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On a Topological Space Generated by Monophonic Eccentric Neighborhoods of a Graph

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Abstract. In this paper, we present a way of constructing a topology on a vertex set of a graph using monophonic eccentric neighborhoods of the graph G. In this type of construction, we characterize those graphs that induce the indiscrete topology, the discrete topology, and a particular point topology.

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Key Words and Phrases: Topology, graph, monophonic distance, monophonic eccentric neighborhood

1. Introduction

A metric or distance function in a non-empty set is known to generate a topology on the set via the family of open balls the metric induces. Indeed, it is well known that every metric space is a topological space. Topologizing a non-empty set can well be done by using a family of subsets of the set (as done in a metric space) that will serve as a base of some topology on the given set. Recently, topologizing the vertex set of a given graph was done to obtain topological spaces from a given graph. Gervacio and Diesto [2] used the standard neighborhoods of a graph to construct a topology on its vertex set. Admittedly, due to its limited circulation, the work is not so popular. This construction, however, was further studied in [3], [6] and [1].

Nianga and Canoy in [8] presented another way of generating a topology on a graph using the hop or 2-step neighborhoods of a graph. They further investigated in [9], the topologies induced by the complement of a graph, the join, corona, composition and the

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Cartesian product of graphs. The same construction was also studied by Canoy and Gimeno [4].

In this paper we construct a topology on a vertex set of a graph using its monophonic eccentric neigbhorhoods and investigate some of the topological structures and properties of the space generated. Under this construction we, among others, characterize those graphs that induced the indiscrete topology, the discrete topology and a particular point topology. For any two vertices u and v in a graph G, the distance $d_G(u,v)$ is the length of a shortest path joining u and v. The *open neighborhood* of a point u is the set $N_G(u)$ consisting of all points v which are adjacent to u. The *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G(A) = \bigcup_{v\in A} N_G(v)$ is called the *open neighborhood of* A and $N_G[A] = N_G(A) \cup A$ is called the *closed neighborhood of* A. The *complement of* $N_G[A]$ is denoted by $F_G[A]$ that is, $F_G[A] = V(G) \backslash N_G[A]$. If $A = \{v\}$, then we write $F_G[A] = F_G[v]$. For each $v \in V(G)$, $N^2_G(v) = \{u \in V(G) : d_G(u,v) = 2\}$ is called the *open hop neighborhood* of v and $N^2_G[v] = \{v\} \cup N^2_G(v)$ is called the *closed hop neighborhood of* v. For any $A \subseteq V(G)$, $N^2_G(A) = \bigcup_{a\in A} N^2_G(a) = \{v \in V(G) : N^2_G(v) \cap A \neq \varnothing\}$ is called the *open hop neighborhood of* A and $N^2_G[A] = A \cup N^2_G(A)$ is the *closed hop neighborhood of* A. Denote by $F^2_G[A]$ the *complement of* $N^2_G[A]$, that is, $F^2_G[A] = V(G) \backslash N^2_G[A]$. Recently, Titus [10] introduced some concepts related to monophonic paths in a graph. A chord of a path P in a graph G is an edge joining two non-adjacent vertices of P. A P in a graph G is called a *monophonic path* if it is chordless. For any two vertices u and v in a connected graph G, the *monophonic distance* $d^m_G(u,v)$ from u to v is defined as the length of a longest u-v monophonic path in G. The *monophonic eccentricity* $e^m_G(v)$ of a vertex v in G is the maximum monophonic distance from v to a vertex of G. The *monophonic radius* $rad_m(G)$ of graph G is $rad_m(G) = min\{e^m_G(v) : v \in V(G)\}$. A vertex w in G is a *monophonic eccentric vertex* of a vertex v in G if $e^m_G(v) = d^m_G(w,v)$. In this case, we say that w is a monophonic eccentric neighbor of v. The set of all monophonic eccentric vertices (neighbors) of v is denoted by $N^{e_m}_G(v)$. That is, $N^{e_m}_G(v) = \{w \in V(G) : d^m_G(w,v) = e^m_G(v)\}$. The *monophonic eccentric open neighborhood* of $A \subseteq V(G)$ given by $N^{e_m}_G(A) = \bigcup_{a\in A} N^{e_m}_G(a)$. The *monophonic eccentric closed neighborhood* of A is $N^{e_m}_G[A] = A \cup N^{e_m}_G(A)$. The complement of $N^{e_m}_G[A]$ is $F^{e_m}_G[A] = V(G) \backslash N^{e_m}_G[A]$. If $A = \{v\}$, we write $F^{e_m}_G[A] = F^{e_m}_G[v]$. For other basic concepts not defined here, we refer the readers to [5] and [7].

2. Results

The first few results show how a topological space from a given graph G is being constructed using the monophonic eccentric neighborhoods of the graph.

Lemma 1. *Let G be any graph and let $A, B \subseteq V(G)$. Then*

$$N^{e_m}_G(A \cup B) = N^{e_m}_G(A) \cup N^{e_m}_G(B).$$

Proof. Clearly, $N_G^{e_m}(A) \subseteq N_G^{e_m}(A \cup B)$ and $N_G^{e_m}(B) \subseteq N_G^{e_m}(A \cup B)$. Hence, $N_G^{e_m}(A) \cup N_G^{e_m}(B) \subseteq N_G^{e_m}(A \cup B)$. Next, let $w \in N_G^{e_m}(A \cup B)$. Then there exists $v \in A \cup B$ such that $d_G^m(w, v) = e_G^m(v)$. Thus, $w \in N_G^{e_m}(A)$ or $w \in N_G^{e_m}(B)$ showing that $N_G^{e_m}(A \cup B) \subseteq N_G^{e_m}(A) \cup N_G^{e_m}(B)$. Therefore, equality holds. □

Lemma 2. *Let G be any graph. If $A, B \subseteq V(G)$ and $A \subseteq B$, then $F_G^{e_m}[B] \subseteq F_G^{e_m}[A]$.*

Proof. Let $v \in F_G^{e_m}[B]$. Then $v \notin B$ and v is not a monophonic eccentric vertex of any vertex in B, that is $d_G^m(v, b) \neq e_G^m(b)$ for all $b \in B$. Since $A \subseteq B, v \notin A$ and v is not a monophonic eccentric vertex of A, that is, $d_G^m(v, a) \neq e_G^m(a)$ for all $a \in A$. Thus, $v \in F_G^{e_m}[A]$. Therefore, $F_G^{e_m}[B] \subseteq F_G^{e_m}[A]$. □

Lemma 3. *Let G be any graph. If $A, B \subseteq V(G)$ then*

$$F_G^{e_m}[A \cup B] = F_G^{e_m}[A] \cap F_G^{e_m}[B].$$

Proof. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B, F_G^{e_m}[A \cup B] \subseteq F_G^{e_m}[A]$ and $F_G^{e_m}[A \cup B] \subseteq F_G^{e_m}[B]$ by Lemma 2. Thus,

$$F_G^{e_m}[A \cup B] \subseteq F_G^{e_m}[A] \cap F_G^{e_m}[B].$$

Now, let $v \in F_G^{e_m}[A] \cap F_G^{e_m}[B]$. Then $v \in F_G^{e_m}[A]$ and $v \in F_G^{e_m}[B]$. It follows that $v \notin A$, $v \notin B, v \notin N_G^{e_m}(A)$ and $v \notin N_G^{e_m}(B)$. Hence, by Lemma 1, $v \notin A \cup B$ and $v \notin N_G^{e_m}(A \cup B)$. Therefore, $v \in F_G^{e_m}[A \cup B]$ and $F_G^{e_m}[A] \cap F_G^{e_m}[B] \subseteq F_G^{e_m}[A \cup B]$. Accordingly,

$$F_G^{e_m}[A \cup B] = F_G^{e_m}[A] \cap F_G^{e_m}[B].$$

□

Note that Lemma 3 can also be proved using Lemma 1. By induction on the number of sets involved, the next is immediate.

Theorem 1. *Let G be any graph. If $A_1, A_2, ..., A_n$ are subsets of $V(G)$, then*

$$F_G^{e_m}\left[\bigcup_{i=1}^{n} A_i\right] = \bigcap_{i=1}^{n} F_G^{e_m}[A_i].$$

Theorem 2. *Let G be any graph. The family $\mathcal{B}_G^{e_m} = \{F_G^{e_m}[A] : A \subseteq V(G)\}$ is a base for some topology on $V(G)$.*

Proof. Note that $N_G^{e_m}[\varnothing] = \varnothing$ and so $F_G^{e_m}[\varnothing] = V(G) \in \mathcal{B}_G^{e_m}$. Now let $A, B \subseteq V(G)$. By Lemma 3, $F_G^{e_m}[A] \cap F_G^{e_m}[B] = F_G^{e_m}[A \cup B] \in B_G^{e_m}$. Therefore, $\mathcal{B}_G^{e_m}$ is a base for some topology on $V(G)$. □

Henceforth, we denote by $\tau_G^{e_m}$ the topology generated by $\mathcal{B}_G^{e_m}$. Also we denote by $\mathcal{I}_G$ and $\mathcal{D}_G$ the indiscrete and the discrete topologies on $V(G)$, respectively.

Theorem 3. *Let G be any graph. The family $\mathcal{S}_G^{e_m} = \{F_G^{e_m}[v] : v \in V(G)\}$ forms a subbase for $\tau_G^{e_m}$.*

Proof. Let $\mathcal{S}_G^{e_m} = \{F_G^{e_m}[v] : v \in V(G)\}$ and let $A = \{a_1, a_2, ..., a_n\}$. By Lemma 3, $F_G^{e_m}[a_1] \cap F_G^{e_m}[a_2] \cap ... \cap F_G^{e_m}[a_n] = F_G^{e_m}[A]$. Thus, every element of $\mathcal{B}_G^{e_m}$ is a finite intersection of members of $\mathcal{S}_G^{e_m}$. Therefore, $\mathcal{B}_G^{e_m}$ is a subbase of $\tau_G^{e_m}$. □

Theorem 4. *Let G be any graph of order $n \geq 1$. Then $\tau_G^{e_m}$ is the indiscrete topology if and only if $G = K_n$.*

Proof. Suppose that $\tau_G^{e_m}$ is the indiscrete topology. Suppose further that $G \neq K_n$. Then there exist $x, y \in V(G)$ such that $d_G^m(x, y) = e_G^m(x) \geq 2$. Let $P = [x_1, x_2, ..., x_k]$, where $x_1 = x$ and $x_k = y$, be an x-y monophonic path. Then $k \geq 3$ and $x_2 \notin N_G^{e_m}[x]$. Hence, $x_2 \in F_G^{e_m}[x] \neq \varnothing$. Since $x, y \notin F_G^{e_m}[x]$, it follows that $F_G^{e_m}[x] \neq V(G)$. Therefore, $\tau_G^{e_m}$ is not the indiscrete topology, a contradiction. Thus, $G = K_n$. Let $G = K_n$ and let A be a non empty subset of $V(G)$. Then $N_G^{e_m}[A] = V(G)$. Hence, $F_G^{e_m}[A] = \varnothing$. Therefore, $\tau_G^{e_m}$ is the indiscrete topology on $V(G)$. □

Theorem 5. *Let G be any graph. Then $\tau_G^{e_m}$ is the discrete topology on $V(G)$ if and only if for each $a \in V(G)$ and for each $v \in V(G)$ with $a \in N_G^{e_m}(v)$, there exists $w \in V(G)\backslash\{a\}$ such that $v \in N_G^{e_m}(w)$ but $a \notin N_G^{e_m}(w)$.*

Proof. Suppose that $\tau_G^{e_m}$ is the discrete topology $\mathcal{D}_G$ on $V(G)$. Let $a \in V(G)$ and let $v \in V(G)$ with $a \in N_G^{e_m}(v)$. Since $\tau_G^{e_m}$ is the discrete topology, $\{a\} \in \mathcal{B}_G^{e_m}$, that is, there exists $A \subseteq V(G)$ such that $F_G^{e_m}[A] = \{a\}$. Since $a \in N_G^{e_m}(v), v \notin A$. Also, $v \notin F_G^{e_m}[A]$ implies that there exists $w \in A$ such that $d_G^m(w, v) = e_G^m(w)$, that is, $v \in N_G^{e_m}(w)$. Moreover, because $a \in F_G^{e_m}[A]$, $a \notin N_G^{e_m}(w)$. Thus, G satisfies the desired property. For the converse, suppose that the given condition is satisfied by G. If $G = K_1$, then clearly, $\tau_G^{e_m} = \mathcal{D}_G$. Suppose $G \neq K_1$. Let $a \in V(G)$ and let $A_a = \{v \in V(G) : a \in N_G^{e_m}(v)\}$. Set $A = V(G)\backslash(A_a \cup \{a\})$. Then, by assumption, $A \neq \varnothing$. Since $a \notin A$ and $a \notin N_G^{e_m}(w)$ for all $w \in A$, it follows that $a \in F_G^{e_m}[A]$. Suppose there exists $q \in F_G^{e_m}[A]\backslash\{a\}$. Then $q \notin A \cup \{a\}$. Hence, $q \in A_a$, that is, $a \in N_G^{e_m}(q)$. By assumption, there exists $w \notin A_a \cup \{a\}$ such that $q \in N_G^{e_m}(w)$, that is, $d_G^m(q, w) = e_G^m(w)$. This contradicts the fact that $q \in F_G^{e_m}[A]$. Therefore, $F_G^{e_m}[A] = \{a\}$. Since a was arbitrarily chosen, it follows that $\{a\} \in \tau_G^{e_m}$ for all $a \in V(G)$. Thus, $\tau_G^{e_m}$ is the discrete topology. □

Corollary 1. *Let $G_1, G_2, ..., G_n$ be graphs such that $\tau_{G_i}^{e_m} = \mathcal{D}_{G_i}$ for each $i \in \{1, 2, ..., n\}$. If $G = \bigcup_{i=1}^{n} G_i$, then $\tau_G^{e_m} = \mathcal{D}_G$.*

Proof. Let $G = \bigcup_{i=1}^{n} G_i$ and let $a, v \in V(G)$ such that $a \in N_G^{e_m}(v)$. Then there exists a unique $i \in \{1, 2, ..., n\}$ such that $a, v \in V(G_i)$. Hence, $a \in N_{G_i}^{e_m}(v)$. Since $\tau_{G_i}^{e_m} = \mathcal{D}_{G_i}$, there exists $w \in V(G_i)\backslash\{a\}$ such that $v \in N_{G_i}^{e_m}(w)$ and $a \notin N_{G_i}^{e_m}(w)$ by Theorem 5. Thus, there exists $w \in V(G)\backslash\{a\}$ such that $v \in N_G^{e_m}(w)$ and $a \notin N_G^{e_m}(w)$. Therefore, $\tau_G^{e_m} = \mathcal{D}_G$. □

Corollary 2. *If* $G = \overline{K}_n$*, then* $\tau_G^{e_m} = \mathcal{D}_G$*.*

Proof. Let $a \in V(G)$. Then $A_a = \{v \in V(G) : a \in N_G^{e_m}(v)\} = \varnothing$. Let

$$A = V(G)\backslash[A_a \cup \{a\}] = V(G)\backslash\{a\}.$$

Then, $F_G^{e_m}[A] = \{a\} \in \tau_G^{e_m}$. Thus, $\tau_G^{e_m} = \mathcal{D}_G$. □

Lemma 4. *Let* $G = C_n = [v_1, v_2, ...v_n, v_1]$ *be a cycle with* $n \geq 3$*. Then* $e_G^m(v) = n - 2$ *for all* $v \in V(G)$*.*

Proof. Suppose $w \in V(C_n)$. Without loss of generality, let $w = v_1$. Since

$$e_{C_n}^m(w) = max\{d_{C_n}^m(w, v) : v \in V(C_n)\},$$

it follows that $e_{C_n}^m(w) = n - 2$. □

Example 1. *Let* $C_5 = [v_1, v_2, v_3, v_4, v_5, v_1]$*. Then by Lemma 4, we have*

$$\begin{aligned}
N_{C_5}^{e_m}[v_1] &= \{v_1, v_3, v_4\} & F_{C_5}^{e_m}[v_1] &= \{v_2, v_5\}\\
N_{C_5}^{e_m}[v_2] &= \{v_2, v_4, v_5\} & F_{C_5}^{e_m}[v_2] &= \{v_1, v_3\}\\
N_{C_5}^{e_m}[v_3] &= \{v_1, v_3, v_5\} & F_{C_5}^{e_m}[v_3] &= \{v_2, v_4\}\\
N_{C_5}^{e_m}[v_4] &= \{v_1, v_2, v_4\} & F_{C_5}^{e_m}[v_4] &= \{v_3, v_5\}\\
N_{C_5}^{e_m}[v_5] &= \{v_2, v_3, v_5\} & F_{C_5}^{e_m}[v_5] &= \{v_1, v_4\}.
\end{aligned}$$

Note that

$$\begin{aligned}
F_{C_5}^{e_m}[v_2] \cap F_{C_5}^{e_m}[v_5] &= \{v_1\} & F_{C_5}^{e_m}[v_3] \cap F_{C_5}^{e_m}[v_5] &= \{v_4\}\\
F_{C_5}^{e_m}[v_1] \cap F_{C_5}^{e_m}[v_3] &= \{v_2\} & F_{C_5}^{e_m}[v_1] \cap F_{C_5}^{e_m}[v_4] &= \{v_5\}.\\
F_{C_5}^{e_m}[v_1] \cap F_{C_5}^{e_m}[v_2] &= \{v_3\}
\end{aligned}$$

Since $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\} \in \mathcal{B}_{C_5}^{e_m}$*, it follows that* $\tau_{C_5}^{e_m} = \mathcal{D}_{C_5}$*.*

Example 2. *Consider now* $C_6 = [v_1, v_2, v_3, v_4, v_5, v_6, v_1]$*. Then by Lemma 4, we have*

$$\begin{aligned}
N_{C_6}^{e_m}[v_1] &= \{v_1, v_3, v_5\} & F_{C_6}^{e_m}[v_1] &= \{v_2, v_4, v_6\}\\
N_{C_6}^{e_m}[v_2] &= \{v_2, v_4, v_6\} & F_{C_6}^{e_m}[v_2] &= \{v_1, v_3, v_5\}\\
N_{C_6}^{e_m}[v_3] &= \{v_1, v_3, v_5\} & F_{C_6}^{e_m}[v_3] &= \{v_2, v_4, v_6\}\\
N_{C_6}^{e_m}[v_4] &= \{v_2, v_4, v_6\} & F_{C_6}^{e_m}[v_4] &= \{v_1, v_3, v_5\}\\
N_{C_6}^{e_m}[v_5] &= \{v_1, v_3, v_5\} & F_{C_6}^{e_m}[v_5] &= \{v_2, v_4, v_6\}\\
N_{C_6}^{e_m}[v_6] &= \{v_2, v_4, v_6\} & F_{C_6}^{e_m}[v_6] &= \{v_1, v_3, v_5\}.
\end{aligned}$$

Note that $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\} \notin \mathcal{B}_{C_6}^{e_m}$*. Hence,* $\tau_{C_6}^{e_m} \neq \mathcal{D}_{C_6}$*.*

Theorem 6. *$\tau^{e_m}_{C_n} \neq \mathcal{D}_{C_n}$ for $n = 3, 4, 6$ and $\tau^{e_m}_{C_n} = \mathcal{D}_{C_n}$ for $n \in \{5, 7, 8, ...\}$.*

Proof. Since $C_3 \cong K_3, \tau^{e_m}_{C_3} = \mathcal{I}_{C_3} \neq \mathcal{D}_{C_3}$ by Theorem 4. Let $C_4 = [v_1, v_2, v_3, v_4, v_1]$ and let $a = v_1$. Set $A_a = \{v \in V(C_4) : a \in N^{e_m}_{C_4}(v)\}$. Then by Lemma 4, $A_a = \{v_3\}$. Note that $v_3 \notin N^{e_m}_G(v_2) \cap N^{e_m}_G(v_4)$. Hence, we could not find $w \neq a$ such that $v_3 \in N^{e_m}_{C_4}(w)$. Therefore, C_4 does not induce the discrete topology. Suppose $C_6 = [v_1, v_2, v_3, v_4, v_5, v_6, v_1]$ and let $a = v_1$. Set $A_a = \{v \in V(C_6) : a \in N^{e_m}_{C_6}(v)\}$. Again, by Lemma 4, $A_a = \{v_3, v_5\}$. Note that the only vertex $w \neq a$ with $v_3 \in N^{e_m}_{C_6}(w)$ is v_5. However, $a = v_1 \in N^{e_m}_{C_6}(v_5)$. Thus, by Theorem 5, C_6 does not induce the discrete topology. Next let $n = 5$ and let $a \in V(C_5)$. We may assume that $a = v_1$. Let $A_a = \{v \in V(C_n) : a \in N^{e_m}_{C_n}(v)\}$. Then $A_a = \{v_3, v_4\}$. Since $v_3 \in N^{e_m}_{C_n}(v_5), v_4 \in N^{e_m}_{C_n}(v_2)$ where $v_2, v_5 \notin A_a$, it follows from Theorem 5 that $\tau^{e_m}_{C_5} = \mathcal{D}_{C_5}$. Suppose $n \geq 7$. Let $a = v_1$. Then, $A_a = \{v \in V(C_n) : a \in N^{e_m}_{C_n}(v)\}$. Thus, $A_a = \{v_3, v_{n-1}\}$. Note that $v_3 \in N^{e_m}_{C_n}(v_5)$ and $v_{n-1} \in N^{e_m}_{C_n}(v_{n-3})$ but $v_1 \notin N^{e_m}_{C_n}(v_5) \cap N^{e_m}_{C_n}(v_{n-3})$. Thus, by Theorem 5, $\tau^{e_m}_{C_n} = \mathcal{D}_{C_n}$. □

Theorem 7. *Let $G = C_n$ be a cycle with ≥ 4. Then*

$$F^{e_m}_G[v_i] = \begin{cases} V(G)\backslash\{v_i,\ v_{i+2},\ v_{i+n-2}\}, & \text{if } i = 1, 2 \\ V(G)\backslash\{v_{i-2},\ v_i,\ v_{i+2}\}, & \text{if } 3 \leq i \leq n-2 \\ V(G)\backslash\{v_{i-n+2},\ v_{i-2},\ v_i\}, & \text{if } i = n, n-1 \end{cases}$$

where $v_{i+2} = v_{i+n-2}$ and $v_{i-2} = v_{i-n+2}$ if $n = 4$.

Proof. Let $i = 1$. By Lemma 4, $e^m_G(v) = 2$. Thus, $N^{e_m}_{C_4}[v_1] = \{v_1, v_3\}$. Hence, $F^{e_m}_{C_4}[v_1] = V(C_4)\backslash\{v_i, v_{i+2}\}$. Similarly, if $i = 2$, then $F^{e_m}_{C_4}[v_2] = V(C_4)\backslash\{v_i, v_{i+2}\}$. If $i = n$, then $N^{e_m}_{C_4}[v_4] = \{v_2, v_4\}$. Thus, $F^{e_m}_{C_4}[v_4] = V(C_4)\backslash\{v_i, v_{i-2}\}$. Similarly, if $i = n-1$, then $F^{e_m}_{C_4}[v_3] = V(C_4)\backslash\{v_i, v_{i-2}\}$. Let $i \in \{1, 2\}$. By Lemma 4, $N^{e_m}_{C_n}[v_1] = \{v_1, v_3, v_{n-1}\}$ and $N^{e_m}_{C_n}[v_2] = \{v_2, v_4, v_n\}$. Thus, $F^{e_m}_{C_n}[v_i] = V(C_n)\backslash\{v_i, v_{i+2}, v_{i+n-2}\}$. Suppose that $i \in \{3, 4, ..., n-2\}$. Then, $N^{e_m}_{C_n}[v_i] = \{v_{i-2}, v_i, v_{i+2}\}$. It follows that

$$F^{e_m}_{C_n}[v_i] = V(C_n)\backslash\{v_{i-2}, v_i, v_{i+2}\}.$$

Next, suppose, $i \in \{n, n-1\}$. By Lemma 4, $N^{e_m}_{C_n}[v_n] = \{v_2, v_{n-2}, v_n\}$ and $N^{e_m}_{C_n}[v_{n-1}] = \{v_1, v_{n-3}, v_{n-1}\}$. Therefore,

$$F^{e_m}_{C_n}[v_i] = V(C_n)\backslash\{v_{i-2}, v_i, v_{i-n+2}\}.$$

This proves the assertion. □

Lemma 5. *$F^{e_m}_{C_3}[v] = \varnothing$ for all $v \in V(C_3)$.*

Theorem 8. *Let $G = P_n = [v_1, v_2, ...v_n]$ be a path of order $n \geq 3$.*

(a) If n is even, then $\tau^{e_m}_G$ has a subbase consisting of all sets of the form

$$F^{e_m}_G[v_i] = \begin{cases} V(G)\backslash\{v_i, v_n\} & \text{if } i \leq \frac{n}{2} \\ V(G)\backslash\{v_1, v_i\} & \text{if } i > \frac{n}{2}. \end{cases}$$

(*b*) *If n is odd, then $\tau_G^{e_m}$ has a subbase consisting of all sets of the form*

$$F_G^{e_m}[v_i] = \begin{cases} V(G)\backslash\{v_i, v_n\} & \text{if } i < \frac{n+1}{2} \\ V(G)\backslash\{v_1, v_i, v_n\} & \text{if } i = \frac{n+1}{2} \\ V(G)\backslash\{v_1, v_i\} & \text{if } i > \frac{n+1}{2}. \end{cases}$$

Proof. Suppose n is even. Let $i \leq \frac{n}{2}$. Then $N_G^{e_m}[v_i] = \{v_i, v_n\}$. Hence, $F_G^{e_m}[v_i] = V(G)\backslash\{v_i, v_n\}$. If $i > \frac{n}{2}$, then $N_G^{e_m}[v_i] = \{v_1, v_i\}$. Thus, $F_G^{e_m}[v_i] = V(G)\backslash\{v_1, v_i\}$. Suppose n is odd. Let $i < \frac{n+1}{2}$. Then $N_G^{e_m}[v_i] = \{v_i, v_n\}$. Thus, $F_G^{e_m}[v_i] = V(G)\backslash\{v_i, v_n\}$. Suppose $i = \frac{n+1}{2}$. Then $N_G^{e_m}[v_i] = \{v_1, v_i, v_n\}$. Hence, $F_G^{e_m}[v_i] = V(G)\backslash\{v_1, v_i, v_n\}$. Let $i > \frac{n+1}{2}$. Then $N_G^{e_m}[v_i] = \{v_1, v_i\}$. Therefore, $F_G^{e_m}[v_i] = V(G)\backslash\{v_1, v_i\}$. □

Theorem 9. *Let $G = P_n = [v_1, v_2, ...v_n]$ be a path of order $n \geq 3$. Then $\{v\} \in \tau_G^{e_m}$ if and only if $v \neq v_1, v_n$.*

Proof. Suppose $\{v\} \in \tau_G^{e_m}$. Suppose further that $v = v_1$. Then there exists $\varnothing \neq A \subseteq V(G)$ such that $F_G^{e_m}[A] = \{v_1\}$. This means that $v_1 \notin A$ and $d_G^m(v_1, a) \neq e_G^m(a)$ for all $a \in A$. Since $N_G^{e_m}(v_n) = \{v_1\}, v_n \notin A$. First, suppose that n is odd. From Theorem 8 (*b*) it follows that $v_i \notin A$ for all $i \geq \frac{n+1}{2}$. Hence,

$$A \subseteq \{v_j : 1 < j < \frac{n+1}{2}\}.$$

Thus, by Theorem 8, $v_{\frac{n+1}{2}} \in F_G^{e_m}[A]$, a contradiction. Suppose n is even. From Theorem 8 (*a*), $v_i \notin A$ for all $i > \frac{n}{2}$. Thus,

$$A \subseteq \{v_j : 1 < j \leq \frac{n}{2}\}.$$

Hence, by Theorem 8, $v_{\frac{n}{2}+1} \in F_G^{e_m}[A]$, a contradiction. Therefore, $\{v_1\} \notin \tau_G^{e_m}$. Similarly, $\{v_n\} \notin \tau_G^{e_m}$. For the converse, suppose that $v \neq v_1, v_n$ and let $v_j \in P_n$. Consider the following cases:
Case 1. $1 < j < \lceil \frac{n}{2} \rceil$. Let $A = V(G)\backslash\{v_j, v_n\}$. Then $F_G^{e_m}[A] = \{v_j\}$.
Case 2. $j = \lceil \frac{n}{2} \rceil$. If n is odd and $j = \frac{n+1}{2}$, then set $B = V(G)\backslash\{v_1, v_j, v_n\}$. Then $F_G^{e_m}[B] = \{v_j\}$. If n is even, set $B = V(G)\backslash\{v_j, v_n\}$.
Case 3. $\lceil \frac{n}{2} \rceil < j < n$. Let $D = V(G)\backslash\{v_1, v_j\}$. Then $F_G^{e_m}[D] = \{v_j\}$. Therefore, $\{v_j\} \in \tau_G^{e_m}$ for all $j \in \{2, 3, ..., n-1\}$. □

Definition 1. *The join $G + H$ of graphs G and H is the graph K with $V(K) = V(G) \cup V(H)$ and $E(K) = E(G) \cup E(H) \cup \{uv : u \in V(G)$ and $v \in V(H)\}$.*

Theorem 10. *Let G be any graph and let $K_1 = \langle v \rangle$.*

(*i*) *If G is connected, then*

$$F_{K_1+G}^{e_m}[w] = \begin{cases} \varnothing, & \text{if } [w \in V(G) \text{ and } e_G^m(w) = 1] \\ & \text{ or } w = v \\ F_G^{e_m}[w] \cup \{v\}, & \text{if } w \in V(G) \text{and} e_G^m(w) \geq 2. \end{cases}$$

(ii) If G is disconnected, then

$$F^{e_m}_{K_1+G}[w] = \begin{cases} \varnothing, & \text{if } w = v \\ N_G(w) \cup \{v\} & \text{if } w \in V(G) \text{ and } \\ & 1 \leq e^m_G(w) \leq 2 \\ F^{e_m}_G[w] \cup \{v\}, & \text{if } w \in V(G) \text{ and } e^m_G(w) \geq 3. \end{cases}$$

Proof. (i) Let G be a connected graph. Suppose $w \in V(G)$ and $e^m_G(w) = 1$. Then, $N^{e_m}_{K_1+G}[w] = V(K_1+G)$. It follows that $F^{e_m}_{K_1+G}[w] = \varnothing$. Clearly, if $w = v$ then $F^{e_m}_{K_1+G}[w] = \varnothing$. Next, suppose $w \in V(G)$ and $e^m_G(w) \geq 2$. Then $[w, v, g]$ is a monophonic path in K_1+G for all $g \notin N_G(w)$. Moreover, since every monophonic path in G is a monophonic path in $K_1 + G$. It follows that $N^{e_m}_{K_1+G}[w] = N^{e_m}_G[w]$. Thus, $F^{e_m}_{K_1+G}[w] = F^{e_m}_G[w] \cup \{v\}$.
(ii) Let G be a disconnected graph. Clearly, if $w = v$, then $N^{e_m}_{K_1+G}[w] = V(K_1 + G)$ and $F^{e_m}_{K_1+G}[w] = \varnothing$. Suppose $w \in V(G)$ and $1 \leq e^m_G(w) \leq 2$. Observe that $N^{e_m}_{K_1+G}[w] = V(G)\backslash N_G(w)$. Hence, $F^{e_m}_{K_1+G}[w] = N_G(w) \cup \{v\}$. Suppose $w \in V(G)$ and $e^m_G(w) \geq 3$. Since every monophonic path in G is a monophonic path in $K_1 + G$. It follows that $N^{e_m}_{K_1+G}[w] = N^{e_m}_G[w]$. Therefore, $F^{e_m}_G[w] \cup \{v\}$. □

Corollary 3. *Let $K_1 = \langle v_0 \rangle$ and let G be any graph. Then,*

(i) $\mathcal{S}_{K_1+G} = \{\varnothing\} \cup \{F^{e_m}_G[w] \cup \{v_0\} : w \in V(G) \text{ and } e^m_G(w) \geq 2\}$ *if G is connected,*

(ii) $\mathcal{S}_{K_1+G} = \{\varnothing\} \cup \{N_G(w) \cup \{v_0\} : w \in V(G) \text{ and } deg_G(w) = 0 \text{ or } 1 \leq e^m_G(w) \leq 2\} \cup \{F^{e_m}_G[w] \cup \{v_0\} : w \in V(G) \text{ and } e^m_G(w) \geq 3\}$ *if G is disconnected.*

(iii) $\{v\} \notin \tau^{e_m}_{K_1+G}$ *for all $v \in V(G)$.*

Proof. Set $H = K_1$. By Theorem 10 (i) and Theorem 10 (ii), (i) and (ii) hold. By (i) and (ii), (iii) holds. □

Lemma 6. *Let $K_1 = \langle v_0 \rangle$ and let G be any graph with $rad_m(G) \geq 2$. Then $\{v_0\} \in \tau^{e_m}_{K_1+G}$.*

Proof. Suppose G is any graph with $rad_m(G) \geq 2$. Then, $e^m_G(z) \geq 2$ for all $z \in V(G)$. Let $v \in V(G)$. Since $v \notin (F^{e_m}_G[v] \cap \{v_0\})$, it follows that $v \notin \bigcap_{z \in V(G)} (F^{e_m}_G[z] \cap \{v_0\})$. Since v was arbitrarily chosen, $\{v_0\} = \bigcap_{z \in V(G)} (F^{e_m}_G[z] \cup \{v_0\})$. By Corollary 3,

$$(F^{e_m}_G[z] \cup \{v_0\}) \in \mathcal{S}^{e_m}_{K_1+G} \subseteq \tau^{e_m}_{K_1+G}.$$

Therefore, $\{v_0\} \in \tau^{e_m}_{K_1+G}$. □

Definition 2. *Let $X \neq \varnothing$ and $p \in X$. The particular point p topology on X is the class $\tau_p = \{\varnothing\} \cup \{A \subseteq X : p \in A\}$.*

Theorem 11. *Let $K_1 = \langle v_0 \rangle$ and let G be a connected graph with $rad_m(G) \geq 2$. Then $\tau^{e_m}_{K_1+G}$ is the particular point topology τ_{v_0} if and only if $\tau^{e_m}_G$ is the discrete topology on $V(G)$.*

Proof. Suppose $\tau^{e_m}_G$ is the discrete topology on $V(G)$. Note that $\{v_0\} \in \tau^{e_m}_{K_1+G}$. Now, since $\{v_0\} \in (F^{e_m}_G[w] \cap \{v_0\})$ for all $w \in V(G)$, it follows that $\{v\} \notin \mathcal{B}^{e_m}_{K_1+G} \subseteq \tau^{e_m}_{K_1+G}$. Next, since $\tau^{e_m}_G$ is a discrete topology, $\{v\} \in \mathcal{B}^{e_m}_G$ for all $v \in V(G)$. Hence, there exist $v_{j_1}, v_{j_2}, ..., v_{j_k} \in V(G)$ such that $\{v\} = \bigcap_{s=1}^{k} F^{e_m}_G[v_{j_s}]$. Therefore,

$$\{v_0, v\} = \left(\bigcap_{s=1}^{k} F^{e_m}_G[v_{j_s}]\right) \cup \{v_0\} = \bigcap_{s=1}^{k} \left(F^{e_m}_G[v_{j_s}] \cup \{v_0\}\right) \in \mathcal{B}^{e_m}_{K_1+G} \subseteq \tau^{e_m}_{K_1+G}.$$

Accordingly, $\tau^{e_m}_{K_1+G} = \tau_{v_0}$. For the converse, suppose that $\tau^{e_m}_{K_1+G} = \tau_{v_0}$. Let $v \in V(G)$. Since $\tau^{e_m}_{K_1+G} = \tau_{v_0}$, $\{v_0, v\} \in \tau^{e_m}_{K_1+G}$. Hence, there exists a basic open set $B \in \mathcal{B}^{e_m}_{K_1+G}$ such that $v \in B \subseteq \{v_0, v\}$. Since $\{v\}$ cannot be a finite intersection of subbasic open sets of the form $F^{e_m}_G[w] \cup \{v_0\}$, it follows that $B \neq \{v\}$. Thus, $B = \{v_0, v\}$. This means that there exist $v_{j_1}, v_{j_2}, ..., v_{j_k} \in V(G)$ such that $\{v_0, v\} = \bigcap_{k=1}^{t} (F^{e_m}_G[v_{j_k}] \cup \{v_0\})$. Therefore, $\{v\} = \bigcap_{k=1}^{t} F^{e_m}_G[v_{j_k}]$, that is, $\{v\} \in \mathcal{B}^{e_m}_G \subseteq \tau^{e_m}_G$. This shows that $\tau^{e_m}_G$ is the discrete topology on $V(G)$. □

Theorem 12. *Let H be a graph with $rad_m(\langle V(H) \backslash \{v_0\} \rangle) \geq 2$ and let $v_0 \in V(H)$. Then $\tau^{e_m}_H = \tau_{v_0}$ if and only if $H = \langle v_0 \rangle + G$ for some graph G such that $rad_m(G) \geq 2$ and $\tau^{e_m}_G$ is the discrete topology on $V(G)$.*

Proof. Suppose $\tau^{e_m}_H = \tau_{v_0}$. Suppose further that there exists $v \in V(H) \backslash \{v_0\}$ such that $v_0 v \notin E(H)$. Then $e^m_H(v_0) \geq 2$. This implies that $N_H(v_0) \neq \varnothing$ and $N_H(v_0) \cap N^{e_m}_H(v_0) = \varnothing$. Hence, $N_H(v_0) \subseteq F^{e_m}_H[v_0]$. This gives a contradiction because $v_0 \notin F^{e_m}_H[v_0]$ and $F^{e_m}_H[v_0] \in \tau^{e_m}_H$. Therefore, $v_0 v \in E(H)$ for all $v \in V(H) \backslash \{v_0\}$. Let $G = \langle V(H) \backslash \{v_0\} \rangle$. Then $H = \langle v_0 \rangle + G$. By Theorem 11, $\tau^{e_m}_G$ is the discrete topology on $V(G)$. For the converse, suppose $H = \langle v_0 \rangle + G$ for some graph G such that $rad_m(G) \geq 2$ and $\tau^{e_m}_G$ is the discrete topology on $V(G)$. Then by Theorem 11, $\tau^{e_m}_H = \tau_{v_0}$. □

Corollary 4. *Let $G = W_n = \langle v_0 \rangle + C_n$, where $n \in \{5, 7, 8, ...\}$. Then $\tau^{e_m}_{W_n}$ is the particular point topology τ_{v_0}.*

Proof. Let $n \in \{5, 7, 8, ...\}$. Then $\tau^{e_m}_{C_n}$ is the discrete topology on $V(C_n)$ by Theorem 6. Thus, by Theorem 11, $\tau^{e_m}_{W_n}$ is the particular point topology τ_{v_0}. □

Theorem 13. *Let $G = F_n = \langle v_0 \rangle + P_n$ $(n \geq 4)$. Then $\{v_0, v\} \in \tau^{e_m}_{F_n}$ for all $v \in V(P_n) \backslash \{v_1, v_n\}$.*

Proof. Suppose $v \in V(P_n)\backslash\{v_1, v_n\}$. By Theorem 9, $\{v\} \in \tau_{P_n}^{e_m}$. Thus, $\{v\} \in \mathcal{B}_{P_n}^{e_m}$. Hence, there exist $v_{i_1}, v_{i_2}, ..., v_{i_k} \in V(P_n)$ such that $\{v\} = \bigcap_{s=1}^{k} F_{P_n}^{e_m}[v_{i_s}]$. Therefore,

$$\{v_0, v\} = \bigcap_{s=1}^{k}(F_{P_n}^{e_m}[v_{i_s}] \cup \{v_0\}) \in \mathcal{B}_{P_n}^{e_m} \subseteq \tau_{F_n}^{e_m},$$

proving our assertion. □

Theorem 14. *If n is a positive integer and $K_1 = \langle v_0 \rangle$, then*

$$\tau_{K_{1,n}}^{e_m} = \begin{cases} \{\varnothing, V(K_{1,n})\} & \text{if } n = 1 \\ \{\varnothing, \{v_0\}, V(K_{1,n})\} & \text{if } n \geq 2. \end{cases}$$

Proof. By Corollary 3,

$$\mathcal{S}_{K_{1,n}}^{e_m} = \begin{cases} \{\varnothing\} & \text{if } n = 1 \\ \{\varnothing\} \cup \{v_0\} & \text{otherwise.} \end{cases}$$

Hence,

$$\tau_{K_{1,n}}^{e_m} = \begin{cases} \{\varnothing, V(K_{1,n})\} & \text{if } n = 1 \\ \{\varnothing, \{v_0\}, V(K_{1,n})\} & \text{if } n \geq 2. \end{cases}$$

This proves the assertion. □

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