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Quadruple g-best Proximity Point for New Contraction in Complete Metric Space

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Abstract. The aim of this manuscript is to propose a contraction to pursue the existence of g-best proximity point results. The finding of this manuscript generalize and unify the results of Rohen and Mlaiki by using the new contraction with P-property and prove the existence and uniqueness of quadruple best proximity point alongwith an example.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Best proximity point, quadruple best proximity point, metric space, contraction

1. Introduction

Fixed point theory is a flourished theory due to its functioning in physics, computer science, engineering etc. As always it is not possible to find fixed point for every self-contractive mappings, then there is possibility of existence of a point with minimum distance between the point and its image. This point is known as best proximity point which was introduced by Fan [8] and extended by Basha [5] and many more researchers.

In 1987, Guo and Lakshmikantham [10], introduced coupled fixed point and proved its related fixed point theorems under appropriate conditions. After that, Lakshmikantham and Ciric in [13] extend these results by defining the g-monotone property. The results of [10] leads to the development of tripled fixed point by Berinde and Borcut [7]. In [7], they proved the existence and uniqueness of the introduced tripled fixed point for non-linear mappings in Partially ordered complete metric space and later on many results exists between coupled and tripled fixed points on different spaces under different contractions. In 2012, tripled fixed point was extended to quadruple fixed point by Karapinar and Luong [11] in complete metric space. Motivated from [18], Rohen and Maliki [17] gave the notion of tripled best proximity points theorem graced with P-property and the developed contraction. See references [15], [2], [9], [14] for further research in coupled best proximity point results.

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Rohen and Maliki [17] and Karapinar and Luong [11], motivated us, in the direction to precede the quadruple best proximity point. We propose the quadruple best proximity point results with P-property and the newly introduce contraction. Also, examples are supplied in favour of our results.

2. Preliminaries

Definition 1. *[3] Let (X,d) be metric space, $\mathcal{Q}$ and $\mathcal{R}$ be two non-empty subset of X. Define*

$$
\begin{aligned}
d(\mathcal{Q},\mathcal{R}) &= \inf\{d(x,y) : x \in \mathcal{Q}, y \in \mathcal{R}\},\\
\mathcal{Q}_0 &= \{x \in \mathcal{Q} : \text{there exists some } y \in \mathcal{R} \text{ such that } d(x,y) = d(\mathcal{Q},\mathcal{R})\},\\
\mathcal{R}_0 &= \{y \in \mathcal{R} : \text{there exists some } x \in \mathcal{Q} \text{ such that } d(x,y) = d(\mathcal{Q},\mathcal{R})\}.
\end{aligned}
$$

In 2011, Basha [4] proved sufficient conditions when $\mathcal{Q}_0$ and $\mathcal{R}_0$ are non-empty.

Definition 2. *[6] Let (X,d) be metric space and $\mathcal{Q} \neq \phi, \mathcal{R} \neq \phi$ are subsets of X. Let $G : \mathcal{Q} \to \mathcal{R}$ be a mapping. Then $x \in \mathcal{Q}$ is said to be best proximity point if and only if $d(x, Gx) = d(\mathcal{Q},\mathcal{R})$.*

Definition 3. *[7] Let $G : X \times X \times X \to X$. An element (x,y,z) is said to be tripled fixed point of G if $G(x,y,z) = x, G(y,x,z) = y$ and $G(z,y,x) = z$.*

Definition 4. *[11] Let $G : X \times X \times X \times X \to X$. An element (x,y,z,t) is said to be quadruple fixed point of G if $G(x,y,z,t) = x, G(y,x,z,t) = y, G(z,y,x,t) = z$ and $G(t,y,z,x) = t$.*

Definition 5. *[16] Let $(\mathcal{Q},\mathcal{R})$ be non-empty pair of subsets of mertic space (X,d) with $\mathcal{Q}_0 \neq \phi$, then the pair $(\mathcal{Q},\mathcal{R})$ has P-property if and only if*

$$
\begin{cases} d(x_1,y_1) = d(\mathcal{Q},\mathcal{R}) \\ d(x_2,y_2) = d(\mathcal{Q},\mathcal{R}) \end{cases} \implies d(x_1,x_2) = d(y_1,y_2),
$$

where $x_1, x_2 \in \mathcal{Q}$ and $y_1, y_2 \in \mathcal{R}$.

Definition 6. *[12] Let $(\mathcal{Q},\mathcal{R})$ be non-empty pair of subsets of mertic space (X,d) . Consider $g : \mathcal{Q} \to \mathcal{Q}$ and $G : \mathcal{Q} \to \mathcal{R}$ be mappings then a point $x \in \mathcal{Q}$ is a best proximity g-point of the pair (g,G) if $d(gx, Gx) = d(\mathcal{Q},\mathcal{R})$.*

Definition 7. *[1] Let Ψ represent the family of functions ψ such that $\psi : [0,\infty) \to [0,\infty)$ which satisfy*

(i) $\psi(x) = 0$ if and only if $x = 0$.

(ii) $\psi(x)$ is continuous and non-decreasing.

Let Θ signify the collection of functions of type $\theta : [0,\infty)^8 \rightarrow [0,\infty)$ such that $\theta(x,y,z,t,a,b,c,u) = \min\{x,y,z,t,a,b,c,u\}$ for all $x,y,z,t,a,b,c,u \in [0,\infty)$.

Definition 8. *[17] Let (X,d) be a complete metric space and $\mathcal{Q} \neq \phi$ and $\mathcal{R} \neq \phi$ are closed subsets. An element $(x,y,z) \in X \times X \times X$ is said to be a tripled best proximity point of $G : X \times X \times X \rightarrow X$ if $x,z \in \mathcal{Q}$ and $y \in \mathcal{R}$ such that $d(x,G(x,y,z)) = d(\mathcal{Q},\mathcal{R}), d(y,G(y,x,y)) = d(\mathcal{Q},\mathcal{R})$ and $d(z,G(z,y,x)) = d(\mathcal{Q},\mathcal{R})$.*

3. Main Results

Definition 9. *Let (X,d) be a complete metric space and $(\mathcal{Q},\mathcal{R})$ be a pair of non-empty subset of X such that $\mathcal{Q}_0$ is non-empty. Consider $g : X \rightarrow X$ and $G : X^4 \rightarrow X$ be two mappings, then (x,y,z,t) is said to be quadruple g-best proximity point of G and g if $d(gx,G(x,y,z,t)) = d(\mathcal{Q},\mathcal{R}), d(gy,G(y,x,z,t)) = d(\mathcal{Q},\mathcal{R}), d(gz,G(z,y,x,t)) = d(\mathcal{Q},\mathcal{R})$ and $d(gt,G(t,y,z,x)) = d(\mathcal{Q},\mathcal{R})$ for all $x,z \in \mathcal{Q}$ and $y,t \in \mathcal{R}$.*

If $g = I$ (Identity mapping) then (x,y,z,t) is said to be quadruple best proximity point of G if $d(x,G(x,y,z,t)) = d(\mathcal{Q},\mathcal{R}), d(y,G(y,x,z,t)) = d(\mathcal{Q},\mathcal{R}), d(z,G(z,y,x,t)) = d(\mathcal{Q},\mathcal{R})$ and $d(t,G(t,y,z,x)) = d(\mathcal{Q},\mathcal{R})$ for all $x,z \in \mathcal{Q}$ and $y,t \in \mathcal{R}$.

Theorem 1. *Let $\mathcal{Q}$ and $\mathcal{R}$ be non-empty subset of complete metric space (X,d) such that $\mathcal{Q}_0$ and $\mathcal{R}_0$ are non-empty and $g : X \rightarrow X$ is an isometry such that $\mathcal{Q}_0 \subseteq g(\mathcal{Q}_0)$ and $\mathcal{R}_0 \subseteq g(\mathcal{R}_0)$, let $G : X^4 \rightarrow X$ be continuous mapping and $\psi, \zeta \in \Psi$ and $\theta \in \Theta$, satisfies the preceeding conditions:*

(i) For every $x,y,z,t,a,b,c,u \in X$

$$\begin{aligned}\psi(d(gx,ga)) =& \psi(d(G(x,y,z,t),G(a,b,c,u)))\\ \leq& \psi\{\max(d(x,a),d(y,b),d(z,c),d(t,u))\}\\ &- \zeta\{\max(d(x,a),d(y,b),d(z,c),d(t,u))\}\\ &+ \theta[d(ga,G(x,y,z,t)) - d(\mathcal{Q},\mathcal{R}), d(gb,G(y,x,z,t)) - d(\mathcal{Q},\mathcal{R}),\\ &\quad d(gc,G(z,y,x,t)) - d(\mathcal{Q},\mathcal{R}), d(gu,G(t,y,z,x)) - d(\mathcal{Q},\mathcal{R}),\\ &\quad d(gx,G(x,y,z,t)) - d(\mathcal{Q},\mathcal{R}), d(gy,G(y,x,z,t)) - d(\mathcal{Q},\mathcal{R}),\\ &\quad d(gz,G(z,y,x,t)) - d(\mathcal{Q},\mathcal{R}), d(gt,G(t,y,z,x)) - d(\mathcal{Q},\mathcal{R})] \quad (1)\end{aligned}$$

(ii) $G(\mathcal{Q}_0,\mathcal{R}_0,\mathcal{Q}_0,\mathcal{R}_0) \subseteq \mathcal{R}_0$

(iii) $G(\mathcal{R}_0,\mathcal{Q}_0,\mathcal{R}_0,\mathcal{Q}_0) \subseteq \mathcal{Q}_0$

(iv) Pair $(\mathcal{Q},\mathcal{R})$ has P-property

then (a,a,a,a) is the unique quadruple g-best proximity point of the pair (g,G).

Proof. Choose $x_0,z_0 \in \mathcal{Q}_0$ and $y_0,t_0 \in \mathcal{R}_0$. Since $G(x_0,y_0,z_0,t_0), G(z_0,y_0,x_0,t_0) \in \mathcal{R}_0$ and $G(y_0,x_0,t_0,z_0)$, $G(t_0,z_0,y_0,x_0) \in \mathcal{Q}_0$, there exist $x_1,z_1 \in \mathcal{Q}$ and $y_1,t_1 \in \mathcal{R}$ such that $d(gx_1,G(x_0,y_0,z_0,t_0)) = d(gy_1,G(y_0,x_0,z_0,t_0)) = d(gz_1,G(z_0,y_0,x_0,t_0)) =$

$d(gt_1, G(t_0, z_0, y_0, x_0)) = d(\mathcal{Q}, \mathcal{R})$. Continuing like this, we get a sequence of $\{gx_n\}, \{gz_n\} \in \mathcal{Q}$ and $\{gy_n\}, \{gt_n\} \in \mathcal{R}$ such that

$$\begin{aligned} &d(gx_{n+1}, G(x_n, y_n, z_n, t_n)) = d(\mathcal{Q}, \mathcal{R}) \\ &d(gy_{n+1}, G(y_n, x_n, t_n, z_n)) = d(\mathcal{Q}, \mathcal{R}) \\ &d(gz_{n+1}, G(z_n, y_n, x_n, t_n)) = d(\mathcal{Q}, \mathcal{R}) \\ &d(gt_{n+1}, G(t_n, z_n, y_n, x_n)) = d(\mathcal{Q}, \mathcal{R}) \text{ for all } n \in \mathbb{N} \cup \{0\} \end{aligned} \tag{2}$$

If $d(gx_n, gx_{n+1}) = d(gy_n, gy_{n+1}) = d(gz_n, gz_{n+1}) = d(gt_n, gt_{n+1}) = 0$ for all $n \in \mathbb{N} \cup \{0\}$ then nothing to prove.

Suppose $d(gx_n, gx_{n+1}) > 0$ or $d(gy_n, gy_{n+1}) > 0$ or $d(gz_n, gz_{n+1}) > 0$ or $d(gt_n, gt_{n+1}) > 0$.

From(1), P-property, and $d(gx_{n+1}, G(x_n, y_n, z_n, t_n)) = d(\mathcal{Q}, \mathcal{R}), d(gx_n, G(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1})) = d(\mathcal{Q}, \mathcal{R})$, we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(G(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), G(x_n, y_n, z_n, t_n)) \\ \psi(d(gx_n, gx_{n+1})) &= \psi(d(G(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), G(x_n, y_n, z_n, t_n))) \\ &\leq \psi\{\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n), d(t_{n-1}, t_n))\} \\ &\quad - \zeta\{\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n), d(t_{n-1}, t_n))\} \\ &\quad + \theta[d(gx_n, G(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}) - d(\mathcal{Q}, \mathcal{R}), \\ &\quad d(gy_n, G(y_{n-1}, x_{n-1}, t_{n-1}, z_{n-1}) - d(\mathcal{Q}, \mathcal{R}), \\ &\quad d(gz_n, G(z_{n-1}, y_{n-1}, x_{n-1}, t_{n-1}) - d(\mathcal{Q}, \mathcal{R}), \\ &\quad d(gt_n, G(t_{n-1}, z_{n-1}, y_{n-1}, x_{n-1}) - d(\mathcal{Q}, \mathcal{R}), \\ &\quad d(gx_{n-1}, G(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}) - d(\mathcal{Q}, \mathcal{R}), \\ &\quad d(gy_{n-1}, G(y_{n-1}, x_{n-1}, t_{n-1}, z_{n-1}) - d(\mathcal{Q}, \mathcal{R}), \\ &\quad d(gz_{n-1}, G(z_{n-1}, y_{n-1}, x_{n-1}, t_{n-1}) - d(\mathcal{Q}, \mathcal{R}), \\ &\quad d(gt_{n-1}, G(t_{n-1}, z_{n-1}, y_{n-1}, x_{n-1}) - d(\mathcal{Q}, \mathcal{R})] \\ &= \psi\{\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n), d(t_{n-1}, t_n))\} \\ &\quad - \zeta\{\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n), d(t_{n-1}, t_n))\} \end{aligned} \tag{3}$$

Similarly for $d(gy_{n+1}, G(y_n, x_n, t_n, z_n)) = d(\mathcal{Q}, \mathcal{R}), d(gy_n, G(y_{n-1}, x_{n-1}, t_{n-1}, z_{n-1})) = d(\mathcal{Q}, \mathcal{R})$,
$d(gz_{n+1}, G(z_n, y_n, x_n, t_n)) = d(\mathcal{Q}, \mathcal{R}), d(gz_n, G(z_{n-1}, y_{n-1}, x_{n-1}, t_{n-1})) = d(\mathcal{Q}, \mathcal{R})$ and
$d(gt_{n+1}, G(t_n, z_n, y_n, x_n)) = d(\mathcal{Q}, \mathcal{R}), d(gt_n, G(t_{n-1}, z_{n-1}, y_{n-1}, x_{n-1})) = d(\mathcal{Q}, \mathcal{R})$, we have

$$\begin{aligned} \psi(d(gy_n, gy_{n+1})) &\leq \psi\{\max(d(y_{n-1}, y_n), d(x_{n-1}, x_n), d(z_{n-1}, z_n), d(t_{n-1}, t_n))\} \\ &\quad - \zeta\{\max(d(y_{n-1}, y_n), d(x_{n-1}, x_n), d(z_{n-1}, z_n), d(t_{n-1}, t_n))\} \end{aligned} \tag{4}$$

$$\begin{aligned} \psi(d(gz_n, gz_{n+1})) &\leq \psi\{\max(d(z_{n-1}, z_n), d(y_{n-1}, y_n), d(x_{n-1}, x_n), d(t_{n-1}, t_n))\} \\ &\quad - \zeta\{\max(d(z_{n-1}, z_n), d(y_{n-1}, y_n), d(x_{n-1}, x_n), d(t_{n-1}, t_n))\} \end{aligned} \tag{5}$$

$$\begin{aligned} \psi(d(gt_n, gt_{n+1})) &\leq \psi\{\max(d(t_{n-1}, t_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n), d(x_{n-1}, x_n))\} \\ &\quad - \zeta\{\max(d(t_{n-1}, t_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n), d(x_{n-1}, x_n))\} \end{aligned} \tag{6}$$

From (3), (4), (5) and (6), we obtain
$\psi[\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), d(gt_n, gt_{n+1})\}]$

$$\leq \psi[\max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n), d(t_{n-1}, t_n)\}]$$
$$- \zeta[\max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n), d(t_{n-1}, t_n)\}]$$
$$= \psi[\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gt_{n-1}, gt_n)\}]$$
$$- \zeta[\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gt_{n-1}, gt_n)\}] \quad (7)$$
$$\leq \psi[\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gt_{n-1}, gt_n)\}]$$

As ψ is continuous function, therefore,

$$\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), d(gt_n, gt_{n+1})\}$$
$$\leq \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gt_{n-1}, gt_n)\}$$

implies $\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), d(gt_n, gt_{n+1})\}$ is a non-increasing sequence of positive real number, it must converge to a positive real number, say τ

$$\implies \lim_{n\to\infty}\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), d(gt_n, gt_{n+1})\} = \tau.$$

Taking limit on both side in (7), we have

$$\psi(\tau) \leq \psi(\tau) - \zeta(\tau)$$
$$\implies \zeta(\tau) = 0$$
$$\tau = 0.$$

Hence,

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \lim_{n\to\infty} d(gy_n, gy_{n+1}) = \lim_{n\to\infty} d(gz_n, gz_{n+1}) = \lim_{n\to\infty} d(gt_n, gt_{n+1}) = 0$$

Now, we prove that $\{gx_n\}, \{gy_n\}, \{gz_n\}$ and $\{gt_n\}$ are Cauchy sequences, i.e. $\max\{d(gx_{n(\iota)}, gx_{m(\iota)}, d(gy_{n(\iota)}, gy_{m(\iota)}), d(gz_{n(\iota)}, gz_{m(\iota)}), d(gt_{n(\iota)}, t_{m(\iota)})\} < \epsilon \,\forall\, m(\iota) > n(\iota) > \iota$. Let if possible sequences are not Cauchy then there exists an $\epsilon > 0$ such that for all $\iota > 0$ there are $m(\iota) > n(\iota) > \iota$ which satisfies the conditions

$$\max\{d(gx_{n(\iota)}, gx_{m(\iota)}, d(gy_{n(\iota)}, gy_{m(\iota)}), d(gz_{n(\iota)}, gz_{m(\iota)}), d(gt_{n(\iota)}, t_{m(\iota)})\} \geq \epsilon$$

and

$$\max\{d(gx_{n(\iota)-1}, gx_{m(\iota)}, d(gy_{n(\iota)-1}, gy_{m(\iota)}), d(gz_{n(\iota)-1}, gz_{m(\iota)}), d(gt_{n(\iota)-1}, t_{m(\iota)})\} < \epsilon.$$

Then, we have

$$\epsilon \leq d(gx_{n(\iota)}, gx_{m(\iota)})$$
$$\leq d(gx_{n(\iota)}, gx_{n(\iota)-1}) + d(gx_{n(\iota)-1}, gx_{m(\iota)}$$
$$\leq d(gx_{n(\iota)}, gx_{n(\iota)-1}) + \epsilon$$

This gives us

$$\epsilon \leq d(gx_{n(\iota)}, gx_{n(\iota)-1}) + \epsilon$$

For $\iota \to \infty$, we have

$$\lim_{\iota\to\infty} d(gx_{n(\iota)}, gx_{m(\iota)}) = \epsilon \quad (8)$$

Also, from triangular inequality, we find

$$d(gx_{n(\iota)-1}, gx_{m(\iota)-1}) \leq d(gx_{n(\iota)-1}, gx_{m(\iota)}) + d(gx_{m(\iota)}, gx_{m(\iota)-1}) \leq \epsilon$$

Hence,

$$d(gx_{n(\iota)-1}, gx_{m(\iota)-1}) \leq \epsilon. \quad (9)$$

Since $d(gx_{n(\iota)}, G(x_{n(\iota)-1}, y_{n(\iota)-1}, z_{n(\iota)-1}, t_{n(\iota)-1})) = d(\mathcal{Q}, \mathcal{R})$

and $d(gx_{m(\iota)}, G(x_{m(\iota)-1}, y_{m(\iota)-1}, z_{m(\iota)-1}, t_{m(\iota)-1})) = d(\mathcal{Q},\mathcal{R})$. From P-property, we have $d(gx_{n(\iota)}, gx_{m(\iota)}) = d(G(x_{n(\iota)-1}, y_{n(\iota)-1}, z_{n(\iota)-1}, t_{n(\iota)-1}), G(x_{m(\iota)-1}, y_{m(\iota)-1}, z_{m(\iota)-1}, t_{m(\iota)-1}))$
Now from (1) and using the continuity of ψ, we obtain

$$\begin{aligned}
&\psi(d(gx_{n(\iota)},gx_{m(\iota)}))\\
&=\psi(d(G(x_{n(\iota)-1}, y_{n(\iota)-1}, z_{n(\iota)-1}, t_{n(\iota)-1}), G(x_{m(\iota)-1}, y_{m(\iota)-1}, z_{m(\iota)-1}, t_{m(\iota)-1})))\\
&\leq \psi[\max\{d(x_{n(\iota)-1}, x_{m(\iota)-1}), d(y_{n(\iota)-1}, y_{m(\iota)-1}),\\
&\quad d(z_{n(\iota)-1}, z_{m(\iota)-1}), d(t_{n(\iota)-1}, t_{m(\iota)-1})\}] - \zeta[\max\{d(x_{n(\iota)-1}, x_{m(\iota)-1}),\\
&\quad d(y_{n(\iota)-1}, y_{m(\iota)-1}), d(z_{n(\iota)-1}, z_{m(\iota)-1}), d(t_{n(\iota)-1}, t_{m(\iota)-1})\}]\\
&\quad + \theta[d(gx_{m(\iota)-1}, G(x_{n(\iota)-1}, y_{n(\iota)-1}, z_{n(\iota)-1}, t_{n(\iota)-1})),\\
&\quad d(gy_{m(\iota)-1}, G(y_{n(\iota)-1}, x_{n(\iota)-1}, t_{n(\iota)-1}, z_{n(\iota)-1})),\\
&\quad d(gz_{m(\iota)-1}, G(z_{n(\iota)-1}, y_{n(\iota)-1}, x_{n(\iota)-1}, t_{n(\iota)-1})),\\
&\quad d(gt_{m(\iota)-1}, G(t_{n(\iota)-1}, z_{n(\iota)-1}, y_{n(\iota)-1}, x_{n(\iota)-1})),\\
&\quad d(gx_{n(\iota)-1}, G(x_{n(\iota)-1}, y_{n(\iota)-1}, z_{n(\iota)-1}, t_{n(\iota)-1})),\\
&\quad d(gy_{n(\iota)-1}, G(y_{n(\iota)-1}, x_{n(\iota)-1}, t_{n(\iota)-1}, z_{n(\iota)-1})),\\
&\quad d(gz_{n(\iota)-1}, G(z_{n(\iota)-1}, y_{n(\iota)-1}, x_{n(\iota)-1}, t_{n(\iota)-1})),\\
&\quad d(gt_{n(\iota)-1}, G(t_{n(\iota)-1}, z_{n(\iota)-1}, y_{n(\iota)-1}, x_{n(\iota)-1}))]\\
&= \psi[\max\{d(x_{n(\iota)-1}, x_{m(\iota)-1}), d(y_{n(\iota)-1}, y_{m(\iota)-1}), d(z_{n(\iota)-1}, z_{m(\iota)-1}),\\
&\quad d(t_{n(\iota)-1}, t_{m(\iota)-1})\}] - \zeta[\max\{d(x_{n(\iota)-1}, x_{m(\iota)-1}), d(y_{n(\iota)-1}, y_{m(\iota)-1}),\\
&\quad d(z_{n(\iota)-1}, z_{m(\iota)-1}), d(t_{n(\iota)-1}, t_{m(\iota)-1})\}].
\end{aligned}$$

Similary, from the same techinque, we obtain

$$\begin{aligned}
&\psi[\max\{d(gx_{n(\iota)}, gx_{m(\iota)}), d(gy_{n(\iota)}, gy_{m(\iota)}), d(gz_{n(\iota)}, gz_{m(\iota)}), d(gt_{n(\iota)}, gt_{m(\iota)})\}]\\
&\quad\leq \psi[\max\{d(gx_{n(\iota)-1}, gx_{m(\iota)-1}), d(gy_{n(\iota)-1}, gy_{m(\iota)-1}), d(gz_{n(\iota)-1}, gz_{m(\iota)-1}),\\
&\quad d(gt_{n(\iota)-1}, gt_{m(\iota)-1})\}] - \zeta[\max\{d(gx_{n(\iota)-1}, gx_{m(\iota)-1}), d(gy_{n(\iota)-1}, gy_{m(\iota)-1}),\\
&\quad d(gz_{n(\iota)-1}, gz_{m(\iota)-1}), d(gt_{n(\iota)-1}, gt_{m(\iota)-1})\}].
\end{aligned}$$

Now, from (8) and (9), we get

$$\begin{aligned}
&\psi(\epsilon) \leq \psi(\epsilon) - \zeta(\epsilon)\\
&\zeta(\epsilon) = 0\\
&\implies \epsilon = 0.
\end{aligned}$$

Thus, for ι tends to infinity, it gives us

$$\lim_{\iota\to\infty}\{d(gx_{n(\iota)}, gx_{m(\iota)}), d(gy_{n(\iota)}, gy_{m(\iota)}), d(gz_{n(\iota)}, gz_{m(\iota)}), d(gt_{n(\iota)}, gt_{m(\iota)})\} = 0,$$

which is contradiction to our suppostion that $\epsilon > 0$. Hence, $\{gx_n\}, \{gz_n\}$ are Cauchy sequences in $\mathcal{Q}$ and $\{gy_n\}, \{gt_n\}$ in $\mathcal{R}$. Since (X, d) is complete metric space, then there exist, $a, b, c, u \in X$ such that

$$\lim_{n\to\infty} gx_n = a, \lim_{n\to\infty} gy_n = b, \lim_{n\to\infty} gz_n = c \text{ and } \lim_{n\to\infty} gt_n = u.$$

As $\mathcal{Q},\mathcal{R}$ are closed subset of X, then $a, c \in \mathcal{Q}$ and $b, u \in \mathcal{R}$. Since G is continuous, then

$$\lim_{n\to\infty} d(gx_n, G(x_n, y_n, z_n, t_n)) = d(\mathcal{Q},\mathcal{R}) \implies d(ga, G(a, b, c, u)) = d(\mathcal{Q},\mathcal{R}).$$

Similarly, $d(gb, G(b,a,c,u)) = d(\mathcal{Q},\mathcal{R}), d(gc, G(c,b,a,u)) = d(\mathcal{Q},\mathcal{R})$ and $d(gu, G(u,b,c,a)) = d(\mathcal{Q},\mathcal{R})$. Thus, (a,b,c,u) is quadruple g-best proximity point of the pair (g,G).
Now, we show that $ga = gb = gc = gu$. Again from P-property, g-isometry and condition (1), we calculate

$$\begin{aligned} d(ga,gc) &= d(G(a,b,c,u), G(c,b,a,u)) \\ \psi(d(ga,gc)) &= \psi(d(G(a,b,c,u), G(c,b,a,u))) \\ &\le \psi(d(a,c)) \\ &= \psi(d(ga,gc)) \\ \implies a &= c. \end{aligned}$$

Therefore, $ga = gb = gc = gu$. To prove the uniqueness of quadruple g-best proximity point , consider q as another point. Now

$$\begin{aligned} d(ga,gq) &= d(G(a,a,a,a), G(q,q,q,q)) \\ \psi(d(ga,gq)) &= \psi(d(G(a,a,a,a), G(q,q,q,q))) \\ &\le \psi(d(a,q)) \\ &= \psi(d(ga,gq)) \\ \implies a &= q. \end{aligned}$$

Hence, the result.

Theorem 2. *Let $\mathcal{Q}$ and $\mathcal{R}$ be non-empty subset of complete metric space (X,d) such that $\mathcal{Q}_0$ and $\mathcal{R}_0$ are non-empty and $g : X \to X$ is an isometry such that $\mathcal{Q}_0 \subseteq g(\mathcal{Q}_0)$ and $\mathcal{R}_0 \subseteq g(\mathcal{R}_0)$, let $G : X^4 \to X$ be continuous mapping and $\psi, \zeta \in \Psi$ and $\theta \in \Theta$, satisfies the preceeding conditions:*

(i) For every $x,y,z,t,a,b,c,u \in X$

$$\begin{aligned} \psi(d(gx,ga)) =&\psi(d(G(x,y,z,t), G(a,b,c,u))) \\ \le&\psi\{\max(d(x,a), d(y,b), d(z,c), d(t,u))\} \\ &- \zeta\{\max(d(x,a), d(y,b), d(z,c), d(t,u))\} \\ &+ \theta[d(ga, G(x,y,z,t)) - d(\mathcal{Q},\mathcal{R}), d(gb, G(y,x,z,t)) - d(\mathcal{Q},\mathcal{R}), \\ &\quad d(gc, G(z,y,x,t)) - d(\mathcal{Q},\mathcal{R}), d(gu, G(t,y,z,x)) - d(\mathcal{Q},\mathcal{R}), \\ &\quad d(gx, G(x,y,z,t)) - d(\mathcal{Q},\mathcal{R}), d(gy, G(y,x,z,t)) - d(\mathcal{Q},\mathcal{R}), \\ &\quad d(gz, G(z,y,x,t)) - d(\mathcal{Q},\mathcal{R}), d(gt, G(t,y,z,x)) - d(\mathcal{Q},\mathcal{R})] \qquad (10) \end{aligned}$$

(ii) $G(\mathcal{Q}_0,\mathcal{Q}_0,\mathcal{Q}_0,\mathcal{Q}_0) \subseteq \mathcal{R}_0$

(iii) $G(\mathcal{R}_0,\mathcal{R}_0,\mathcal{R}_0,\mathcal{R}_0) \subseteq \mathcal{Q}_0$

(iv) Pair $(\mathcal{Q},\mathcal{R})$ has P-property

then (a,a,a,a) is the unique quadruple g-best proximity point of the pair (g,G).

Proof. Consider $x_0,\ y_0,\ z_0,\ t_0 \in \mathcal{Q}_0$ then $G(x_0,y_0,z_0,t_0),\ G(y_0,x_0,z_0,t_0),\ G(z_0,y_0,x_0,t_0)$ and $G(t_0,y_0,z_0,x_0) \in \mathcal{R}_0$. Then by the same process as in theorem (1), we obtain (a,a,a,a) as the unique quadruple g-best proximity point of the pair (g,G).

Corollary 1. *Let $\mathcal{Q}$ be non-empty subset of complete metric space (X,d) such that $\mathcal{Q}_0$ is non-empty and $g : X \to X$ be mapping such that $\mathcal{Q} \subseteq g(\mathcal{Q})$ and $\mathcal{R} \subseteq g(\mathcal{R})$, let $G : X^4 \to X$ be continuous mapping and $\psi, \zeta \in \Psi$ and $\theta \in \Theta$, satisfies the preceeding conditions:*

(i) For every $x, y, z, t, a, b, c, u \in X$

$$\begin{aligned}\psi(d(gx, ga)) =&\psi(d(G(x,y,z,t), G(a,b,c,u)))\\ \le&\psi\{\max(d(x,a), d(y,b), d(z,c), d(t,u))\}\\ &-\zeta\{\max(d(x,a), d(y,b), d(z,c), d(t,u))\}\\ &+\theta[d(ga, G(x,y,z,t)) - d(\mathcal{Q},\mathcal{R}), d(gb, G(y,x,z,t)) - d(\mathcal{Q},\mathcal{R}),\\ &d(gc, G(z,y,x,t)) - d(\mathcal{Q},\mathcal{R}), d(gu, G(t,y,z,x)) - d(\mathcal{Q},\mathcal{R}),\\ &d(gx, G(x,y,z,t)) - d(\mathcal{Q},\mathcal{R}), d(gy, G(y,x,z,t)) - d(\mathcal{Q},\mathcal{R}),\\ &d(gz, G(z,y,x,t)) - d(\mathcal{Q},\mathcal{R}), d(gt, G(t,y,z,x)) - d(\mathcal{Q},\mathcal{R})]\end{aligned} \tag{11}$$

(ii) $G(\mathcal{Q},\mathcal{Q},\mathcal{Q},\mathcal{Q}) \subseteq \mathcal{Q}$

(iii) g is an isometry

then (a,a,a,a) is the unique quadruple g-fixed point of the pair (g,G).

Proof. By taking $\mathcal{Q} = \mathcal{R}$ in Theorem (1), we have the desired result.

Example 1. *Consider $X = [1,5]$ with $d(x,y) = \|x-y\|$for all $x, y \in X$. Let $\mathcal{Q} = [2,3]$ and $\mathcal{R} = [3,4]$ be subsets of X and $G : X^4 \to X, g : \mathcal{Q} \to \mathcal{Q}$ both are continuous mappings given by $G(x,y,z,t) = \frac{1}{2}(x - y + z + t)$ and $g(x) = x$ respectively for all $x, y, z, t \in X$. Consider $\theta : [0,\infty)^8 \to [0,\infty)$ defined by $\theta(x,y,z,t,a,b,c,u) = \min\{x,y,z,t,a,b,c,u\}$ and $\psi, \zeta : [0,\infty) \to [0,\infty)$ are given by $\psi(q) = \frac{1}{2}q, \zeta(q) = \frac{1}{3}q$.*

Here $\mathcal{Q}_0 = \{3\}$ and $\mathcal{R}_0 = \{3\}$ with $d(\mathcal{Q},\mathcal{R}) = 0$. Taking $x_0, z_0 \in \mathcal{Q}_0$ and $y_0, t_0 \in \mathcal{R}_0$, then $G(\mathcal{Q}_0,\mathcal{R}_0,\mathcal{Q}_0,\mathcal{R}_0) \subseteq \mathcal{R}_0$, $G(\mathcal{R}_0,\mathcal{Q}_0,\mathcal{R}_0,\mathcal{Q}_0) \subseteq \mathcal{Q}_0$. Also, the remaining conditions of the theorem are satisfied. Hence by the theorem (1), $(3,3,3,3)$ is the quadruple g-best proximity point of g and G.

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