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Boundedness of non regular pseudo-differential operators on variable exponent Triebel-Lizorkin-Morrey spaces

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Abstract. In this paper, we study the boundedness of non regular pseudo-differential operators on variable exponent Besov-Morrey spaces $\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ with symbols $a(x,\xi)$ belonging to $C^{\ell}_{*}S^{m}_{1,\delta}$. For these symbols x-regularity is measured in Hölder-Zygmund spaces.

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1. Introduction

Pseudo-differential calculus is a well-established tool for the analysis of partial differential equations, especially non-linear ones. Indeed, in [16] one can find many applications of the calculus of non regular pseudo-differential operators to non-linear differential equations. The boundedness of these operators has been extensively addressed in several works. For boundedness on Lebesgue spaces, Besov spaces, Triebel-Lizorkin spaces and Sobolev spaces, we refer to [2], [6], [12] and [13].

The boundedness of pseudo-differential operators in Triebel-Lizorkin-Morrey spacses with constant exponents denoted $\mathcal{E}^{s}_{p,u,q}$ was studied by Yoshihiro Sawano in [15].

Our focus in this paper concerns the boundedness of pseudo-differential operators on Triebel-Lizorkin-Morrey spaces with variable exponents denoted $\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ (see [4]) with symbols in the class $C^{\ell}_{*}S^{m}_{1,\delta}$.

The results of this paper are certainly relevant because they generalize those of [15].

Our approach is as follows. To treat the boundedness of these operators with non-regular symbols belonging to $C^{\ell}_{*}S^{m}_{1,\delta}$ we use elementary symbols as it was done in [2], [12], [14]

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and [15].
Indeed, the symbol reduction method, due to Coifman and Meyer[6], makes it possible to be limited to symbols $a(x,\xi) \in C_*^\ell S_{1,\delta}^m$ of the form $a(x,\xi) = \sum_{j\geq 0} \sigma_j(x)\psi_j(\xi)$(see [14] and [2]). Then, we rewrite the symbol as a sum of three parts, a "low-high", a "high-high", and a "high-low" part. Thus, the operator $a(x,D)$ with symbol a can be resolved into three operators $a_1(x,D)$, $a_2(x,D)$ and $a_3(x,D)$ with symbols a_1, a_2 and a_3. Now it remains to study the boundedness of each elementary operators.
We structure this paper in **4** sections as follows. In Section **2** we give the preliminaries, where we recall the definitions of Morrey spaces and Besov-Morrey spaces with variable exponents. In Section **3**, we recall necessary tools for the proofs of the lemmas and the main result that we give in Section **4**.

2. Preliminaries

We denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{Z}$ stands for the set of all integer numbers. We write $B(x,r)$ for the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$. We use c as a generic positive constant, i.e. a constant whose value may change with each appearance. The expression $f \lesssim g$ means that $f \leq cg$ for some independent constant c, and $f \approx g$ means $f \lesssim g \lesssim f$. Throughout the paper we denote by $\mathcal{M}(\mathbb{R}^n)$ the family of all complex or extended real-valued measurable functions on $\mathbb{R}^n$.
By supp$f$ we denote the support of the function f , i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then χ_E denotes its characteristic function.
We denote by $\mathcal{S}(\mathbb{R}^n)$ the set of all Schwartz functions on $\mathbb{R}^n$. We denote by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. The Fourier transform of a tempered distribution f is denoted by $\mathcal{F}f$ or $\hat{f}$ while its inverse transform is denoted by $\mathcal{F}^{-1}f$ or $\check{f}$.

2.1. Variable exponents

For more information on the results of this paragraph, see [11] and [7].
• By $\mathcal{P}(\mathbb{R}^n)$ we denote the set of all measurable functions $p : \mathbb{R}^n \to (0,+\infty]$ (called variable exponents) which are essentially bounded away from zero. We denote $p^+_{\mathbb{R}^n}:=$ ess $\sup_{\mathbb{R}^n} p(x)$ and $p^-_{\mathbb{R}^n} :=$ ess $\inf_{\mathbb{R}^n} p(x)$; we abbreviate $p^+ = p^+_{\mathbb{R}^n}$ and $p^- = p^-_{\mathbb{R}^n}$.
• The function ϕ_p is defined as follows:

$$\phi_{p(x)}(t) = \begin{cases} t^{p(x)} & \text{if } p(x) \in (0,+\infty), \\ 0 & \text{if } p(x) = +\infty \text{ and } t \in [0,1], \\ +\infty & \text{if } p(x) = +\infty \text{ and } t \in (1,+\infty]. \end{cases}$$

The variable exponent modular associated to $p(\cdot)$ is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \phi_{p(x)}(|f(x)|)\mathrm{d}x.$$

The variable exponent Lebesgue space $L_{p(\cdot)} := L_{p(\cdot)}(\mathbb{R}^n)$ is the family of (equivalence classes of) functions $f \in \mathcal{M}(\mathbb{R}^n)$ such that $\varrho_{p(\cdot)}(f/\lambda)$ is finite for some $\lambda > 0$.
$L_{p(\cdot)}$ is a quasi-Banach space equipped with the quasinorm

$$\|f\|_{p(\cdot)} := \inf\left\{\mu > 0 : \varrho_{p(\cdot)}\left(\frac{1}{\mu}f\right) \leq 1\right\}.$$

• We say that a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C^{log}_{loc}(\mathbb{R}^n)$, if there exists $c_{log}(g) \geq 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{log}(g)}{\log(e + 1/|x-y|)} \quad \text{for all } x, y \in \mathbb{R}^n. \tag{1}$$

The function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be *globally* log-*Hölder continuous*, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if it is locally log-Hölder continuous and there exists $g_\infty \in \mathbb{R}$ and $c_\infty(g) \geq 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_\infty(g)}{\log(e + |x|)} \quad \text{for all } x \in \mathbb{R}^n.$$

We write $g \in \mathcal{P}^{log}(\mathbb{R}^n)$ if $0 < g^- \leq g(x) \leq g^+ \leq +\infty$ with $\dfrac{1}{g} \in C^{log}(\mathbb{R}^n)$.

We define $\dfrac{1}{g_\infty} := \lim\limits_{|x|\to+\infty} \dfrac{1}{g(x)}$ and we use the convention $\dfrac{1}{\infty} = 0$.

2.2. Variable exponent Triebel-Lizorkin-Morrey spaces

We refer to the papers [4], [18], [3], [5], [17] and [9], for further results on Triebel-Lizorkin-Morrey spaces and variable exponent Triebel-Lizorkin-Morrey spaces.

• **Morrey spaces**

Definition 1. *For $p, u \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \leq p(x) \leq u(x) \leq +\infty$, the variable exponent Morrey space $M_{p(\cdot),u(\cdot)} := M_{p(\cdot),u(\cdot)}(\mathbb{R}^n)$ consists of all functions $f \in \mathcal{M}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p(\cdot),u(\cdot)}} := \sup_{x\in\mathbb{R}^n,\ r>0} r^{\frac{n}{u(x)} - \frac{n}{p(x)}} \left\|f\chi_{B(x.r)}\right\|_{L_{p(\cdot)}}. \tag{2}$$

By the definition of the $L_{p(\cdot)}$ quasinorm, (2) can also be written as

$$\|f\|_{M_{p(\cdot),u(\cdot)}} := \sup_{x\in\mathbb{R}^n,\ r>0} \inf\left\{\lambda > 0 :\ \varrho\left(r^{\frac{n}{u(x)} - \frac{n}{p(x)}}\frac{f}{\lambda}\chi_{B(x.r)}\right) \leq 1\right\}.$$

Definition 2. *Let $p, q, u \in \mathcal{P}(\mathbb{R}^n)$ with $p(x) \leq u(x)$. The mixed space $M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})$ consists of all sequences $(f_\nu)_\nu \subset \mathcal{M}(\mathbb{R}^n)$ such that,*

$$\|(f_\nu)_\nu\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} := \left\|\left(\sum_{\nu=0}^{+\infty} |f_\nu(\cdot)|^{q(\cdot)}\right)^{1/q(\cdot)}\right\|_{M_{p(\cdot),u(\cdot)}} < +\infty. \tag{3}$$

Remark 1. *[4] Note that* $\|\cdot\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$ *defined a quasinorm on* $M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})$*. It is a norm when* $\min(p^-,q^-)\geq 1$*.*

Proposition 1. *Let* f *and* g *be two measurable functions with* $0\leq f(x)\leq g(x)$ *for a.e.* $x\in\mathbb{R}^n$*. Then it holds*

$$\|f\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}\leq \|g\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.$$

Proposition 2. *Let* $p,q,u\in\mathcal{P}(\mathbb{R}^n)$ *with* $p(x)\leq u(x)$ *and* $0<t<+\infty$*.*
Let $(f_\nu)_\nu\subset\mathcal{M}(\mathbb{R}^n)$

$$\left\|(|f_\nu|^t)_\nu\right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}\left(\ell_{\frac{q(\cdot)}{t}}\right)}=\|(f_\nu)_\nu\|^t_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$$

with the usual modification every time $q(x)=+\infty$*.*

•**Triebel-Lizorkin-Morrey spaces**.
We first recall a *Littlewood-Paley partition of unity* $\{\psi_\nu\}$, $\nu\geq 0$.
The functions ψ_ν are defined as follows. Let $\psi_0\in C_0^\infty(\mathbb{R}^n)$ such that $\psi_0\equiv 1$ on $B(0;1)$ and $\text{supp}\psi_0\subset B(0;2)$.
Set

$$\psi_\nu(\xi)=\psi_0(2^{-\nu}\xi)-\psi_0(2^{-\nu+1}\xi)\ \text{ for all } \nu\in\mathbb{N}.$$

Then ψ_ν is supported on the dyadic shell

$$D_\nu=\left\{\xi\in\mathbb{R}^n:\ 2^{\nu-1}\leq|\xi|\leq 2^{\nu+1}\right\}.$$

If $f\in\mathcal{S}'$, then

$$f=\sum_{\nu\geq 0}\psi_\nu f.$$

The Fourier multiplier $\psi_j(D)$ with symbol ψ_j is defined as

$$\psi_\nu(D)f(x)=\mathcal{F}^{-1}(\psi_\nu\cdot\hat{f})(x)=\int_{\mathbb{R}^n}\psi_\nu(\xi)\hat{f}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi.$$

Definition 3. *Let* $\{\psi_\nu\}$ *be the usual Littlewood-Paley partition of unity. Let* $s:\mathbb{R}^n\to\mathbb{R}$*,* $p,q\in\mathcal{P}^{log}(\mathbb{R}^n)$ *and* $u\in\mathcal{P}(\mathbb{R}^n)$ *such that* $0<p^-\leq p(x)\leq u(x)\leq\sup u<+\infty$ *and* $q^-,q^+\in(0,+\infty)$*. The Triebel-Lizorkin-Morrey spaces* $\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ *consists of all distributions* $f\in\mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}:=\|\psi_0(D)f\|_{M_{p(\cdot),u(\cdot)}}+\left\|\left(2^{\nu s(\cdot)}\psi_\nu(D)f_\nu\right)_{\nu\geq 1}\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}<+\infty. \quad (4)$$

Remark 2. *[4](remark4.4) Note that* $\|\cdot\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}$ *defined a quasinorm on* $\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$*. It is a norm when* $\min(p^-,q^-)\geq 1$*.*

3. Basic tools

In this section we present some useful results for the last section. At First, we recall the η-functions defined by

$$\eta_{\nu,m}(x) = 2^{n\nu}\left(1 + 2^{\nu}|x|\right)^{-m}, \quad \nu \in \mathbb{N}_0,\ m > 0.$$

Note that $\eta_{\nu,m} \in L_1$ for $m > n$ and the corresponding L_1-norm does not depend on ν.

The following lemma is from [8](Lemma19) and [10](Lemma6.1)

Lemma 1. *Let $\alpha \in C^{log}_{loc}(\mathbb{R}^n)$ and let $m \geq 0,\ R \geq c_{log}(\alpha)$, where c_{log} is the constant from (1) for α.*
Then

$$2^{\nu\alpha(x)}\eta_{\nu,m+R}(x-y) \leq c2^{\nu\alpha(y)}\eta_{\nu,m}(x-y)$$

with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $\nu \in \mathbb{N}_0$.

The following lemma is from [10](lemma A.6).

Lemma 2. *Let $t > 0, \nu \in \mathbb{N}_0$ and $m > n$. Then there exists $c = c(t, m, n)$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$ with $supp\mathcal{F}g \subset \left\{\xi \in \mathbb{R}^n :\ |\xi| \leq 2^{\nu+1}\right\}$, We have*

$$|g(x)| \leq c\left(\eta_{\nu,m} * |g|^t(x)\right)^{1/t}, \quad x \in \mathbb{R}^n.$$

The following lemma is from[4](theorem3.3).

Lemma 3. *Let $p, q \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $u \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p^- \leq p(x) \leq u(x) \leq \sup u < +\infty$ and $q^-, q^+ \in (1, +\infty)$. If*

$$m > n + n\max\left\{0, \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\},$$

then there exists $c > 0$ such that for all sequences $(f_\nu)_\nu \subset M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})$.

$$\left\|(\eta_{\nu,m} * f_\nu)_\nu\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \leq c\left\|(f_\nu)_\nu\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.$$

The following lemma is from[1](Corollary 4.8.)

Lemma 4. *Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $u \in \mathcal{P}$ with $1 < p^- \leq p(x) \leq u(x) \leq \sup u < +\infty$. If $m > n + n\max\left\{0, \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\}$.*
Then there exists $c > 0$ such that

$$\|\eta_{\nu,m} * f\|_{M_{p(\cdot),u(\cdot)}} \leq c\|f\|_{M_{p(\cdot),u(\cdot)}}.$$

The following lemma is from[4](Lemma 3.7).

Lemma 5. *Let $p, u, q \in \mathcal{P}(\mathbb{R}^n)$ with $p(x) \leq u(x)$. Let $\delta > 0$. For any sequence $(g_j)_{j\in\mathbb{N}_0}$ of non negative measurable functions on $\mathbb{R}^n$, we denote*

$$G_\nu(x) := \sum_{j=0}^{+\infty} 2^{-|\nu-j|\delta} g_j(x), \quad x \in \mathbb{R}^n, \ \nu \in \mathbb{N}_0.$$

Then it holds $\|(G_\nu)_\nu\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} \leq c(\delta, q) \left\|(g_j)_j\right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}$ *where*

$$c(\delta, q) = \max\left(\sum_{\nu\in\mathbb{Z}} 2^{-|\nu|\delta}, \left[\sum_{\nu\in\mathbb{Z}} 2^{-|\nu|\delta q^-}\right]^{1/q^-}\right).$$

4. Boundedness of pseudo-differential operators

We will use symbols for which x-regularity is measured in Hölder-Zygmund spaces.

Definition 4. *[14] The function $a(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ belongs to the symbol class $C_*^\ell S_{1,\delta}^m$, $\delta \in [0, 1]$, $\ell > 0$ if it is smooth in ξ and satisfies the following estimates:*

$$\begin{cases} \left\|\partial_\xi^\alpha a(\cdot, \xi)\right\|_{C_*^\ell S_{1,\delta}^m} \leq c_\alpha \langle\xi\rangle^{m-|\alpha|+\ell\delta} \ \text{and} \\ \left|\partial_\xi^\alpha a(x, \xi)\right| \leq c'_\alpha \langle\xi\rangle^{m-|\alpha|} \end{cases} \tag{5}$$

In (5), $\langle\xi\rangle$ stand for $\left(1 + |\xi|^2\right)^{1/2}$.
A pseudo-differential operator on $\mathcal{E}_{p(\cdot),u(\cdot),q(\cdot)}^{s(\cdot)}$ with symbol $a \in C_*^\ell S_{1,\delta}^m$ is defined by

$$a(x, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x, \xi)\mathcal{F}f(\xi)d\xi, \quad f \in \mathcal{E}_{p(\cdot),u(\cdot),q(\cdot)}^{s(\cdot)}.$$

Definition 5. *We call elementary symbol in the class $C_*^\ell S_{1,\delta}^m$, $\delta \in [0, 1]$, $\ell > 0$ an expression of the form*

$$a(x, \xi) = \sum_{j\geq 0} a_j(x)\psi_j(\xi)$$

where ψ_0 is smooth supported on the ball $B(0, 2)$, $\psi_j(\xi) = \psi(2^{-j}\xi)$ and $\psi \in C_0^\infty$ is supported on the dyadic shell $D_0 = \{\xi \in \mathbb{R}^n : \ 1/2 \leq |\xi| \leq 2\}$, while a_j is uniformly bounded sequence such that

$$\|a_j\|_{C_*^\ell S_{1,\delta}^m} \leq c2^{j(m+\ell\delta)}.$$

Since $a(x, D)$ and $\psi_j(D)$ do not commute, to study boundedness of $a(x, D)$, the symbol reduction method due to Coifman and Meyer[6] makes it possible to be limited to elementary symbols.
Therefore, the operator $a(x, D)$ with symbol a can be resolved into "elementary operators" $a_k(x, D)$ with symbols a_k. This idea has been exploited to establish continuity of pseudo-differential operators with non-regular symbols in inhomogeneous Sobolev spaces $H^{s,p}$ and Hölder-Zygmund spaces C_*^ℓ (see [12] and [2]).

Lemma 6. *[14] Let $f = \sum_{j\geq 0} f_j$ in $\mathcal{S}'$, with supp$\hat{f}_j \subset B(0, A2^j)$ for some $A > 0$. Then, for $\ell > 0$,*

$$\|f\|_{C^\ell_*} \leq c(A) \sup_{j\geq 0} \left\{ 2^{j\ell} \|f_j\|_{L_\infty} \right\}. \tag{6}$$

The following lemmas plays a fundamental role in the proof of the boundedness of pseudo-differential operators on $\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$.

Lemma 7. *Let $c_1, c_2 > 0$, $s \in C^{log}_{loc}$, $p, q \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $u \in \mathcal{P}(\mathbb{R}^n)$ such that $0 < p^- \leq p(x) \leq u(x) \leq \sup u < \infty$ and $q^-, q^+ \in (0, +\infty)$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of tempered distributions such that*

$$supp\mathcal{F} f_0 \subset B(0, 2c_2)$$

and

$$supp\mathcal{F} f_k \subset \left\{ \xi \in \mathbb{R}^n : \ c_1 2^{k-1} < |\xi| < c_2 2^{k+1} \right\} \quad \text{for } k > 0$$

Then

$$\left\| \sum_{k=0}^{+\infty} f_k \right\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \left\| \left(2^{ks(\cdot)} f_k \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.$$

Proof. Let $\{\psi_j\}$ be the Littlewood-Paley partition of unity defined above. By hypothesis, $\psi_j, j \geq 1$ are supported on the dyadic shell D_j, while ψ_0 is supported on the ball $B(0; 2)$. Hence, there is $N_1, N_2 \in \mathbb{N}_0$ such that

$$\begin{aligned} \psi_0(D) \left(\sum_{k=0}^{+\infty} f_k \right) &= \psi_0(D) \left(\sum_{k=0}^{N_1} f_k \right) \\ \text{and } \psi_j(D) \left(\sum_{k=0}^{+\infty} f_k \right) &= \psi_j(D) \left(\sum_{k=j-N_1}^{j+N_2} f_k \right) \end{aligned}$$

Then

$$\left\| \sum_{k=0}^{+\infty} f_k \right\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} = \left\| \sum_{k=0}^{N_1} \check{\psi}_0 * f_k \right\|_{M_{p(\cdot),u(\cdot)}} + \left\| \left\{ 2^{js(\cdot)} \sum_{k=j-N_1}^{j+N_2} \check{\psi}_j * f_k \right\}_{j\geq N_1} \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \tag{7}$$

- Let us first estimate $\left\| \left\{ 2^{js(\cdot)} \sum_{k=j-N_1}^{j+N_2} \check{\psi}_j * f_k \right\}_{j\geq 1} \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$

Since $\check{\psi}_j * f_k \in \mathcal{S}'$ and supp$\mathcal{F}\left(\check{\psi}_j * f_k\right) \subset \left\{\xi \in \mathbb{R}^n : \ |\xi| \leq 2^{j+1}\right\}$, then, by lemma 2,

$$|\check{\psi}_j * f_k| \lesssim \left(\eta_{j,m} * |f_k|^t\right)^{1/t}, \quad k = j - N_1, \ldots, j + N_2.$$

for any $m > n + c_{log}(s) + n\max\left\{0, \sup_{x\in\mathbb{R}^n}\left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\}$ and any $t > 0$.
Thus

$$\left\|\left\{2^{js(\cdot)}\sum_{k=j-N_1}^{j+N_2}\check{\psi}_j * f_k\right\}_{j\geq N_1}\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \lesssim \left\|\left\{\sum_{k=j-N_1}^{j+N_2} 2^{js(\cdot)}\left(\eta_{j,m} * |f_k|^t\right)^{1/t}\right\}_j\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$$

By lemma 1, we can move $2^{js(\cdot)}$ inside the convolution

$$2^{js(\cdot)}\left(\eta_{j,m} * |f_k|^t\right)^{1/t} \lesssim \left(\eta_{j,m-c_{log}(s)} * 2^{js(\cdot)t}|f_k|^t\right)^{1/t}.$$

Then

$$\begin{aligned}
&\left\|\left\{2^{js(\cdot)}\sum_{k=j-N_1}^{j+N_2}\check{\psi}_j * f_k\right\}_{j\geq N_1}\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \\
&\lesssim \left\|\left\{\sum_{k=j-N_1}^{j+N_2}\left(\eta_{j,m-c_{log}(s)} * 2^{js(\cdot)t}|f_k|^t\right)^{1/t}\right\}_j\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \\
&= \left\|\left\{\sum_{k=j-N_1}^{j+N_2}\left(\eta_{j,m-c_{log}(s)} * 2^{js(\cdot)t}|f_k|^t\right)\right\}_j\right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})}.
\end{aligned}$$

With $t \in (0, \min\{1, p^-, q^-\})$, lemma 4 yields

$$\left\|\left\{2^{js(\cdot)}\sum_{k=j-N_1}^{j+N_2}\check{\psi}_j * f_k\right\}_{j\geq N_1}\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \lesssim \sum_{k=0}^{N_1+N_2}\left\|\left(2^{js(\cdot)t}|f_{j+k-N_1}|^t\right)_j\right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})}.$$

Then $\left\|\left\{2^{js(\cdot)}\sum_{k=j-N_1}^{j+N_2}\check{\psi}_j * f_k\right\}_{j\geq 1}\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \lesssim \left\|\left(2^{ks(\cdot)}f_k\right)_k\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$.

• Now we estimate the first term .
Since $\mathrm{supp}\mathcal{F}\left(\check{\psi}_0 * f_k\right) \subset \{\xi \in \mathbb{R}^n : \ |\xi| \leq 2\}$, then by lemma 2, $\left|\check{\psi}_0 * f_k\right| \lesssim |f_k|$.
Thus

$$\begin{aligned}
\left\|\sum_{k=0}^{N_1}\check{\psi}_0 * f_k\right\|_{M_{p(\cdot),u(\cdot)}} &\lesssim \sum_{k=0}^{N_1}\|f_k\|_{M_{p(\cdot),u(\cdot)}} \\
&= \sum_{k=0}^{N_1}\|(0,\ldots,f_k,0,\ldots)\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}
\end{aligned}$$

$$\lesssim \left\| \left(2^{ks(\cdot)} f_k\right)_k \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}.$$

The proof is completed. □

Lemma 8. *Let $c > 0$, $s \in C^{log}_{loc}$, $p, q \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $u \in \mathcal{P}(\mathbb{R}^n)$ such that $0 < p^- \le p(x) \le u(x) \le \sup u < +\infty$, $s^- > 0$ and $q^-, q^+ \in (0, +\infty)$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of tempered distributions such that*

$$supp\mathcal{F} f_k \subset B(0, c2^{k+1})$$

Then

$$\left\| \sum_{k=0}^{+\infty} f_k \right\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \left\| \left(2^{ks(\cdot)} f_k\right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$$

Proof. In view of the hypothesis on Suppψ_j, there is $N \in \mathbb{N}_0$ such that

$$\left\| \sum_{k=0}^{+\infty} f_k \right\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} = \left\| \psi_0(D) \left(\sum_{k=0}^{+\infty} f_k \right) \right\|_{M_{p(\cdot),u(\cdot)}} + \left\| \left\{ 2^{js(\cdot)} \psi_j(D) \left(\sum_{k=j-N}^{+\infty} f_k \right) \right\}_{j \ge N} \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}. \tag{8}$$

(i) At first we estimate $\left\| \left\{ 2^{js(\cdot)} \psi_j(D) \left(\sum_{k=j-N}^{+\infty} f_k \right) \right\}_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$.

We have

$$\left\| \left\{ 2^{js(\cdot)} \psi_j(D) \left(\sum_{k=j-N}^{+\infty} f_k \right) \right\}_{j \ge N} \right\|_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})} = \left\| \left\{ \sum_{k=j-N}^{+\infty} 2^{js(\cdot)} \left(\check{\psi}_j * f_k\right) \right\}_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$$

Since

$$\begin{cases} \text{supp}\mathcal{F}\left(\check{\psi}_j * f_k\right) \subset \left\{\xi \in \mathbb{R}^n : \ |\xi| \le 2^{j+1}\right\} \\ \text{supp}\mathcal{F}\left(\check{\psi}_j * f_k\right) \subset \left\{\xi \in \mathbb{R}^n : \ |\xi| \le 2^{k+1}\right\}, \end{cases}$$

by lemma 2 ,

$$\begin{cases} 2^{js(\cdot)} \left(\check{\psi}_j * f_k\right) \lesssim 2^{js(\cdot)} \left(\eta_{j,m} * |f_k|^t\right)^{1/t} \\ 2^{js(\cdot)} \left(\check{\psi}_j * f_k\right) \lesssim 2^{js(\cdot)} \left(\eta_{k,m} * |f_k|^t\right)^{1/t}. \end{cases}$$

for $m > n + c_{log}(1/q) + c_{log}(s) + n \max\left\{0, \sup_{x\in\mathbb{R}^n} \left(\frac{1}{p(x)} - \frac{1}{u(x)}\right) - \frac{1}{p_\infty}\right\}$ and $t > 0$.

Therefore

$$\left\| \left\{ 2^{js(\cdot)} \psi_j(D) \left(\sum_{k=j-N}^{+\infty} f_k \right) \right\}_{j \ge N} \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \lesssim \left\| \left\{ \sum_{k=j-N}^{j} 2^{js(\cdot)} \left(\eta_{j,m} * |f_k|^t\right)^{1/t} \right\}_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$$

$$+ \left\| \left\{ \sum_{k=j+1}^{+\infty} 2^{-(k-j)s(\cdot)} 2^{ks(\cdot)} \left(\eta_{k,m} * |f_k|^t \right)^{1/t} \right\}_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$$

Let us estimate each one of the two terms on the right-hand side.
Using lemmas 1 we can move $2^{\nu s(\cdot)}$ inside the convolution $2^{\nu s(\cdot)} \left(\eta_{\nu,m} * |f_k|^t \right)^{1/t}$. And we have

$$2^{\nu s(\cdot)} \left(\eta_{\nu,m} * |f_k|^t \right)^{1/t} \lesssim \left(\eta_{\nu,m_0} * 2^{\nu s(\cdot)t} |f_k|^t \right)^{1/t} \; , \nu = j \text{ or } k \quad \text{where } m_0 = m - c_{log}(s)$$

Thus

- $$\left\| \left\{ \sum_{k=j-N}^{j} 2^{js(\cdot)} \left(\eta_{j,m} * |f_k|^t \right)^{1/t} \right\}_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} = \left\| \left\{ \sum_{k=j-N}^{j} \eta_{j,m_0} * 2^{js(\cdot)t} |f_k|^t \right\}_j \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})}$$
$$\lesssim \sum_{k=-N}^{0} \left\| \left\{ \eta_{j,m_0} * 2^{js(\cdot)t} |f_{k+j}|^t \right\}_j \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})} .$$

For $t \in (0, \min\{p^-, q^-\})$, lemma 3 yields

$$\sum_{k=-N}^{0} \left\| \left\{ \eta_{j,m_0} * 2^{js(\cdot)t} |f_{k+j}|^t \right\}_j \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})} \lesssim \sum_{k=-N}^{0} \left\| \left\{ 2^{js(\cdot)t} |f_{k+j}|^t \right\}_j \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})} .$$

Then

$$\left\| \left\{ \sum_{k=j-N}^{j} 2^{js(\cdot)} \left(\eta_{j,m} * |f_k|^t \right)^{1/t} \right\}_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \lesssim \left\| \left(2^{js(\cdot)} f_j \right)_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} .$$

And

- $$\left\| \left\{ \sum_{k=j+1}^{+\infty} 2^{-(k-j)s(\cdot)} 2^{ks(\cdot)} \left(\eta_{k,m} * |f_k|^t \right)^{1/t} \right\}_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$$

$$\lesssim \left\| \left\{ \sum_{k=j+1}^{+\infty} 2^{-|j-k|s(\cdot)} \left(\eta_{k,m_0} * 2^{ks(\cdot)t} |f_k|^t \right) \right\}_j \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})}$$

$$\lesssim \left\| \left\{ \sum_{k=j+1}^{+\infty} 2^{-|j-k|s^-} \left(\eta_{k,m_0} * 2^{ks(\cdot)t} |f_k|^t \right) \right\}_j \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})}$$

$$\lesssim \left\| \left\{ \sum_{k=0}^{+\infty} 2^{-|j-k|s^-} \left(\eta_{k,m_0} * 2^{ks(\cdot)t} |f_k|^t \right) \right\}_j \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})}$$

By lemma 5 ,

$$\left\| \left\{ \sum_{k=0}^{+\infty} 2^{-|j-k|s^-} \left(\eta_{k,m_0} * 2^{ks(\cdot)t} |f_k|^t \right) \right\}_j \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}\left(\ell_{\frac{q(\cdot)}{t}}\right)} \lesssim \left\| \left(\eta_{k,m_0} * 2^{ks(\cdot)t} |f_k|^t \right)_k \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})}.$$

For $t \in (0, \min p^-, q^-)$, lemma 3 yields

$$\left\| \left(\eta_{k,m_0} * 2^{ks(\cdot)t} |f_k|^t \right)_k \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})} \lesssim \left\| \left(2^{ks(\cdot)} f_k \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.$$

Then

$$\left\| \left\{ \sum_{k=j+1}^{+\infty} 2^{-(k-j)s(\cdot)} 2^{ks(\cdot)} \left(\eta_{k,m_0} * |f_k|^t \right)^{1/t} \right\}_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \lesssim \left\| \left(2^{ks(\cdot)} f_k \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.$$

(ii) Now we estimate $\left\| \psi_0(D) \left(\sum_{k=0}^{+\infty} f_k \right) \right\|_{M_{p(\cdot),u(\cdot)}}$.

Since

$$\text{supp}\mathcal{F}\left(\check{\psi}_0 * f_k \right) \subset \left\{ \xi \in \mathbb{R}^n : \ |\xi| \leq 2^{k+1} \right\},$$

Then

$$\begin{aligned} \left\| \psi_0(D) \left(\sum_{k=0}^{+\infty} f_k \right) \right\|_{M_{p(\cdot),u(\cdot)}} &= \left\| \sum_{k=0}^{N} \psi_0 * f_k \right\|_{M_{p(\cdot),u(\cdot)}} \\ &\lesssim \left\| \sum_{k=0}^{\infty} \left(\eta_{k,m} * |f_k|^t \right)^{1/t} \right\|_{M_{p(\cdot),u(\cdot)}}, \end{aligned}$$

for $m > n + c_{log}(1/q) + c_{log}(s) + n \max \left\{ 0, \sup_{x\in\mathbb{R}^n} \left(\frac{1}{p(x)} - \frac{1}{u(x)} \right) - \frac{1}{p_\infty} \right\}$ by lemma 2.
Then by lemma 1

$$\begin{aligned} \left\| \psi_0(D) \left(\sum_{k=0}^{+\infty} f_k \right) \right\|_{M_{p(\cdot),u(\cdot)}} &\lesssim \left\| \sum_{k=0}^{\infty} 2^{-ks^-} \left(\eta_{k,m-c_{log}(s)} * 2^{ks(\cdot)t} |f_k|^t \right) \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}} \\ &= \left\| \left\{ \sum_{k=0}^{+\infty} 2^{-ks^-} \left(\eta_{k,m-c_{log}(s)} * 2^{ks(\cdot)t} |f_k|^t \right) \right\}_j \right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})} \end{aligned}$$

Thus

$$\begin{aligned}\left\|\psi_0(D)\left(\sum_{k=0}^{+\infty} f_k\right)\right\|_{M_{p(\cdot),u(\cdot)}} &\lesssim \left\|\left(\eta_{k,m-c_{log}(s)} * 2^{ks(\cdot)t}|f_k|^t\right)_k\right\|_{M_{\frac{p(\cdot)}{t},\frac{u(\cdot)}{t}}(\ell_{\frac{q(\cdot)}{t}})} \\ &\lesssim \left\|\left(2^{ks(\cdot)} f_k\right)_k\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \quad \text{by lemma 5 and lemma 2.}\end{aligned}$$

The proof is completed. □

Theorem 1. *Let $a(x,\xi) \in C_*^\ell S_{1,\delta}^m$ where $m \in \mathbb{R}$, $\delta \in [0,1]$ and $\ell > 0$. Let $1 \leq p^- \leq p(x) \leq u(x) \leq \sup u < +\infty$ and $q^-, q^+ \in [1, +\infty)$. Let $s \in C_{loc}^{log}$ such that $0 < s^- \leq s^+ < \ell$. Then*

$$a(x,D) : \mathcal{E}_{p(\cdot),u()\cdot,q(\cdot)}^{s(\cdot)+m} \longrightarrow \mathcal{E}_{p(\cdot),u(\cdot),q(\cdot)}^{s(\cdot)}$$

is bounded.

Proof. We recall that the symbol reduction method, due to Coifman and Meyer[6], makes it possible to be limited to symbols $a(x,\xi) \in C_*^\ell S_{1,\delta}^m$ of the form (see [14] and [2])

$$a(x,\xi) = \sum_{j\geq 0} \sigma_j(x)\psi_j(\xi)$$

where σ_j satisfies

$$\|\sigma_j\|_{C_*^\ell} \leq c2^{j(m+\ell\delta)} \tag{9}$$

$$\text{and} \quad \|\sigma_j\|_{L_\infty} \leq c \tag{10}$$

with c depending on δ and ℓ but not on j. And ψ_j is exactly a Littlewood-Paley function. We have

$$\begin{aligned}\sigma_j(x) &= \sum_{k=0}^{+\infty} \psi_j(D)\sigma_j(x). \\ \text{Then} \quad \sigma_j(x)\psi_j(\xi) &= \left(\sum_{k=0}^{+\infty} \psi_j(D)\sigma_j(x)\psi_j(\xi)\right).\end{aligned}$$

$$\text{Therefore} \quad a(x,\xi) = \sum_{j=0}^{+\infty}\left(\sum_{k=0}^{+\infty} \psi_k(D)\sigma_j(x)\right)\psi_j(\xi).$$

Set $a_{kj} = \psi_k(D)\sigma_j$. Then

$$a(x,\xi) = \sum_{j=0}^{+\infty}\left(\sum_{k=0}^{+\infty} a_{kj}\right)\psi_j(\xi). \tag{11}$$

(i) At first, it's necessary to estimate $\|a_{kj}\|_{L_\infty}$.
We recall the quasinorm of C^ℓ_*: $\|\psi_k(D)\sigma_j\|_{C^\ell_*} = \sup_k 2^{k\ell}\|\psi_k(D)\sigma_j\|_{L_\infty}$.

$$\text{Since} \quad \|\psi_k(D)\sigma_j\|_{C^\ell_*} \leq c\|\sigma_j\|_{C^\ell_*}.$$

$$\text{Then} \quad \sup_k 2^{k\ell}\|\psi_k(D)\sigma_j\|_{L_\infty} \leq c\|\sigma_j\|_{C^\ell_*}.$$

Using (9), we obtain

$$\|a_{kj}\|_{L_\infty} \leq c2^{j(m+\ell\delta)}2^{-k\ell}. \tag{12}$$

Note that $(1-\Delta)^{\frac{m}{2}}$, $m \in \mathbb{R}$ is an isomorphism that composes well with pseudo-differential operators (see[14] and [15]). Therefore, it is enough to examine the case $m = 0$. If $m = 0$ then

$$\|a_{kj}\|_{L_\infty} \leq c2^{j\ell\delta}2^{-k\ell} \tag{13}$$

(ii) Now we rewrite the symbol as a sum of three parts

$$\begin{aligned} a(x,\xi) &= \sum_{j\geq 0}\left(\sum_{k=0}^{j-4} a_{kj}(x) + \sum_{k=j-3}^{j+3} a_{kj}(x) + \sum_{k=j+4}^{\infty} a_{kj}(x)\right)\psi_j(\xi) \\ &= a_1(x,\xi) + a_2(x,\xi) + a_3(x,\xi) \end{aligned}$$

where

$$\begin{aligned} a_1(x,D)f &= \sum_{j=0}^{+\infty}\left(\sum_{k=0}^{j-4} a_{kj}\psi_j(D)f\right), \\ a_2(x,D)f &= \sum_{j=0}^{+\infty}\left(\sum_{k=j-3}^{j+3} a_{kj}\psi_j(D)f\right), \\ a_3(x,D)f &= \sum_{j=0}^{+\infty}\left(\sum_{k=j+4}^{\infty} a_{kj}\psi_j(D)f\right). \end{aligned}$$

•We have

$$\begin{aligned} \mathcal{F}\left(\sum_{k=0}^{j-4} a_{kj}f_j\right) &= \sum_{k=0}^{j-4} \mathcal{F}(\psi_k(D)\sigma_j) * \mathcal{F}(\psi_j(D)f) \\ &= \sum_{k=0}^{j-4}(\psi_k\mathcal{F}\sigma_j) * (\psi_j\mathcal{F}f). \end{aligned}$$

Using the fact that $\mathrm{supp}(f*g) \subset \mathrm{supp}f + \mathrm{supp}g$ for all compactly supported distributions $f, g \in \mathcal{S}'$, we have $\mathrm{supp}\mathcal{F}\left(\sum_{k=0}^{j-4} a_{kj}f_j\right) \subset \left\{\xi \in \mathbb{R}^n : \ c_1 2^{j-1} \leq |\xi| \leq c_2 2^{j+1}\right\}$ with $c_1, c_2 > 0$.
Then lemma 7 yields

$$\begin{aligned}
\|a_1(x,D)f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &= \left\|\sum_{j=0}^{+\infty}\left(\sum_{k=0}^{j-4} a_{kj}\psi_j(D)f\right)\right\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \\
&\lesssim \left\|\left(2^{js(\cdot)}\sum_{k=0}^{j-4} a_{kj}\psi_j(D)f\right)_j\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \\
&\lesssim \left\|\left(\sum_{k=0}^{j-4}\|\sigma_j\|_{L_\infty} 2^{js(\cdot)}\psi_j(D)f\right)_j\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \\
&\lesssim \left\|\left(2^{js(\cdot)}\psi_j(D)f\right)_j\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.
\end{aligned}$$

Then

$$\|a_1(x,D)f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \|f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}.$$

•For the second part $\|a_2(x,D)f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} = \left\|\sum_{j=0}^{+\infty}\left(\sum_{k=j-3}^{j+3} a_{kj}f_j\right)\right\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}$,

we observe that

$$\begin{aligned}
\mathcal{F}\left(\sum_{k=j-3}^{j+3} a_{kj}f_j\right) &= \sum_{k=j-3}^{j+3}\mathcal{F}\left(\psi_k(D)\sigma_j\right) * \mathcal{F}\left(\psi_j(D)f\right) \\
&= \sum_{k=j-3}^{j+3}\left(\psi_k\mathcal{F}\sigma_j\right) * \left(\psi_j\mathcal{F}f\right).
\end{aligned}$$

Then $\mathcal{F}\left(\sum_{k=j-3}^{j+3} a_{kj}f_j\right)$ is supported on the ball $B(0,2^{j+4})$.
By lemma 8,

$$\begin{aligned}
\|a_2(x,D)f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &\lesssim \left\|\left(2^{js(\cdot)}\sum_{k=j-3}^{j+3} a_{kj}f_j\right)_j\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \\
&\le 2^{-m}\left\|\left(\sum_{k=j-3}^{j+3}\|a_{kj}\|_{L_\infty} 2^{js(\cdot)}\psi_j(D)f\right)_j\right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.
\end{aligned}$$

$$\text{One have} \quad \sum_{k=j-3}^{j+3} \|a_{kj}\|_{L_\infty} \lesssim \sum_{k=-3}^{3} 2^{-k\ell} < +\infty \quad (\text{with } \delta = 1).$$

Then

$$\begin{aligned}\|a_2(x,D)f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &\lesssim \left\| \left(2^{js(\cdot)} \psi_j(D) f \right)_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \\ &\lesssim \|f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}}.\end{aligned}$$

• Now let us estimate last part. Since $\mathcal{F}\left(\sum_{k=j+4}^{+\infty} a_{kj} f_j\right)$ is not supported on any ball or shell, we cannot directly use neither lemma7 nor lemma8.
However, in $\mathcal{S}'$ we can write

$$\sum_{j=0}^{+\infty} \sum_{k=j+4}^{+\infty} a_{kj} f_j = \sum_{k=4}^{+\infty} \sum_{j=0}^{k-4} a_{kj} f_j.$$

We have

$$\mathcal{F}\left(\sum_{j=0}^{k-4} a_{kj} f_j\right) = \sum_{j=0}^{k-4} (\psi_k \mathcal{F} a_j) * (\psi_j \mathcal{F} f).$$

We have $\text{supp}\mathcal{F}\left(\sum_{k=0}^{j-4} a_{kj} f_j\right) \subset \left\{\xi \in \mathbb{R}^n | \ \ c_1 2^{j-1} \leq |\xi| \leq c_2 2^{j+1}\right\}$ with $c_1, c_2 > 0$.
Thus we can use lemma 7.

$$\begin{aligned}\|a_3(x,D)f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} &= \left\| \sum_{k=4}^{+\infty} \left(\sum_{j=0}^{k-4} a_{kj} f_j \right) \right\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \\ &\lesssim \left\| \left(2^{ks(\cdot)} \sum_{j=0}^{k-4} a_{kj} f_j \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \\ &\lesssim \left\| \left(\sum_{j=0}^{k-4} \|a_{kj}\|_{L_\infty} 2^{ks(\cdot)} \psi_j(D) f \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.\end{aligned}$$

If we use(13) with $\delta = 1$, we have

$$\|a_3(x,D)f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \left\| \left(\sum_{j=0}^{k-4} 2^{j\ell} 2^{-k\ell} 2^{ks(\cdot)} \psi_j(D) f \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}$$

$$
\begin{aligned}
&= \left\| \left(\sum_{j=0}^{k-4} 2^{(k-j)(s(\cdot)-\ell)} 2^{js(\cdot)} \psi_j(D) f \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \\
&\leq \left\| \left(\sum_{j=0}^{k-4} 2^{-|k-j||s^- - \ell|} 2^{js(\cdot)} \psi_j(D) f \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \\
&\leq \left\| \left(\sum_{j=0}^{+\infty} 2^{-|k-j||s^- - \ell|} 2^{js(\cdot)} \psi_j(D) f \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} .
\end{aligned}
$$

By hypothesis $|s^- - \ell| > 0$. Therefore, by lemma 5

$$
\left\| \left(\sum_{j=0}^{k-4} 2^{-|k-j||s^- - \ell|} 2^{js(\cdot)} \psi_j(D) f \right)_j \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \lesssim \left\| \left(2^{js(\cdot)} \psi_j(D) f \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} .
$$

Then

$$
\|a_3(x,D)f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \|f\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} .
$$

The proof is completed. □

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