EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 1, 2022, 229-237
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Definite Integrals involving Logarithmic Powers, Binomials and Polynomials expressed in terms of the Lerch Function

Robert Reynolds[1,*], Allan Stauffer[1]

[1] *Department of Mathematics and Statistics, Faculty of Science, York University, Toronto, Ontario, Canada, M3J1P3*

Abstract. Closed expressions using the Lerch function for a definite integral are derived and evaluated. Some of these closed expressions are given in Gradshteyn and Ryzhik. Some special cases of the integral are derived and discussed. The majority of the results in this work are new.

2020 Mathematics Subject Classifications: 30E20, 33-01, 33-03, 33-04, 33-33B

Key Words and Phrases: Entries of Gradshteyn and Ryzhik; Lerch function; analytic continuation

1. Introduction

In this manuscript we focus on the derivation of the definite integral given by

$$\int_0^1 \frac{x^m \log^k(ax) - x^{-m} \log^k\left(\frac{a}{x}\right)}{x^2-1} dx, \tag{1}$$

which has a closed form solution in terms of the Lerch function. In our case the parameters in the formula are general complex numbers subject to the restrictions given below. This integral and its closed form solution are important because it allows us to provide derivations for integrals in the books of Gradshteyn and Rhyzik [6] and Birens de haan [8]. We also derive new forms of definite integrals such as $\tan^{-1}(\log(x))$ not available in current literature. Since equation (1) is expressed in terms of the Lerch function, all solutions of the integrals are analytically continued which widens the range of computation. The derivations follow the method used by us in [10]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}} dw. \tag{2}$$

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i1.4153

Email addresses: `milver@my.yorku.ca` (R. Reynolds), `stauffer@yorku.ca` (A. Stauffer)

where C is in general an open contour in the complex plane where the bilinear concomitant [10] has the same value at the end points of the contour. Then we multiply both sides by a function and take a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of equation (2) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2. Definite integral of the contour integral

We use the method in [10]. Here we use the contour of Figure 2 in [10] but for the z-plane where $z = m + w$ except we replace the vertical lines ± 0 by $\pm\Re(a)$. Note Figure 2 represents a Hankel contour which is in the z-plane, with the cut along the positive y-axis and the contour on opposite sides of the cut but along the y-axis. Using a generalization of Cauchy's integral formula we first replace y by $\log(ax)$ then y by $\log(a/x)$ takig their difference followed by multiplying both sides by $\frac{1}{x^2-1}$ then taking the definite integral with respect $x \in [0,1]$ to get

$$\begin{aligned}\int_0^1 \frac{x^m \log^k(ax) - x^{-m}\log^k\left(\frac{a}{x}\right)}{x^2-1}dx &= \frac{1}{2\pi i}\int_0^1\int_C \frac{a^w w^{-k-1}\left(x^{m+w} - x^{-m-w}\right)}{x^2-1}dwdx \\ &= \frac{1}{2\pi i}\int_C\int_0^1 \frac{a^w w^{-k-1}\left(x^{m+w} - x^{-m-w}\right)}{x^2-1}dxdw \\ &= \frac{1}{2\pi i}\int_C \frac{1}{2}\pi a^w w^{-k-1}\tan\left(\frac{1}{2}\pi(m+w)\right)dw,\end{aligned} \tag{3}$$

from (3.269.3) in [6] where the digamma function $\psi^0(x)$ can be written out using equation (44:5:3) in [9] and $-1 < \Re(m+w) < 1$. The logarithmic function is given for example in section (4.1) in [1]. We are able to switch the order of integration over $z = w + m$ and x using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C} \times [0,1]$.

3. The Lerch function

The Lerch function [3] has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n, \tag{4}$$

where $|z|< 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{5}$$

where $\Re(v) > 0$, and either $|z|\leq 1, z \neq 1, \Re(s) > 0$, or $z = 1, \Re(s) > 1$.

4. Infinite sum of the contour integral

Using equation (2) and replace y by $\log(a) + i\pi(y+1)$ then multiply both sides by $-i\pi(-1)^y e^{i\pi m(y+1)}$ followed by taking the infinite sum over $y \in [0, \infty)$, simplify to get

$$\begin{aligned}
&-\frac{(i\pi)^{k+1} e^{i\pi m} \Phi\left(-e^{im\pi}, -k, 1 - \frac{i\log(a)}{\pi}\right)}{\Gamma(k+1)} \\
&= -\frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_C i\pi(-1)^y w^{-k-1} \exp(w(\log(a) + i\pi(y+1)) + i\pi m(y+1)) dw \\
&= -\frac{1}{2\pi i} \int_C \sum_{y=0}^{\infty} i\pi(-1)^y w^{-k-1} \exp(w(\log(a) + i\pi(y+1)) + i\pi m(y+1)) dw \\
&= \frac{1}{2\pi i} \int_C \left(\frac{1}{2}\pi a^w w^{-k-1} \tan\left(\frac{1}{2}\pi(m+w)\right) - \frac{1}{2} i\pi a^w w^{-k-1}\right) dw.
\end{aligned} \tag{6}$$

from equation (1.232.1) in [6], where $\Im(m+w) > 0$ in order for the sum to converge.

5. The additional contour integral

Using equation (2) and replace y by $\log(a)$ followed by multiplying both sides by $\frac{\pi}{2i}$ to get

$$-\frac{i\pi \log^k(a)}{2\Gamma(k+1)} = -\frac{1}{2\pi i} \int_C \frac{1}{2} i\pi a^w w^{-k-1} dw, \tag{7}$$

6. A Note on the Hypergeometric function

In this manuscript we will derive definite integrals in terms of the Lerch function which simplify to the Hypergeometric function by equation (1.11.10) in [4].

$$\Phi(z, 1, v) = \sum_{n=0}^{\infty} \frac{z^n}{n+v} = v^{-1} {}_2F_1(1, v, 1+v; z). \tag{8}$$

7. The definite integral in terms of the Lerch function

Since the right-hand sides of equations (3), (6) and (7) are equal we may equate the left hand sides simplify to get

$$\int_0^1 \frac{x^m \log^k(ax) - x^{-m} \log^k\left(\frac{a}{x}\right)}{x^2-1} dx = \frac{1}{2} i\pi \left(\log^k(a) - 2(i\pi)^k e^{i\pi m} \Phi\left(-e^{im\pi}, -k, 1 - \frac{i\log(a)}{\pi}\right)\right), \tag{9}$$

where $-1 < \Re(m) < 1$.

8. Derivation of entry 4.282.13 in [6]

Using equation (9) first replacing a by e^{qi} then setting $k = -1$, we then replace m by p and $-p$ to get a second equation, then taking the difference of these two equations simplify we get

$$\int_0^1 \frac{(x^p - x^{-p})}{(x^2-1)\left(q^2 + \log^2(x)\right)} dx = \frac{ie^{-i\pi p}}{2q}\left(\Phi\left(-e^{-ip\pi}, 1, \frac{q+\pi}{\pi}\right) - e^{2i\pi p}\Phi\left(-e^{ip\pi}, 1, \frac{q+\pi}{\pi}\right)\right)$$

$$= \frac{i\pi e^{-i\pi p}}{2q(q+\pi)}\left({}_2F_1\left(1, \frac{q+\pi}{\pi}; \frac{q}{\pi}+2; -e^{-ip\pi}\right) - e^{2i\pi p}\,{}_2F_1\left(1, \frac{q+\pi}{\pi}; \frac{q}{\pi}+2; -e^{ip\pi}\right)\right). \tag{10}$$

This solution represents the analytic continuation of the integral in [6]. The solution listed in [6] is slowly convergent and limited in the variable domain of evaluation.

9. Derivation of entry 4.282.4 in [6]

In this section we will use the formula ${}_2F_1(1,2;3;z) = -\frac{2(z+\log(1-z))}{z^2}$ where $z = -1$, which is derived from section (15.2) (relations between contiguous functions) in [1]. Using equation (10) then taking the first partial derivative with respect to p followed by setting $q = \pi$ and $p = 0$ we get

$$\int_0^1 \frac{\log(x)}{(x^2-1)\left(\log^2(x) + \pi^2\right)} dx = \frac{1}{4}(\log(4) - 1). \tag{11}$$

10. Derivation of entry 4.282.8 in [6]

In this section we will use the formula ${}_2F_1\left(1, \frac{3}{2}; \frac{5}{2}; z\right) = -\frac{3}{z} + \frac{3\tanh^{-1}\left(\sqrt{z}\right)}{z^{3/2}}$ where $z = -1$, which is derived from section (15.2) (relations between contiguous functions) in [1]. Using equation (10) then taking the first partial derivative with respect to p followed by setting $q = \pi/2$ and $p = 0$ we get

$$\int_0^1 \frac{\log(x)}{(x^2-1)\left(4\log^2(x) + \pi^2\right)} dx = \frac{1}{16}(\pi - 2). \tag{12}$$

11. Derivation of entry 4.282.10 in [6]

In this section we will use the formula ${}_2F_1\left(1,\frac{5}{4};\frac{7}{4};z\right)=\frac{3B_z\left(\frac{3}{4},\frac{1}{2}\right)}{4\sqrt{1-z}z^{3/4}}$ where $z=-1$, which is derived from section (15.2) (relations between contiguous functions) in [1]. Using equation (10) then taking the first partial derivative with respect to p followed by setting $q=\pi/4$ and $p=0$ we get

$$\int_0^1 \frac{\log(x)}{(x^2-1)\left(16\log^2(x)+\pi^2\right)}dx=\frac{1}{64}\left(-4+\sqrt{2}\pi+2\sqrt{2}\log\left(\cot\left(\frac{\pi}{8}\right)\right)\right). \tag{13}$$

12. A special case in terms of the Hypergeometric function

Using equation (9) and first replacing a and e^{ai} then setting $k=-1$, and replacing m by $-m$ to form a second equation and adding both simplify to get

$$\int_0^1 \frac{x^{-p}\left(x^{2p}+1\right)\log(x)}{(x^2-1)\left(q^2+\log^2(x)\right)}dx$$

$$=\frac{1}{2}\pi\left(\frac{1}{q}-\frac{e^{-i\pi p}}{q+\pi}\left({}_2F_1\left(1,\frac{q+\pi}{\pi};\frac{q}{\pi}+2;-e^{-ip\pi}\right)+e^{2i\pi p}\,{}_2F_1\left(1,\frac{q+\pi}{\pi};\frac{q}{\pi}+2;-e^{ip\pi}\right)\right)\right). \tag{14}$$

13. A special case in terms of the Polylogarithm function

Using equation (9) and first setting $a=1$ and replacing m by $-m$ to form a second equation and subtracting both simplify to get

$$\int_0^1 \frac{x^{-m}\left(x^{2m}-1\right)\log^k(x)}{x^2-1}dx=-\frac{1}{2}ie^{i\pi k}\pi^{k+1}\sec\left(\frac{\pi k}{2}\right)\left(\mathrm{Li}_{-k}\left(-e^{-im\pi}\right)-\mathrm{Li}_{-k}\left(-e^{im\pi}\right)\right), \tag{15}$$

from equation (6) in [7].

14. A special case in terms of the Lerch function

Using equation (9) and first setting $k=-2$ and replacing a by e^{qi} then replacing m by $-m$ to form a second equation and subtracting both simplify to get

$$\int_0^1 \frac{x^{-m}\left(x^{2m}-1\right)\left(q^2-\log^2(x)\right)}{(x^2-1)\left(q^2+\log^2(x)\right)^2}dx$$

$$= \frac{ie^{-i\pi m}}{2\pi}\left(\Phi\left(-e^{-im\pi},2,\frac{q+\pi}{\pi}\right)-e^{2i\pi m}\Phi\left(-e^{im\pi},2,\frac{q+\pi}{\pi}\right)\right). \quad (16)$$

15. Definite integral of nested logarithm function in terms of the derivative of the Polylogarithm function

Using equation (9) and first setting $k=-2$ and replacing a by e^{qi} then replacing m by $-m$ to form a second equation and subtracting both simplify to get

$$\int_0^1 \frac{1}{\sqrt{x}(x+1)\left(\log^2(x)+\pi^2\right)}dx = \frac{\log(2)}{2\pi} \quad (17)$$

and

$$\int_0^1 \frac{i\pi\log\left(\log^2(x)+\pi^2\right)+\log(x)\log\left(\frac{\pi+i\log(x)}{\pi-i\log(x)}\right)}{\sqrt{x}(x+1)\left(\log^2(x)+\pi^2\right)}dx = i\mathrm{Li}_1'(-i)+i\mathrm{Li}_1'(i)+i\log(2)\log(\pi), \quad (18)$$

from equation (27) in [2].

16. Derivation of entry BI(131)(3) in [8]

In this section we will use the formula ${}_2F_1(1,2;3;z) = -\frac{2(z+\log(1-z))}{z^2}$ where $z=-1$, which is derived from section (15.2) (relations between contiguous functions) in [1]. Using equation (14) and setting $q=\pi$ simplify we get

$$\int_0^1 \frac{x^{-p}\left(x^{2p}+1\right)\log(x)}{(x^2-1)\left(\log^2(x)+\pi^2\right)}dx = \frac{1}{2}(\pi p\sin(\pi p)+\cos(\pi p)\log(2(\cos(\pi p)+1))-1). \quad (19)$$

17. Derivation of entry BI(131)(4) in [8]

In this section we will use the formula ${}_2F_1(1,1;2;z) = -\frac{\log(1-z)}{z}$ which is derived from section (15.2) (relations between contiguous functions) in [1]. Using equation (10) and setting $q=\pi$ simplify we get

$$\begin{aligned}\int_0^1 \frac{x^{-p}\left(x^{2p}-1\right)}{(x^2-1)\left(\log^2(x)+\pi^2\right)}dx &= \frac{i\left(e^{-i\pi p}\log\left(1+e^{i\pi p}\right)-e^{i\pi p}\log\left(1+e^{-i\pi p}\right)\right)}{2\pi}\\ &= \frac{\sin(\pi p)\log(2(\cos(\pi p)+1))-\pi p\cos(\pi p)}{2\pi}.\end{aligned} \quad (20)$$

18. Derivation of arctangent logarithmic integrals

In this section we will look at deriving definite integrals of the arctangent of the logarithmic function. We will also derive integrals in terms of π and the loggmma function.

Using (9) and setting $m = 0$ simplifying we get

$$\int_0^1 \frac{\log^k(ax) - \log^k\left(\frac{a}{x}\right)}{x^2-1}dx$$
$$= \frac{1}{2}\left((2i\pi)^{k+1}\left(\zeta\left(-k, 1-\frac{i\log(a)}{2\pi}\right) - \zeta\left(-k, \frac{\pi - i\log(a)}{2\pi}\right)\right) + i\pi\log^k(a)\right), \quad (21)$$

from equations (64:5:3) in [9] and (25.14.2) in [5]. Then we take the first partial derivative with respect to k then set $k = 0$ and replace $a = e^a$ simplifying to get

$$\int_0^1 \frac{\tanh^{-1}\left(\frac{\log(x)}{a}\right)}{x^2-1}dx = \frac{1}{8}\pi\left(-4i\text{log}\Gamma\left(-\frac{ia}{2\pi}\right) + 4i\text{log}\Gamma\left(-\frac{ia+\pi}{2\pi}\right)\right.$$
$$\left. -4i\log(-ia) + 2i\log(a) + 4i\log(-\pi - ia) + \pi - 2i\log(2\pi)\right) \quad (22)$$

Next we replace a by $-\frac{1}{ai}$ simplifying to get

$$\int_0^1 \frac{\tan^{-1}(a\log(x))}{x^2-1}dx = \frac{1}{8}i\pi\left(\pi + 4i\log\left(\frac{\sqrt{\frac{i}{a}}\Gamma\left(\frac{\pi+\frac{1}{a}}{2\pi}\right)}{\sqrt{2\pi}\Gamma\left(1+\frac{1}{2\pi a}\right)}\right)\right) \quad (23)$$

where $Re(a) > 0$.

18.1. Example 1

Using equation (23) and setting $a = 1$ simplifying we get

$$\int_0^1 \frac{\tan^{-1}(\log(x))}{x^2-1}dx = \frac{1}{4}\pi\log\left(\frac{2\pi\Gamma\left(1+\frac{1}{2\pi}\right)^2}{\Gamma\left(\frac{1+\pi}{2\pi}\right)^2}\right) \quad (24)$$

18.2. Example 2

Using equation (23) and setting $a = 1/\pi$ simplifying we get

$$\int_0^1 \frac{\tan^{-1}\left(\frac{\log(x)}{\pi}\right)}{x^2-1}dx = \frac{1}{4}\pi\log\left(\frac{\pi}{2}\right) \quad (25)$$

18.3. Example 3

Using equation (23) and setting $a = 1/(2\pi)$ simplifying we get

$$\int_0^1 \frac{\cot^{-1}\left(\frac{2\pi}{\log(x)}\right)}{x^2-1}dx = \frac{1}{4}\pi \log\left(\frac{4}{\pi}\right). \tag{26}$$

19. Discussion

In comparing our results with Table 4.282 in [6], our formulae have a wider range of the parameters than are listed in the Gradshteyn and Ryzhik book [6] due to the use of the Lerch function in the derivation of these integrals. We also provided correct formula for an integral supplied by Bierens de Haan. We will be looking at other integrals using this contour integral method for future work.

20. Conclusion

In this paper, we have presented a novel method for deriving some interesting definite integrals not previously published in literature using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

Acknowledgements

This research is supported by NSERC Canada under grant 504070.

References

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables.* Courier Corporation, 01 1965.

[2] D.H. Bailey and J.M. Borwein. Crandall's computation of the incomplete gamma function and the hurwitz zeta function, with applications to dirichlet l-series. *Applied Mathematics and Computation*, 268:462–477, 10 2015.

[3] Eugenio Balanzario and Jorge Sánchez-Ortiz. Riemann-siegel integral formula for the lerch zeta function. *Mathematics of Computation*, 81:2319–2333, 11 2011.

[4] Harry Bateman. *Higher Transcendental Functions V.1.* McGraw-Hill, 1953.

[5] Nist digital library of mathematical functions. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[6] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007.

[7] Jesús Guillera and Jonathan Sondow. Double integrals and infinite products for some classical constants via analytic continuations of lerch's transcendent. *The Ramanujan Journal*, 16:247–270, 07 2008.

[8] D. Bierens de (David Bierens) Haan. Nouvelles tables d'integrales definies. *jscholarship.library.jhu.edu*, 1867.

[9] Keith B. Oldham, Jan Myland, and Jerome Spanier. *An Atlas of Functions: with Equator, the Atlas Function Calculator*. Springer Science & Business Media, 07 2010.

[10] Robert Reynolds and Allan Stauffer. A method for evaluating definite integrals in terms of special functions with examples. *International Mathematical Forum*, 15:235–244, 2020.