*. A $k$-fair total dominating set of cardinality $\gamma_{kftd}(G)$ is called a *minimum $k$-fair total dominating set* or a *$\gamma_{kftd}$-set*. We investigate the notion of $k$-fair total domination in this paper. We also characterize the $k$-fair total dominating sets in the join, corona, lexicographic product and Cartesian product of graphs and determine the exact values or sharp bounds of their corresponding $k$-fair total domination number.

**2020 Mathematics Subject Classifications**: 05C69, 05C76

**Key Words and Phrases**: $k$-fair domination, $k$-fair total domination, Join, Corona, Lexicographic product, Cartesian product

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## 1. Introduction

Let $G = (V(G), E(G))$ be a simple graph and $v \in V(G)$. The *open neighborhood* of $v$ in $G$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. For $X \subseteq V(G)$, the *open neighborhood* of $X$ in $G$ is the set $N_G(X) = \bigcup\limits_{v \in X} N_G(v)$ and its *closed neighborhood* is the set $N_G[X] = N_G(X) \cup X$. A set $S \subseteq V(G)$ is a *dominating set* in $G$ if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The minimum cardinality of a dominating set in $G$, denoted by $\gamma(G)$, is the *domination number* of $G$. Any dominating set in $G$ of cardinality $\gamma(G)$ is referred to as a *$\gamma$-set* in $G$. For a connected graph $G$, a set $S \subseteq V(G)$ is a *total*

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*dominating set* in $G$ if $N_G(S) = V(G)$.

A domination variant called fair domination was introduced by Caro, Hansberg and Henning [2] in 2012. For an integer $k \geq 1$, a *k-fair dominating set* (*kfd-set*) is a dominating set $S \subseteq V(G)$ such that $|N_G(u) \cap S| = k$ for every $u \in V(G) \backslash S$. The *k-fair domination number* of $G$, denoted by $\gamma_{kfd}(G)$, is the minimum cardinality of a *kfd-set.*

In 2014, Maravilla et al.[5] characterized the $k$-fair dominating sets in the join, corona, lexicographic product, and Cartesian product of graphs and determined the bounds or exact values of the $k$-fair domination numbers of these graphs. Two variants of $k$-fair domination, namely connected $k$-fair domination and neighborhood connected $k$-fair domination, were studied by Bent-Usman et al. [1, 6] in 2018 and 2019, respectively. Recently, Ortega and Isla [7] introduced and investigated the concepts of semitotal $k$-fair domination and independent $k$-fair domination in graphs.

Maravilla et al. [4] introduced the notion of $k$-fair total domination in graphs. For a non-empty graph $G$ and an integer $k \geq 1$, a *k-fair total dominating set* (*kftd-set*) is a total dominating set $S \subseteq V(G)$ such that $|N_G(u) \cap S| = k$ for every $u \in V(G) \backslash S$. The *k-fair total domination number* of $G$, denoted by $\gamma_{kftd}(G)$, is the minimum cardinality of a *kftd-set.* A $k$-fair total dominating set of cardinality $\gamma_{kftd}(G)$ is called a *minimum k-fair total dominating set* or a $\gamma_{kftd}$*-set.* In this paper, we investigate the concept of $k$-fair total domination and characterize the $k$-fair total dominating sets in graphs under some binary operations. We also determine the exact values or sharp bounds of their corresponding $k$-fair total domination number.

A comprehensive treatment of the theoretical, algorithmic, and application (e.g., facility location) aspects of domination in graphs was provided by Haynes et al.[3] in 1998.

The *join* $G + H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of two graphs $G$ and $H$, denoted by $G \circ H$, is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the $i$-th vertex of $G$ to every vertex in the $i$-th copy of $H$. For every $v \in V(G)$, we denote by $H^v$ the copy of $H$ whose vertices are joined or attached to the vertex $v$. For each $v \in V(G)$, the subgraph $\langle v \rangle + H^v$ of $G \circ H$ will be denoted by $v + H^v$. The *lexicographic product* of two graphs $G$ and $H$, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The *Cartesian product* of two graphs $G$ and $H$, denoted by $G\Box H$, is the graph with vertex-set $V(G\Box H) = V(G) \times V(H)$ and edge-set $E(G\Box H)$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G\Box H)$ if and only if either $u_1u_2 \in E(G)$ and $v_1 = v_2$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

## 2. Preliminary Results

**Remark 1.** *For any connected graph $G$ of order $n \geq 2$ and a positive integer $k$ , $\gamma_{kfd}(G) \leq \gamma_{kftd}(G)$ and $\gamma_{kftd}(G) \geq 2$.*

**Remark 2.** *Any $kftd$-set is a $kfd$-set, where $k$ is a positive integer.*

**Theorem 1.** *Let $n$ and $r$ be positive integers where $n \geq 2$ and $r \geq 1$. Then*

$$\gamma_{1ftd}(P_n) = \begin{cases} 2, & n = 2,3 \\ 2r, & n = 4r \\ 2r+1, & n = 4r+1 \\ 2r+2, & \textit{otherwise.} \end{cases}$$

*Proof.* Let $G = P_n = \{v_1, v_2, v_3, ..., v_n\}$. If $n = 2$ or $n = 3$, then clearly, $\gamma_{1ftd}(P_n) = 2$ . Let $n \geq 4$ and consider the following cases:

Case 1: $n = 4r$

Group the first $4r$ vertices of $P_n$ into $r$ disjoint subsets.

$$\begin{aligned} S_1 &= \{v_1, v_2, v_3, v_4\} \\ S_2 &= \{v_5, v_6, v_7, v_8\} \\ S_3 &= \{v_9, v_{10}, v_{11}, v_{12}\} \\ &\vdots \\ S_{r-1} &= \{v_{4r-7}, v_{4r-6}, v_{4r-5}, v_{4r-4}\} \\ S_r &= \{v_{4r-3}, v_{4r-2}, v_{4r-1}, v_{4r}\} \end{aligned}$$

For every induced subgraph $\langle v_i, v_{i+1}, v_{i+2}, v_{i+3} \rangle$ of $P_n$, where $i = 1, 5, 9, ..., 4r-3$, the vertices $v_{i+1}$ and $v_{i+2}$ are in a 1-fair total dominating set of $P_n$. Thus, the set $T = \{v_2, v_3, v_6, v_7, ..., v_{4r-2}, v_{4r-1}\}$ is a 1-fair total dominating set of $P_n$. Since $|T| = 2r$, $\gamma_{1ftd}(P_n) \leq 2r$. Note that every pair of adjacent vertices in $P_n$ can dominate at most 2 vertices. Thus, every 1-fair total dominating set of $P_n$ contains at least $\lceil \frac{n}{2} \rceil$ vertices. Hence, $\gamma_{1ftd}(P_n) \geq \lceil \frac{n}{2} \rceil = 2r$ since $n = 4r$. Thus, $\gamma_{1ftd}(P_n) = 2r$.

Case 2: $n = 4r + 1$

The set $T$ in Case 1 is no longer a $\gamma_{1ftd}$-set of $T_n$ here since $v_{4r+1}$ is not adjacent to any vertex in $T$, but clearly, $T \cup \{v_{4r}\}$ is a $\gamma_{1ftd}$-set. Thus, $\gamma_{1ftd}(P_n) = 2r + 1$.

Case 3. $n = 4r + 2$

The set $S = T \cup \{v_{4r}\}$ is not a $\gamma_{1ftd}$-set of $P_n$ here since $v_{4r+2}$ is not adjacent to any vertex in $S$, but $T \cup \{v_{4r}, v_{4r+1}\}$ is clearly a $\gamma_{1ftd}$-set. Hence,$\gamma_{1ftd}(P_n) = 2r + 2$.

Case 4. $n = 4r + 3$

Consider the 1-fair total dominating set $T$ in Case 1. Add $v_{4r+2}$ and $v_{4r+3}$ to the vertices in $T$ so that $T \cup \{v_{4r+2}, v_{4r+3}\}$ is a $\gamma_{1ftd}$-set of $P_n$. Hence, $\gamma_{1ftd}(P_n) = 2r + 2$. □

**Theorem 2.** *Let $n$ and $r$ be positive integers where $n \geq 3$ and $r \geq 1$. Then*

$$\gamma_{1ftd}(C_n) = \begin{cases} 3, & n = 3 \\ 2r, & n = 4r \\ 2r+1, & n = 4r+1 \\ 2r+2, & n = 4r+2 \\ 2r+3, & n = 4r+3. \end{cases}$$

*Proof.* Suppose that $C_n = [v_1, v_2, ..., v_n, v_1]$. If $n = 3$, then clearly, $\gamma_{1ftd}(C_3) = 3$. The proof for $n = 4r$, $n = 4r + 1$, and $n = 4r + 2$ is similar to the proof of Cases 1 to 3 of Theorem 1. When $n = 4r + 3$, let $T = \{v_2, v_3, v_6, v_7, ...., v_{4r-2}, v_{4r-1}\}$. It can be verified that $T \cup \{v_{4r}, v_{4r+1}, v_{4r+2}\}$ is a $\gamma_{1ftd}$-set of $C_n$. Thus, $\gamma_{1ftd}(C_n) = 2r + 3$. □

**Lemma 1.** [5] *Let $K_n$ be the complete graph of order $n$ and $k$ a positive integer with $k \leq n$. Then $\gamma_{kfd}(K_n) = k$.*

**Theorem 3.** *Let $n$ and $k$ be positive integers, $2 \leq k \leq n$. Then, $\gamma_{kftd}(K_n) = k$.*

*Proof.* Clearly, $\gamma_{2ftd}(K_2) = 2$, $\gamma_{2ftd}(K_3) = 2$, and $\gamma_{3ftd}(K_3) = 3$. Let $n > 3$. Let $V(K_n) = \{v_1, v_2, ..., v_n\}$, and $S = \{v_1, v_2, ..., v_k\}$. Note that each vertex in $S$ is adjacent to the remaining $k - 1$ vertices in $S$. Moreover, for each $v_i \in V(K_n)\backslash S$, that is, for each $v_i$, $k + 1 \leq i \leq n$, $|N(v_i) \cap S| = k$. Thus, $S$ is a $kftd$-set in $K_n$ and $\gamma_{kftd}(K_n) \leq k$. However, $\gamma_{kftd}(K_n) \geq \gamma_{kfd}(K_n) = k$ by Remark 1 and Lemma 1. Thus, $\gamma_{kftd}(K_n) = k$. □

**Theorem 4.** *Let $a$ and $b$ be positive integers such that $a \leq b$. Then there exists a connected graph $G$ such that $\gamma_{1fd}(G) = a$ and $\gamma_{1ftd}(G) = b$.*

*Proof.* Consider the following cases:

Case 1. $a = b$

Let $G = G_1$ be the graph shown in Figure 1.

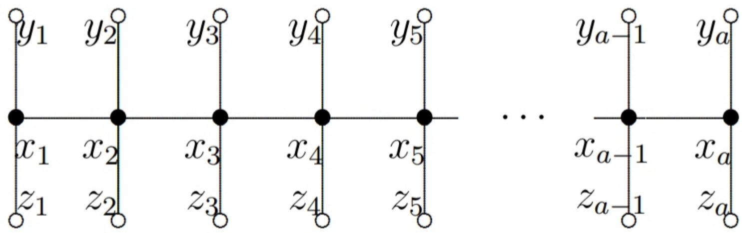


Figure 1: A graph $G$ with $\gamma_{1fd}(G) = \gamma_{1ftd}(G) = a$

It is clear that the set $A = \{x_i : i = 1, 2, ...a\}$ is both a $\gamma_{1fd}$-set and a $\gamma_{1ftd}$-set in $G_1$. It follows that $\gamma_{1fd}(G_1) = \gamma_{1ftd}(G_1) = a$.

Case 2. $a < b$

Let $G = G_2$ be the graph shown in Figure 2.

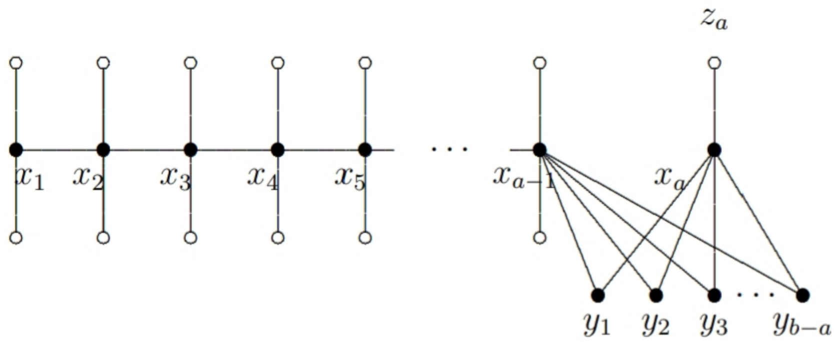


Figure 2: A graph $G$ with $\gamma_{1fd}(G) = a < \gamma_{1ftd}(G) = b$

Let $A = \{x_1, x_2, ..., x_{a-1}\}$. It is clear that the set $B = A \cup \{z_a\}$ is a $\gamma_{1fd}$-set and the set $C = A \cup \{x_a\} \cup \{y_1, y_2, ..., y_{b-a}\}$ is a $\gamma_{1ftd}$-set in $G_2$. It follows that $\gamma_{1fd}(G_2) = |B| = a$ and $\gamma_{1ftd}(G_2) = |C| = b$. □

**Corollary 1.** *$\gamma_{1ftd} - \gamma_{1fd}$ can be made arbitrarily large.*

## 3. Known Results

The following characterizations of $k$-fair dominating sets in the join, corona, and lexicographic product of two nontrivial, connected graphs are found in Maravilla et al. [5].

**Theorem 5.** [5] *Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively, and $k$ a positive integer with $1 \leq k \leq max\{m, n\}$. Then $S \subseteq V(G+H)$ is a $kfd$-set in $G + H$ if and only if one of the following holds:*

(a) *$S = V(G + H)$.*

(b) *$S \subseteq V(G)$, $|S| = k$ and $S$ is a $kfd$-set in $G$.*

(c) *$S \subseteq V(H)$, $|S| = k$ and $S$ is a $kfd$-set in $H$.*

(d) *$S = S_G \cup S_H$, where $S_G$ is a $(k - |S_H|)fd$-set in $G$ and $S_H$ is a $(k - |S_G|)fd$-set in $H$.*

(e) *$S = V(G) \cup T$, where $|V(G)| = m < k$ and $T$ is a $(k - m)fd$-set in $H$.*

(f) *$S = D \cup V(H)$, where $|V(H)| = n < k$ and $D$ is a $(k - n)fd$-set in $G$.*

**Theorem 6.** [5] *Let $G$ and $H$ be nontrivial connected graphs and let $k$ be a positive integer with $k \leq |V(H)|$. Then $C \subseteq V(G \circ H)$ is a $kfd$-set in $G \circ H$ if and only if one of the following holds:*

(a) $C = V(G) \cup B$, *where* $B = \varnothing$ *or* $B = \bigcup_{v \in V(G)} S_v$, *where each* $S_v$ *is a* $(k-1)fd$*-set in* $H^v$.

(b) $C = \bigcup_{v \in V(G)} S_v$, *where each* $S_v$ *is a* $kfd$*-set in* $H^v$ *and* $|S_v| = k$.

**Theorem 7.** [5] *Let* $G$ *and* $H$ *be nontrivial connected graphs. Then* $C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ *is a* $kfd$*-set in* $G[H]$ *if and only if the following hold:*

(i) $S$ *is a dominating set in* $G$.

(ii) *For each* $x \in S \cap N_G(S)$, $T_x = V(H)$ *and* $|V(H)| = r \leq k$ *whenever* $C \neq V(G[H])$ *or* $T_x$ *is an* $rfd$*-set and* $\sum_{z \in N_G(x) \cap S} |T_z| = k - r$.

(iii) *For each* $x \in S \backslash N_G(S)$, $T_x = V(H)$ *and* $|V(H)| \leq k$ *or* $|T_x| = k$ *and* $T_x$ *is a* $kfd$*-set in* $H$.

(iv) *For each* $y \in V(G) \backslash S$, $\sum_{v \in N_G(y) \cap S} |T_v| = k$.

## 4. Main Results

We characterize the $k$-fair total dominating sets in the join, corona, and lexicographic product of graphs in this section, as well as some such sets in the Cartesian product of graphs. We also determine the $k$-fair total domination number of the join and corona of any two connected graphs and establish sharp bounds of the $k$-fair total domination number of the lexicographic and Cartesian products of graphs.

**Theorem 8.** *Let* $G$ *and* $H$ *be nontrivial connected graphs of orders* $m$ *and* $n$*, respectively, and* $k$ *a positive integer with* $2 \leq k \leq max\{m, n\}$. *Then* $S \subseteq V(G + H)$ *is a* $kftd$*-set in* $G + H$ *if and only if one of the following holds:*

(*a*) $S = V(G + H)$.

(*b*) $S \subseteq V(G)$, $|S| = k$ *and* $S$ *is a* $kftd$*-set in* $G$.

(*c*) $S \subseteq V(H)$, $|S| = k$ *and* $S$ *is a* $kftd$*-set in* $H$.

(*d*) $S = S_G \cup S_H$, *where* $S_G$ *is a* $(k - |S_H|)fd$*-set in* $G$ *and* $S_H$ *is a* $(k - |S_G|)fd$*-set in* $H$.

(*e*) $S = V(G) \cup T$, *where* $|V(G)| = m < k$ *and* $T$ *is a* $(k - m)fd$*-set in* $H$.

(*f*) $S = D \cup V(H)$, *where* $|V(H)| = n < k$ *and* $D$ *is a* $(k - n)fd$*-set in* $G$.

*Proof.* Suppose that $S \subseteq V(G+H)$ is a $kftd$-set in $G+H$, where $k \geq 2$. Then $S$ is a $kfd$-set in $G+H$. Suppose further that $S \neq V(G+H)$. If $S \subseteq V(G)$, then $|S| = k$ and $S$ is a $kfd$-set in $G$ by Theorem 5. Since $S$ is a total dominating set in $G+H$, it is a total dominating set in $G$. Hence, $S$ must be a $kftd$-set in $G$. Similarly, if $S \subseteq V(H)$, then $|S| = k$ and $S$ is a $kftd$-set in $H$. Suppose $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$. Then by Theorem 5, $S = S_G \cup S_H$, where $S_G$ is a $(k-|S_H|)fd$-set in $G$ and $S_H$ is a $(k-|S_G|)fd$-set in $H$, or $S = V(G) \cup T$, where $|V(G)| = m < k$ and $T$ is a $(k-m)fd$-set in $H$, or $S = D \cup V(H)$, where $|V(H)| = n < k$ and $D$ is a $(k-n)fd$-set in $G$.

Conversely, suppose one of Statements $(a)$ to $(f)$ holds. Then $S$ is a $kfd$-set in $G+H$ by Theorem 5. If Statement $(a)$ holds, then $S$ is clearly a $kftd$-set in $G+H$. Suppose Statement $(b)$ holds. Since $S$ is a $kftd$-set in $G$, $S$ is a $kftd$-set in $G+H$. Similarly, if Statement $(c)$ holds, then the same conclusion follows. If Statement $(d)$ is satisfied, then every vertex in $S_G$ is adjacent to each vertex in $S_H$ and vice versa, hence $S = S_G \cup S_H$ is a $kftd$-set in $G+H$. If Statement $(e)$ holds, then every vertex in $T$ is adjacent to each of the vertices in $G$ and each vertex in $G$ is adjacent to some vertex in $G$ and to each of the vertices in $T$, hence $S = V(G) \cup T$ is a $kftd$-set in $G+H$. Similarly, if Statement $(f)$ holds, then $S = D \cup V(H)$ is a $kftd$-set in $G+H$. This proves the assertion. □

**Corollary 2.** *Let $G$ and $H$ be connected nontrivial graphs of orders $m$ and $n$, respectively, and $k$ a positive integer with $2 \leq k \leq max\{m,n\}$. If $G$ or $H$ has a $kftd$-set $S$ with $|S| = k$, then $\gamma_{kftd}(G+H) = k$.*

**Theorem 9.** *Let $G$ be a nontrivial connected graph and $H$ a nontrivial graph, and let $k$ be a positive integer with $k \leq |V(H)|$. Then $C \subseteq V(G \circ H)$ is a $kftd$-set in $G \circ H$ if and only if one of the following holds:*

*(a)* $C = V(G) \cup B$, *where* $B = \varnothing$ *when* $k = 1$ *and* $B = \bigcup_{v \in V(G)} S_v$, *where each* $S_v$ *is a* $(k-1)fd$*-set in* $H^v$ *when* $k \geq 2$.

*(b)* $C = \bigcup_{v \in V(G)} S_v$, *where each* $S_v$ *is a* $kftd$*-set in* $H^v$ *and* $|S_v| = k$.

*Proof.* Suppose that Statement $(a)$ holds. Then by Theorem 6, $C$ is a $kfd$-set in $G \circ H$ when $k = 1$. If $B = \varnothing$, then $C = V(G)$ is clearly a $kftd$-set in $G \circ H$ when $k = 1$. Suppose $B = \bigcup_{v \in V(G)} S_v$, where each $S_v$ is a $(k-1)fd$-set in $H^v$. Each vertex $v$ in $V(G)$ is adjacent to some vertex $u$ in $V(G)$, and each $x \in S_v$ is adjacent to $v$. Thus, $C = V(G) \cup B$ is a $kftd$-set in $H^v$. Suppose Statement $(b)$ holds. Since each $S_v$ is a $kfd$-set in $H^v$ and $|S_v| = k$, $C = \bigcup_{v \in V(G)} S_v$ is a $kfd$-set in $G \circ H$ by Theorem 6. Moreover, since each $S_v$ is a $kftd$-set in $H^v$, it follows that $C$ is a $kftd$-set in $G \circ H$.

Conversely, suppose $C \subseteq V(G \circ H)$ is a $kftd$-set in $G \circ H$. Then $C$ is a $kfd$-set in $G \circ H$ and by Theorem 6, either Statement $(a)$ holds, or $C = \bigcup_{v \in V(G)} S_v$, where each $S_v$ is a $kfd$-set in $H^v$ and $|S_v| = k$. Suppose that there is a vertex $x$ in $S_v$ that is not adjacent to another vertex in $S_v$. Then $C$ is not a $kftd$-set in $V(G \circ H)$, contrary to assumption. Thus, each $S_v$ must be a $kftd$-set in $H^v$ and Statement $(b)$ holds. □

The next result is an immediate consequence of Theorem 9.

**Corollary 3.** *Let $G$ be a nontrivial connected graph of order $m$ and let $H$ be a nontrivial graph of order $n$, and let $k$ be a positive integer with $1 \leq k \leq n$. Then*

$$\gamma_{kftd}(G \circ H) = \begin{cases} m, & \text{if } k = 1 \\ mk, & \text{if } k \geq 2 \text{ and } H \text{ has a kftd-set } S \text{ with } |S| = k \\ m(1 + \gamma_{(k-1)fd}(H)), & \text{if } k \geq 2 \text{ and } H \text{ has no kftd-set } S \text{ with } |S| = k \ . \end{cases}$$

**Theorem 10.** *Let $G$ and $H$ be nontrivial connected graphs and let $k \geq 2$. Then $C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ is a $kftd$-set in $G[H]$ if and only if the following hold:*

- $(i)$ *$S$ is a dominating set in $G$,*
- $(ii)$ *for each $x \in S \cap N_G(S)$ such that $T_x \neq V(H)$, $T_x$ is an $rfd$-set and $\sum_{z \in N_G(x) \cap S} |T_z| = k - r$,*
- $(iii)$ *for each $x \in S \backslash N_G(S)$ with $T_x \neq V(H)$, $|T_x| = k$ and $T_x$ is a $kftd$-set in $H$, and*
- $(iv)$ *for each $y \in V(G) \backslash S$, $\sum_{v \in N_G(y) \cap S} |T_v| = k$.*

*Proof.* Suppose $C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ is a $kftd$-set in $G[H]$. Then $C$ is a $kfd$-set in $G[H]$ and by Theorem 7, Statements $(i)$, $(ii)$, and $(iv)$ hold. Moreover, for each $x \in S \backslash N_G(S)$, $T_x = V(H)$ and $|V(H)| \leq k$ or $|T_x| = k$ and $T_x$ is a $kfd$-set in $H$. Suppose there is a vertex $a \in T_x$ which is not adjacent to any other vertex in $T_x$. Then $(x, a)$ is not adjacent to any vertex in $C$, contrary to assumption. Hence, $T_x$ is a $kftd$-set in $H$ and Statement $(iii)$ holds.

Conversely, suppose Statements $(i)$ to $(iv)$ hold. Then $T_x$ is a $kfd$-set in $H$. Thus, $C$ is a $kfd$-set in $G[H]$ by Theorem 7. Suppose $C \neq V(G[H])$. Let $(x, a) \in C$. Consider the following cases.

Case 1: $x \in S \cap N_G(S)$

If $T_x = V(H)$ where $|V(H)| = r \leq k$, then there exists a $b \in T_x$ such that $ab \in E(H)$ since $H$ is a nontrivial connected graph. It follows that $(x, b) \in C$ and $(x, a)(x, b) \in E(G[H])$. If $T_x$ is an $rfd$-set and $\sum_{z \in N_G(x) \cap S} |T_z| = k - r$, then there is a $z \in N_G(x) \cap S$ and there is a $d \in T_z$ such that $(z, d) \in C$. Clearly, $(x, a)(z, d) \in E(G[H])$.

Case 2: $x \in S \backslash N_G(S)$

If $T_x = V(H)$ where $|V(H)| \leq k$, then similar to Case 1, there exists a $b \in T_x$ such that $ab \in E(H)$, $(x, b) \in C$, and $(x, a)(x, b) \in E(G[H])$. If $|T_x| = k$ and $T_x$ is a $kftd$-set in $H$, then there exists a $d \in T_x$ such that $ad \in E(H)$, $(x, d) \in C$, and $(x, a)(x, d) \in E(G[H])$. Therefore, in both cases, $C$ is a $kftd$-set in $G[H]$. □

**Corollary 4.** *Let $G$ and $H$ be nontrivial connected graphs with $\gamma_{1fd}(H) = 1$. If $G$ has a $\gamma_{2ftd}$-set $S$ with $|N_G(x) \cap S| = 1$ for all $x \in S$, then $\gamma_{2ftd}(G[H]) \leq \gamma_{2ftd}(G)$.*

*Proof.* For each $x \in S$, let $T_x = \{a\}$, where $\{a\}$ is a $\gamma_{1fd}$-set of $H$, and let $C = \bigcup_{x \in S}[\{x\} \times T_x]$. Since $S$ is a total dominating set, $|N_G(x) \cap S| = 1$ and $|T_x| = 1$ for all $x \in S$, Conditions $(i)$, $(ii)$, and $(iii)$ of Theorem 10 are satisfied. Moreover, since $S$ is a $\gamma_{2fd}$-set, $|N_G(y) \cap S| = 2$ for each $y \in V(G) \backslash S$. Hence, Condition $(iv)$ of Theorem 10 is also satisfied. Therefore, by Theorem 10, $C$ is a $2ftd$-set of $G[H]$. Accordingly, $\gamma_{2ftd}(G[H]) \leq |C| = \sum_{x \in S} |T_x| = |S| = \gamma_{2ftd}(G)$. □

**Remark 3.** *The bound given in Corollary 4 is sharp.*

To see this, consider the graph $P_5[P_3]$ shown in Figure 3. The shaded vertices in $P_5[P_3]$ form a $\gamma_{2ftd}$-set. Thus, $\gamma_{2ftd}(P_5[P_3]) = 4 = \gamma_{2ftd}(P_5)$.

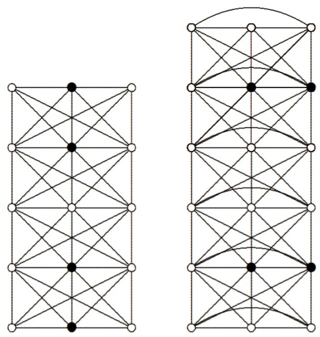


Figure 3: The graphs $P_5[P_3]$ and $P_6[C_3]$

**Corollary 5.** *Let $G$ and $H$ be nontrivial connected graphs such that $|V(H)| \geq 3$ and $\gamma_{2ftd}(H) = 2$. If $G$ has a $\gamma$-set $S$ such that $N_G(S) \cap S = \varnothing$ and $|N_G(y) \cap S| = 1$ for all $y \in V(G) \backslash S$, then $\gamma_{2ftd}(G[H]) \leq 2\gamma(G)$.*

*Proof.* Let $\{a, b\}$ be a $\gamma_{2ftd}$-set of $H$ and let $T_x = \{a, b\}$ for each $x \in S$. Let $C = \bigcup_{x \in S}[\{x\} \times T_x]$. Since $N_G(S) \cap S = \varnothing$, $S \backslash N_G(S) = S$, $|T_x| = 2$ and $T_x$ is a $2ftd$-set of $H$ for each $x \in S$, Conditions $(i)$, $(ii)$, and $(iii)$ of Theorem 10 are satisfied. Also, since $|N_G(y) \cap S| = 1$ for each $y \in V(G) \backslash S$, Condition $(iv)$ of Theorem 10 is also satisfied. Thus, by Theorem 10, $C$ is a $2ftd$-set of $G[H]$. Therefore, $\gamma_{2ftd}(G[H]) \leq |C| = \sum_{x \in S} |T_x| = 2|S| = 2\gamma(G)$. □

**Remark 4.** *The bound given in Corollary 5 is sharp.*

To see this, consider the graph $P_6[C_3]$ shown in Figure 3. The shaded vertices in $P_6[C_3]$ form a $\gamma_{2ftd}$-set. Thus, $\gamma_{2ftd}(P_6[C_3]) = 4 = 2\gamma(P_6)$.

**Theorem 11.** *Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively. Then $C_1 = S_1 \times V(H)$ and $C_2 = V(G) \times S_2$ are $kftd$-sets in $G\Box H$ if and only if $S_1$ and $S_2$ are $kfd$-sets in $G$ and $H$, respectively.*

*Proof.* Suppose $S_1$ is a $kfd$-set in $G$ and $C_1 = S_1 \times V(H)$. Let $(x,a) \in (G\Box H)\backslash C_1$. Then $x \notin S_1$. Since $S_1$ is a $kfd$-set in $G$, $|N_G(x) \cap S_1| = k$. Since $N_{G\Box H}((x,a)) \cap C = \bigcup_{y\in N_G(x)\cap S} [\{y\} \times \{a\}]$, it follows that $|N_{G\Box H}(x,a) \cap C_1| = |N_G(x) \cap S| = k$, showing that $C_1$ is a $k$-fair dominating set in $G\Box H$. Let $(z,c) \in C_1$. Since $H$ is a nontrivial connected graph, there exists $d \in V(H)$ such that $cd \in E(H)$. Thus, $(z,d) \in C_1$ and $(z,c)(z,d) \in E\langle C_1\rangle$. Hence, $C_1$ is a $kftd$-set in $G\Box H$. Similarly, $C_2 = V(G) \times S_2$, where $S_2$ is a $kfd$-set in $H$, is a $kftd$-set in $G\Box H$.

For the converse, suppose that $C_1 = S_1 \times V(H)$ is a $kftd$-set in $G\Box H$. Suppose further that $S_1$ is not a $kfd$-set in $G$. If $S_1$ is not a dominating set in $G$, then there exists an $x \in V(G)\backslash S_1$ such that $xy \notin E(G)$ for every $y \in S_1$. Let $a \in V(H)$. Then $(x,a) \in V(G\Box H)\backslash C_1$ and $(x,a)(y,a) \notin E(G\Box H)$ for any $(y,a) \in C_1$, contrary to the assumption that $C_1$ is a $kftd$-set, hence a dominating set. Thus, $S_1$ is a dominating set. If $S_1$ is not a $kfd$-set, then there exists a $u \in V(G)\backslash S_1$ such that $|N_G(u) \cap S_1| = r \neq k$. Let $a \in V(H)$. Then $(u,a) \in V(G\Box H)\backslash C_1$ and $|N_{G\Box H}(u,a) \cap C_1| = r \neq k$, contrary to the assumption that $C_1$ is a $kftd$-set. Therefore, $S_1$ is a $kfd$-set in $G$.

Similarly, if $C_2 = V(G) \times S_2$ is a $kftd$-set in $G\Box H$, then $S_2$ is a $kfd$-set in $H$. □

**Corollary 6.** *Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively, and $k$ a positive integer with $k \leq min\{m,n\}$. Then*

$$\gamma_{kftd}(G\Box H) \leq min\{m \cdot \gamma_{kfd}(H), n \cdot \gamma_{kfd}(G)\}.$$

**Remark 5.** *The bound given in Corollary 6 is sharp.*

To see this, consider the graphs shown in Figure 4. The shaded vertices in each graph form a $\gamma_{kftd}$-set. Thus, $\gamma_{1ftd}(P_4\Box C_3) = 4 = min\{4,6\} = \{4 \cdot 1, 3 \cdot 2\} = min\{m \cdot \gamma_{1fd}(C_3), n \cdot \gamma_{1fd}(P_4)\} = m \cdot \gamma_{1fd}(C_3)$, and $\gamma_{2ftd}(P_5\Box P_3) = 9 = min\{10,9\} = \{5 \cdot 2, 3 \cdot 3\} = min\{m \cdot \gamma_{2fd}(P_3), n \cdot \gamma_{2fd}(P_5)\} = n \cdot \gamma_{2fd}(P_5)$.

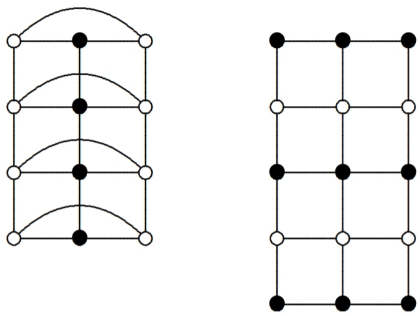

Figure 4: The graphs $P_4 \square C_3$ and $P_5 \square P_3$


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