EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 3, 2021, 969-979
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Generalized iteration method for the solution of fourth order BVP via Green′s function

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Abstract. The aim of this paper is to extend and generalize Picard-Green's fixed point iteration method for the solution of fourth-order Boundary Value Problems. Several numerical applications to linear and nonlinear fourth-order Boundary Value Problems are discussed to illustrate the main results.

2020 Mathematics Subject Classifications: 47J26, 41A25

Key Words and Phrases: Boundary value problem, fixed point iteration method, Green's function, integral operator, rate of convergence

1. Introduction

An iterative method is an important tool for solving linear and nonlinear Boundary Value Problems (BVPs). It has been used in the research areas of mathematics and several branches of science and other fields. In recent years, the study of generalization for the solution of third and fourth-order BVPs has attracted many scientists' interest. For instance, in mathematical modeling applications of a two-dimensional channel with porous walls, deformation of elastic beams, fluid dynamics. These problems have been studied by several mathematicians(see [11], [1], [4], [7], [2], [3], [8], [5], [15], [14] and the references therein. [11] converted the BVP to Initial Value Problem (IVP) and solved the linear equation while [13] solved the problem by transforming the fourth-order BVP to third-order BVP and applied by contracting mapping principle. Meanwhile, [6] applied the contracting mapping principle to the fourth-order BVP. Both of them showed the existence, uniqueness, and convergence of the proposed methods.

[12] introduced his method for IVP by showing the existence and uniqueness for ordinary differential equations, whereas [10] introduced his method.

[9] considered a cantilever beam problem in equilibrium position as follows:

$$u^{''''}(t) = f(t, u(t), u^{'}(t), u^{''}(t), u^{'''}(t)) \tag{1}$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v14i3.4020

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subject to the boundary conditions:

$$u(0) = 0, \quad u^{'}(0) = 0, \quad u^{''}(1) = 0, \quad u^{'''}(1) = 0 \tag{2}$$

where $t \in (0,1)$ and $f : [0,1] \times R_{+}^{3} \times R_{-} \to R_{+}$ is continuous. By assuming that $f(t,u,x,y,z)$ is superlinear on u, x, y, z and satisfying a Nagumo-type condition, the existence and uniqueness of the problem Eq. $(1) - (2)$ are provided in the same article. However, Quanq in [6] showed that the theorems in [9] are not sufficient for solving resembling problems and provided some examples.

In this paper, we extend and generalize Picard-Green's known iteration method to fourth-order BVP. We proved the existence and uniqueness of theorems satisfying necessary conditions for convergence. Furthermore, we give some numerical examples, including linear and nonlinear BVPs, to illustrate the main results. Finally, we confirm better approximation with minimum errors than existing solutions by MATLAB.

2. Green's function and methodology

Consider the following fourth-order BVP,

$$L[u] = p(t)u^{''''}(t) + q(t)u^{'''}(t) + r(t)u^{''}(t) + h(t)u^{'}(t) + g(t)u(t) = f(t) \tag{3}$$

with the boundary conditions

$$\begin{aligned} B_a[u] &= \alpha_1 u(a) + \alpha_2 u^{'}(a) + \alpha_3 u^{''}(a) + \alpha_4 u^{'''}(a) = \alpha \\ B_b[u] &= \beta_1 u(b) + \beta_2 u^{'} u(b) + \beta_3 u^{''}(b) + \beta_4 u^{'''}(b) = \beta \\ B_c[u] &= \gamma_1 u(c) + \gamma_2 u^{'}(c) + \gamma_3 u^{''}(c) + \gamma_4 u^{'''}(c) = \gamma \\ B_d[u] &= \omega_1 u(c) + \omega_2 u^{'}(c) + \omega_3 u^{''}(c) + \omega_4 u^{'''}(c) = \omega \end{aligned} \tag{4}$$

where $t \in (a,b)$, α, β, γ and ω are constants and either $c = a$ or $c = b$ and either $d = a$ or $d = b$. The existence and uniqueness results for the solution of the problems Eq. (3)-(4) are given in [13],[6] and [3].

The Green's function $G(t,s)$ corresponding to linear term in Eq. (3)

$$G(t,s) = \begin{cases} a_1 u_1 + b_1 u_2 + c_1 u_3 + d_1 u_4, & a < t < s \\ a_2 u_1 + b_2 u_2 + c_2 u_3 + d_2 u_4, & s < t < b \end{cases} \tag{5}$$

where $t \neq s$, u_1, u_2, u_3 and u_4 are linearly independent solutions of $L[u]$ and a_i, b_i, c_i and d_i, $(i = 1,2)$ are constants. We follow five conditions to find the constants and final version of Green's function of the fourth-order boundary value problem.

1. G satisfies the homogeneous boundary conditions;

$$B_a[G(t,s)] = B_b[G(t,s)] = B_c[G(t,s)] = B_d[G(t,s)] = 0, \tag{6}$$

2. G is continuous at $t = s$;

$$a_1u_1(s) + b_1u_2(s) + c_1u_3(s) + d_1u_4(s) = a_2u_1(s) + b_2u_2(s) + c_2u_3(s) + d_2u_4(s), \quad (7)$$

3. $G^{'}$ is continuous at $t = s$;

$$a_1u_1^{'}(s) + b_1u_2^{'}(s) + c_1u_3^{'}(s) + d_1u_4^{'}(s) = a_2u_1^{'}(s) + b_2u_2^{'}(s) + c_2u_3^{'}(s) + d_2u_4^{'}(s), \quad (8)$$

4. $G^{''}$ is continuous at $t = s$;

$$a_1u_1^{''}(s) + b_1u_2^{''}(s) + c_1u_3^{''}(s) + d_1u_4^{''}(s) = a_2u_1^{''}(s) + b_2u_2^{''}(s) + c_2u_3^{''}(s) + d_2u_4^{''}(s). \quad (9)$$

5. $G^{'''}$ has jumping discontinuous at $t = s$;

$$a_1u_1^{'''}(s) + b_1u_2^{'''}(s) + c_1u_3^{'''}(s) + d_1u_4^{'''}(s) + 1/p(s) = a_2u_1^{'''}(s) + b_2u_2^{'''}(s) + c_2u_3^{'''}(s) + d_2u_4^{'''}(s). \quad (10)$$

As a consequence of these calculations, the Green's function can be written in the following form

$$u_p = \int_a^b G(t,s)f(s)ds, \quad (11)$$

where u_n is the particular solution of Eq. (3).

3. Picard-Green's fixed-point iterative method (PGEM)

Let us define the following linear integral operator to implement the proposed Picard-Green's fixed point iteration method.

$$T[u] = u_h + \int_a^b G(t,s)(p(s)u^{''''}(s) + q(s)u^{'''}(s) + r(s)u^{''}(s) + h(s)u^{'}(s) + g(s)u(s))ds, \quad (12)$$

By using Eq. (12), we obtain

$$T[u] = u_h + \int_a^b G(t,s)(p(s)u^{''''}(s) + q(s)u^{'''}(s) + r(s)u^{''}(s) + h(s)u^{'}(s) + g(s)u(s) - f(s))ds + \int_a^b G(t,s)f(s)ds. \quad (13)$$

Using Eq. (11), we get

$$T[u] = u + \int_a^b G(t,s)(p(s)u^{''''}(s) + q(s)u^{'''}(s) + r(s)u^{''}(s) + h(s)u^{'}(s) + g(s)u(s) - f(s))ds. \quad (14)$$

Let the starting function u_0 be the homogeneous solution of $L[u] = 0$ and $u_{n+1} = T[u_n]$, for all $n \geq 0$, then Picard-Green's fixed point iteration method for Eq. (3) is defined as:

$$u_{n+1} = u_n + \int_a^b G(t,s)(p(s)u_n^{''''}(s) + q(s)u_n^{'''}(s) + r(s)u_n^{''}(s) + h(s)u_n^{'}(s) + g(s)u_n(s) - f(s))ds. \quad (15)$$

4. Mann-Green's fixed point iterative method (MGEM)

The operator for Mann-Green's fixed point iteration method is $u_{n+1} = (1-\alpha_n)u_n + \alpha_n T[u_n]$, for $n \geq 0$. We generalize the Mann-Green's method by applying the solutions of Eq. (12)-(14) as follows:

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n \int_a^b G(t,s)(p(s)u_n^{''''}(s) + q(s)u_n^{'''}(s) + r(s)u_n^{''}(s) + h(s)u_n^{'}(s) + g(s)u_n(s) - f(s))ds \\ &= u_n + \alpha_n \int_a^b G(t,s)(p(s)u_n^{''''}(s) + q(s)u_n^{'''}(s) + r(s)u_n^{''}(s) + h(s)u_n^{'}(s) + g(s)u_n(s) - f(s))ds. \quad (16) \end{aligned}$$

Here, the choice of starting point is similar to the PGEM.

5. Convergence analysis

In this section, we analyze the convergence and determine the convergence rate. The convergence analysis will be provided by using nonlinear differential equations and by the contraction principle.

Consider the fourth-order BVP

$$u^{''''}(t) = f(t, u(t), u^{'}(t), u^{''}(t), u^{'''}(t)), \tag{17}$$

with the boundary conditions

$$u(0) = \alpha, \quad u^{'}(0) = \beta, \quad u^{''}(1) = \gamma, \quad u^{'''}(1) = \omega. \tag{18}$$

In particular, the solution of the problem Eq. $(17) - (18)$ is as follows:

$$u_p = \int_0^1 G(t,s) f(s, u_p, u_p^{'}, u_p^{''}, u_p^{'''})ds. \tag{19}$$

Note that Eq. (19) cannot be replaced in Eq. (13). Due to this, Eq. (12) should be modified as

$$T[u_p] = \int_a^b G(t,s)(p(s)u_p^{''''}(s) + q(s)u_p^{'''}(s) + r(s)u_p^{''}(s) + h(s)u_p^{'}(s) + g(s)u_p(s))ds. \tag{20}$$

From Eq. (19) and Eq. (20), we get

$$\begin{aligned} T[u_p] = u_p + \int_a^b G(t,s)(p(s)u_p^{''''}(s) &+ q(s)u_p^{'''}(s) + r(s)u_p^{''}(s) + h(s)u_p^{'}(s) \\ &+ g(s)u_p(s) - f(s, u_p, u_p^{'}, u_p^{''}, u_p^{'''}))ds. \end{aligned} \tag{21}$$

Let $u_p = u$. The Green's function of Eq. $(1) - (2)$ is as follows:

$$G(t,s) = \begin{cases} \frac{-t^3}{6} + \frac{st^2}{2}, & 0 < t < s, \\ \frac{-s^3}{6} + \frac{ts^2}{2}, & s < t < 1, \end{cases} \tag{22}$$

and the adjoint of Eq. (22) is as follows:

$$G^*(t,s) = -\begin{cases} \frac{-s^3}{6} + \frac{ts^2}{2}, & 0 < s < t, \\ \frac{-t^3}{6} + \frac{st^2}{2}, & t < s < 1. \end{cases} \tag{23}$$

By using Eq. (23) and Eq. (15), we have

$$u_{n+1} = u_n + \int_a^b G^*(t,s)(u_n^{''''}(s) - f(s, u_n(s), u_n^{'}(s), u_n^{''}(s), u_n^{'''}(s))ds. \tag{24}$$

In particular, we have

$$\begin{aligned} u_{n+1} = u_n - \int_0^t (\frac{-s^3}{6} + \frac{ts^2}{2})(u_n^{''''}(s) - f(s, u_n(s), u_n^{'}(s), u_n^{''}(s), u_n^{'''}(s))ds \\ - \int_t^1 (\frac{-t^3}{6} + \frac{st^2}{2})(u_n^{''''}(s) - f(s, u_n(s), u_n^{'}(s), u_n^{''}(s), u_n^{'''}(s))ds. \end{aligned} \tag{25}$$

Theorem 1. *Let $f(t,u,u',u'',u''')$ be a continuous function with bounded derivative u and*

$$K := 1/8L_c < 1,$$

where

$$L_c = \max_{[0,1]\times R^3} |\frac{\partial(f)}{\partial(u)}|.$$

Then, the iterative sequence $u_n(t)_{(n=1)}^{\infty}$ given in Eq. (26) converges uniformly to the solution of Eq. (1) − (2) for any bounded function $f(t,u,u',u'',u''')$ on [0,1].

Proof. Let $\|u\| = \max_{0\leq\ t\leq 1}|u(t)|$ in $C[0,1]$. By integrating (26) by parts, we get

$$\begin{aligned} u_{n+1}\ (t) = u_n\ (t) + G^*(t,1)u_n^{'''}(1) - G^*(t,0)u_n^{'''}(0) - G_s^*(t,1)u_n^{''}(1) + G_s^*(t,0)u_n^{''}(0) \\ + G_{ss}^*(t,1)u_n^{'}(1) - G_{ss}^*(t,0)u_n^{'}(0) - G_{sss}^*(t,1)u_n(1) + G_{sss}^*(t,0)u_n(0) \\ + \int_0^1 G_{ssss}^*(t,s)u_n(s)ds - \int_0^1 G^*(t,s)f(s,u_n(s),u_n^{'}(s),u_n^{''}(s),u_n^{'''}(s))ds. \end{aligned} \tag{26}$$

By using (18) and the conditions of Green's function in Eq. (26), we obtain

$$u_{n+1}\ (t) = \frac{t^3\omega}{6} + \frac{\gamma+\omega}{2}t^2 + \beta t - \alpha - \int_0^1 G^*(t,s)(f(s,u_n(s),u_n^{'}(s),u_n^{''}(s),u_n^{'''}(s))ds. \tag{27}$$

Let $u_{n+1} = T_G(u)$ where $T_G : C[0,1] \to C[0,1]$, then

$$T_G(u) \equiv \frac{\omega}{6}t^3 + \frac{\gamma+\omega}{2}t^2 + \beta t - \alpha - \int_0^1 G^*(t,s)(f(s,u_n(s),u_n^{'}(s),u_n^{''}(s),u_n^{'''}(s))ds. \tag{28}$$

Theorem 2. *Let $T_G(u)$ be a contractive mapping, then the fixed-point iteration method is convergent.*

From Eq. (28),

$$|T_G(u) - T_G(z)| = |\int_0^1 G^*(t,s)(f(s,u,u',u'',u''') - f(s,z,z',z'',z'''))ds|, \tag{29}$$

and by using the fact that

$$\max_{0\leq t\leq 1} |\int_0^1 G^*(t,s)ds| = 1/8, \tag{30}$$

we get

$$|T_G(u) - T_G(z)| \leq \frac{1}{8}\int_0^1 |f(s,u,u',u'',u''') - f(s,z,z',z'',z''')|ds. \tag{31}$$

By applying Mean Value Theorem, we obtain the following inequality:

$$\begin{aligned} |T_G(u) - T_G(z)| \leq \frac{1}{8}\max_{[0,1]} |f(t,u(t),u'(t),u''(t),u'''(t)) \\ -f(t,z(t),z'(t),z''(t),z'''(t))| \leq \frac{1}{8}L_c\|u-z\|, \end{aligned} \tag{32}$$

where $\|u - z\| = \max_{0\leq t\leq 1}|u(t) - z(t)|$ and $L_c = \max_{[0,1]\times R^3}|\frac{\partial f(t,u,u',u'',u''')}{\partial u}|$. From Theorem 1, we get

$$\|T_G(u) - T_G(z)\| \leq K\|u - z\|, \tag{33}$$

where $K \in (0,1)$. This proves that T_G is a contraction mapping. On the other hand, the rate of convergence can be estimated as follows. Consider

$$\|u_{n+1} - u_n\| = \|T_G(u_n) - T_G(u_{n-1})\| \leq K\|u_n - u_{n-1}\| \leq K^n\|u_1 - u_0\|. \tag{34}$$

If $m > n > 0$, from Eq. (34), we obtain

$$\begin{aligned} \|u_m - u_n\| = \|u_m - u_{m-1}\| + ... + \|u_{n+1} - u_n\| \leq (K^{m-1} + ... + K^n)\|u_1 - u_0\| \\ \leq K^n(1 + K + K^2 + K^3 + ...)\|u_1 - u_0\| = \frac{K^n}{1-K}\|u_1 - u_0\|. \end{aligned} \tag{35}$$

The estimated error is

$$\|u^* - u_n\| \leq \frac{K^n}{1-K}\|u_1 - u_0\| \tag{36}$$

while $n \to \infty$.

We establish the convergence of MGEM analogously.

6. Numerical Examples

In this section, we give numerical examples to confirm the applicability of the main results.

Example 1. *Consider the fourth order BVP*

$$u''''(t) = -3\frac{u'u'''}{1152} + \frac{(u'')^2}{576} + \frac{t}{4} + \frac{95}{4}, \tag{37}$$

with the boundary conditions

$$u(0) = u'(0) = u''(1) = u'''(1) = 0. \tag{38}$$

The exact solution of Eq. (37) − (38) *is*

$$u(t) = t^4 - 4t^3 + 6t^2, \tag{39}$$

and the Green's function is

$$G(t,s) = \begin{cases} \frac{-t^3}{6} + \frac{st^2}{2}, & 0 < t < s \\ \frac{-s^3}{6} + \frac{ts^2}{2}, & s < t < 1. \end{cases} \tag{40}$$

By applying PGEM, we get

$$\begin{aligned} u_{n+1} = u_n &- \int_0^t \left(\frac{-s^3}{6} + \frac{ts^2}{2}\right)\left(u_n''''(s) + 3\frac{u_n' u_n'''}{1152} - \frac{(u_n'')^2}{576} - \frac{s}{4} - \frac{95}{4}\right)ds \\ &- \int_t^1 \left(\frac{-t^3}{6} + \frac{st^2}{2}\right)\left(u_n''''(s) + 3\frac{u_n' u_n'''}{1152} - \frac{(u_n'')^2}{576} - \frac{s}{4} - \frac{95}{4}\right)ds, \end{aligned} \tag{41}$$

where the starting function is $u_0 = 0$*. The absolute error of the problem is estimated by*

$$Err = |u(t) - u_n(t)|. \tag{42}$$

Table 1 shows the maximum errors obtained by PGEM for several iterations, which indicate high accuracy. Meanwhile, Table 2 lists the errors of the results obtained by PGEM, MGEM, and contraction mapping for comparison. Here, it is assumed that $\alpha = 0.99$*. From Table 2 it is clear that PGEM has a better rate of convergence than other methods.*

Table 1: Maximum Errors of Exercise 4.1.

Number of Iterations	Maximum Error
3	$1.75E-07$
5	$3.42E-12$
7	$6.68E-17$

Table 2: Error comparisons of Exercise 4.1.

	Picard	Contraction Mapping	Mann
t	Error (7)	Error (7)	Error(7)
0.1	$1.64E-18$	$4.44E-16$	$9.76E-15$
0.2	$5.85E-18$	$1.33E-15$	$3.51E-14$
0.3	$1.17E-17$	$1.89E-15$	$7.10E-14$
0.4	$1.87E-17$	$3.55E-15$	$1.14E-13$

Example 2. *Consider the following fourth-order BVP*

$$u^{''''}(t) = \frac{u(t)}{6}(u(t) + u^{'}(t) + u^{''}(t) - u^{'''}(t)) + 1, \tag{43}$$

with the boundary conditions

$$u(0) = u^{'}(0) = u^{''}(1) = u^{'''}(1) = 0. \tag{44}$$

Note that problem Eq. (43) − (44) *does not have an exact solution. On the other hand, the Green's function of the problem is*

$$G(t,s) = \begin{cases} \frac{-t^3}{6} + \frac{st^2}{2}, & 0 < t < s \\ \frac{-s^3}{6} + \frac{ts^2}{2}, & s < t < 1. \end{cases} \tag{45}$$

Using Eq. (45)*and PGEM,*

$$\begin{aligned} u_{n+1} = u_n - \int_0^t (\frac{-s^3}{6} + \frac{ts^2}{2})(u_n^{''''}(s) - \frac{u_n(s)}{6}(u_n(s) + u_n^{'}(s) + u_n^{''}(s) - u_n^{'''}(s)) - 1)ds \\ - \int_t^1 (\frac{-t^3}{6} + \frac{st^2}{2})(u_n^{''''}(s) - \frac{u_n(s)}{6}(u_n(s) + u_n^{'}(s) + u_n^{''}(s) - u_n^{'''}(s)) - 1)ds \end{aligned} \tag{46}$$

is obtained. Here, the starting function is $u_0 = 0$. *In Table 3, the errors obtained by PGEM and MGEM are illustrated. For MGEM,* $\alpha = 0.99$ *and* $\alpha = 0.80$ *are chosen, respectively.From Table 3, it is clear that the closer* α *to 1, the higher accuracy for MGEM is obtained. Moreover, Picard-Green's method is the special case of Mann-Green's method, when* $\alpha = 1$.

Example 3. *Consider the fourth-order BVP*

$$u^{''''}(t) + 4u(t) = 1, \tag{47}$$

with the boundary conditions

$$u(-1) = u(1) = u^{'}(-1) = u^{'}(1) = 0. \tag{48}$$

The Green's function of the problem Eq.(47) − (48)*is*

$$G(t,s) = \begin{cases} \frac{s^3-3s+2}{24}(-1-t)^3 + \frac{s^3-s^2-s+1}{8}(-1-t)^2, & -1 < s < t \\ \frac{s^3-3s-2}{24}(1-t)^3 + \frac{-s^3-s^2+s+1}{8}(1-t)^2, & t < s < 1. \end{cases} \tag{49}$$

Table 3: Error comparisons of Exercise 4.2.

	Picard	Mann	
t	Error (7)	Error (7)	Error (7)
0.1	$5.89E-15$	$2.63E-13$	$1.55E-07$
0.3	$4.72E-14$	$1.89E-12$	$1.22E-06$
0.5	$1.16E-13$	$4.63E-12$	$2.96E-06$
0.7	$1.98E-13$	$7.94E-12$	$5.04E-06$
0.9	$2.86E-13$	$1.15E-11$	$7.26E-06$

When Eq.(49) *is embedded into PGEM, the iterative method is obtained as*

$$u_{n+1} = u_n - \int_{-1}^{t} \left(\frac{s^3-3s-2}{24}(1-t)^3 + \frac{-s^3-s^2+s+1}{8}(1-t)^2\right)(u_n^4(s)+4u_n(s)-1)ds$$
$$- \int_{t}^{1} \left(\frac{s^3-3s+2}{24}(-1-t)^3 + \frac{s^3-s^2-s+1}{8}(-1-t)^2\right)(u_n^4(s)+4u_n(s)-1)ds. \quad (50)$$

In Eq.(50) *the starting function* $u_0 = 0$*. The numerical results obtained by PGEM at 20th iteration are presented with the errors in Table 4. In this case, to apply MGEM,* $\alpha = 0.95$ *and* $\alpha = 0.70$ *were chosen. The Figure 1 below represents the 20th iteration by PGEM which is very close to the exact solution of the problem Eq.* $(47)-(48)$ *.*

Table 4: Error comparisons of Exercise 4.3.

		Picard	Mann	
t	Numerical Value	Error (20)	Error(20)	Error (20)
-1.0	0.0	$3.49E-18$	$3.49E-18$	$3.49E-18$
-0.8	0.004829301786589	$9.19E-19$	$9.13E-19$	$1.12E-14$
-0.6	0.015199277038320	$5.81E-19$	$6.01E-19$	$1.80E-14$
-0.4	0.026098892616535	$1.14E-18$	$1.17E-18$	$8.75E-15$
-0.2	0.034019288419593	$8.72E-19$	$9.19E-19$	$6.17E-15$
0.0	0.036887761992940	$1.01E-19$	$5.05E-20$	$1.31E-14$

7. Conclusion

In this paper, we present the Picard-Green's method, one of the most popular methods, generalized and extended for the fourth-order nonlinear and linear BVP by embedding Green's function into the Picard-Green's Mann-Green's fixed point iteration methods. Different examples were solved to demonstrate its accuracy. The obtained results were compared to other numerical and analytical solutions earlier found in the literature during the solution. Moreover, we proved the convergence and found the rate of convergence.

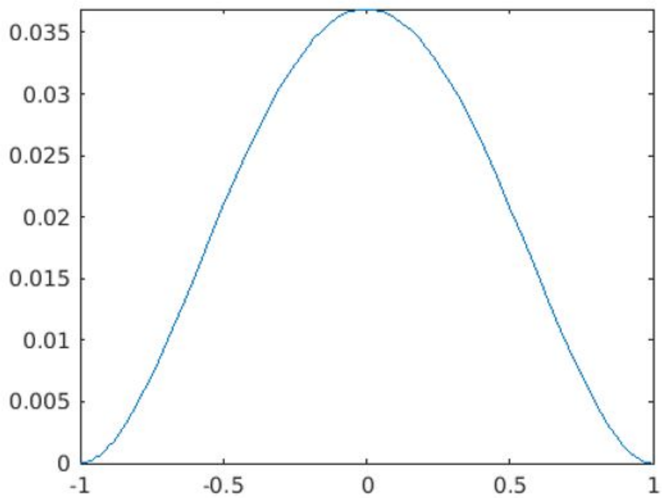

Figure 1: 20th iteration of Example 3

Acknowledgements

The authors thank the referees for their positive recommendations.

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