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Triple Integral involving the Product of the Logarithmic and Bessel Functions expressed in terms of the Lerch Function

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Abstract. The aim of the present document is to evaluate a triple integral involving the product a general class of logarithmic, special and exponential functions. Importance of our results lies in the fact that they involve the Bessel function of the First Kind, which is used in a wide range of areas spanning Science and Engineering. Further we establish some special cases.

2020 Mathematics Subject Classifications: 30E20, 33-01, 33-03, 33-04, 33-33B

Key Words and Phrases: Triple integral, Bessel function, Catalan's constant, Apéry's constant, Cauchy integral

1. Significance Statement

Triple integrals whose kernels feature special functions are tabled in the book of Prudnikov et al. [9], in evaluating Euler type integrals involving a general class of polynomials, special functions and multivariable A-function [4], in the study of celestial mechanics or Hamiltonian dynamics, as applied to the ellipsoidal components of galaxies [1], in the theory of Eisentein series for the Group $SL(3,\mathbb{R})$ and its applications to a binary problem, and in the theory of automorphic forms, which are defined arithmetically on any reductive Lie group, which have been studied intensively for many years [2].

Based on current literature triple integrals of Special functions is of high importance, researched and used widely. One feature of current work on these integrals which is not present is a closed form solution where possible. In our present work we derive a triple integral whose kernel involves the Bessel function of the first kind $J_v(t)$ and expressed it in terms of the Hurwitz-Lerch zeta function. The Bessel function itself is a very important function and are a set of solutions to a second-order differential equation that can appear in a variety of contexts [6].

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2. Introduction

In this paper we derive the triple definite integral given by

$$\int_0^\infty \int_0^\infty \int_0^\infty t^m x^{v-m} y^{\frac{1}{2}(-m-v-1)} J_v(t) e^{-bx^2-cy} \log^k\left(\frac{at}{x\sqrt{y}}\right) dxdydt \tag{1}$$

where the parameters k, a, b, c, v, m are general complex numbers and $Re(b) > 0, Re(c) > 0, Re(v) > 0, Re(m) < -1$. This definite integral will be used to derive special cases in terms of special functions and fundamental constants. The derivations follow the method used by us in [10]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}} dw. \tag{2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x, y and t, then take a definite triple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (2) by another function of y take the infinite sum of both sides such that the contour integral of both equations are the same.

3. Definite Integral of the Contour Integral

We use the method in [10]. The variable of integration in the contour integral is $s = w+m+v$. The cut and contour are in the first or second quadrant of the complex s-plane depending on the sign of s. The cut approaches the origin from the interior of the first or second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the triple integral by replacing y by $\log\left(\frac{at}{x\sqrt{y}}\right)$ and multiplying by $t^m x^{v-m} y^{\frac{1}{2}(-m-v-1)} J_v(t) e^{-bx^2-cy}$ then taking the definite triple integral with respect to $x \in [0,\infty)$ and $y \in [0,\infty)$ and

$t \in [0,\infty)$ to obtain

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\int_0^\infty\int_0^\infty\int_0^\infty t^m x^{v-m} y^{\frac{1}{2}(-m-v-1)} J_v(t) e^{-bx^2-cy} \\
&\log^k\left(\frac{at}{x\sqrt{y}}\right)dxdydt \\
&= \frac{1}{2\pi i}\int_0^\infty\int_0^\infty\int_0^\infty\int_C a^w w^{-k-1} t^{m+w} J_v(t) e^{-bx^2-cy} x^{-m+v-w} \\
&y^{\frac{1}{2}(-m-v-w-1)} dwdxdydt \\
&= \frac{1}{2\pi i}\int_C\int_0^\infty\int_0^\infty\int_0^\infty a^w w^{-k-1} t^{m+w} J_v(t) e^{-bx^2-cy} x^{-m+v-w} \\
&y^{\frac{1}{2}(-m-v-w-1)} dxdydtdw \\
&= \frac{1}{2\pi i}\int_C \pi a^w w^{-k-1} 2^{m+w-1} b^{\frac{1}{2}(m-v+w-1)} c^{\frac{1}{2}(m+v+w-1)} \\
&\sec\left(\frac{1}{2}\pi(m+v+w)\right)dw
\end{aligned} \tag{3}$$

from equation (10.22.43) in [3] and (3.326.2) in [5] where $Re(w+m+v) > -1, Re(w+m) < -1/2$ and using the reflection formula (8.334.3) in [5] for the Gamma function. We are able to switch the order of integration over w, x, y and t using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C}\times[0,\infty)\times[0,\infty)\times[0,\infty)$

4. The Hurwitz-Lerch zeta Function and Infinite Sum of the Contour Integral

In this section we use Equation (2) to derive the contour integral representations for the Hurwitz-Lerch zeta function.

4.1. The Hurwitz-Lerch zeta Function

The Hurwitz-Lerch zeta function (25.14) in [3] has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s} z^n \tag{4}$$

where $|z| < 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{5}$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

4.2. Infinite sum of the Contour Integral

Using equation (2) and replacing y by $\log(a)+\frac{\log(b)}{2}+\frac{\log(c)}{2}+\frac{1}{2}i\pi(2y+1)+\log(2)$ then multiplying both sides by $\pi 2^m(-1)^y b^{\frac{1}{2}(m-v-1)}c^{\frac{1}{2}(m+v-1)}e^{\frac{1}{2}i\pi(2y+1)(m+v)}$ taking the infinite sum over $y\in[0,\infty)$ and simplifying in terms of the Hurwitz-Lerch zeta function we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\pi^{k+1}2^m b^{\frac{1}{2}(m-v-1)}c^{\frac{1}{2}(m+v-1)}e^{\frac{1}{2}i\pi(k+m+v)}\\
&\quad\Phi\left(-e^{i\pi(m+v)},-k,\frac{-2i\log(2a)-i\log(b)-i\log(c)+\pi}{2\pi}\right)\\
&\quad=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C \pi(-1)^y a^w w^{-k-1}2^{m+w}b^{\frac{1}{2}(m-v+w-1)}c^{\frac{1}{2}(m+v+w-1)}e^{\frac{1}{2}i\pi(2y+1)(m+v+w)}dw\\
&\quad=\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty} \pi(-1)^y a^w w^{-k-1}2^{m+w}b^{\frac{1}{2}(m-v+w-1)}c^{\frac{1}{2}(m+v+w-1)}e^{\frac{1}{2}i\pi(2y+1)(m+v+w)}dw\\
&\quad=\frac{1}{2\pi i}\int_C \pi a^w w^{-k-1}2^{m+w-1}b^{\frac{1}{2}(m-v+w-1)}c^{\frac{1}{2}(m+v+w-1)}\sec\left(\frac{1}{2}\pi(m+v+w)\right)dw
\end{aligned} \tag{6}$$

from equation (1.232.2) in [5] where $Im\left(\frac{1}{2}\pi(m+v+w)\right)>0$ in order for the sum to converge.

5. Definite Integral in terms of the Hurwitz-Lerch zeta Function

Theorem 1. *For all $k,a\in\mathbb{C}, Re(b)>0, Re(c)>0, Re(v)>0, Re(m)<-1$,*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty t^m x^{v-m}y^{\frac{1}{2}(-m-v-1)}J_v(t)e^{-bx^2-cy}\log^k\left(\frac{at}{x\sqrt{y}}\right)dxdydt\\
&=\pi^{k+1}2^m b^{\frac{1}{2}(m-v-1)}c^{\frac{1}{2}(m+v-1)}e^{\frac{1}{2}i\pi(k+m+v)}\\
&\Phi\left(-e^{i\pi(m+v)},-k,\frac{-2i\log(2a)-i\log(b)-i\log(c)+\pi}{2\pi}\right)
\end{aligned} \tag{7}$$

Proof. The right-hand sides of relations (3) and (6) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion.

Example 1. *The degenerate case.*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty t^m x^{v-m}y^{\frac{1}{2}(-m-v-1)}J_v(t)e^{-bx^2-cy}dxdydt\\
&=\pi 2^{m-1}b^{\frac{1}{2}(m-v-1)}c^{\frac{1}{2}(m+v-1)}\sec\left(\frac{1}{2}\pi(m+v)\right)
\end{aligned} \tag{8}$$

Proof. Use equation (7) and set $k = 0$ and simplify using entry (2) in Table below (64:12:7) in [8].

Example 2.

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{t^m e^{-x^2-y} x^{v-m} y^{\frac{1}{2}(-m-v-1)} J_v(t)}{\log\left(-\frac{t}{2x\sqrt{y}}\right)} dxdydt = 2^{m+1} e^{-\frac{1}{2}i\pi(2m+2v+1)} \left(e^{\frac{1}{2}i\pi(m+v)} - \tan^{-1}\left(e^{\frac{1}{2}i\pi(m+v)}\right)\right) \tag{9}$$

Proof. Use equation (7) and set $k = -1, a = -1/2, b = c = 1$ and simplify using entry (3) in Table below (64:12:7) in [8].

Example 3.

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^3 J_{\frac{5}{3}}(t) e^{-x^2-y}}{t^{4/3} y^{2/3} \left(\log^2\left(\frac{t}{2x\sqrt{y}}\right) + \pi^2\right)} dxdydt = -\frac{\pi + \sqrt{3}(\log(3) - 4)}{8\sqrt[3]{2}\pi} \tag{10}$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^3 J_{\frac{5}{3}}(t) e^{-x^2-y} \log\left(\frac{t}{2x\sqrt{y}}\right)}{t^{4/3} y^{2/3} \left(\log^2\left(\frac{t}{2x\sqrt{y}}\right) + \pi^2\right)} dxdtdt = \frac{-4 + \sqrt{3}\pi - \log(3)}{8\sqrt[3]{2}} \tag{11}$$

Proof. Use equation (9) and set $m = -4/3, v = 5/3$ rationalize the denominator and simplify.

Example 4. *The Polylogarithm function* $Li_k(z)$,

$$\int_0^\infty \int_0^\infty \int_0^\infty t^m e^{-x^2-y} x^{v-m} y^{\frac{1}{2}(-m-v-1)} J_v(t) \log^k\left(\frac{it}{2x\sqrt{y}}\right) dxdydt = \pi^{k+1}\left(-2^m\right) e^{\frac{1}{2}i\pi(k+m+v)-i\pi(m+v)} Li_{-k}\left(-e^{i\pi(m+v)}\right) \tag{12}$$

Proof. Use equation (7) and set $a = i/2, b = c = 1$ and simplify using equation (64:12:2) in [8].

Example 5. *Catalan's constant* G

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^2 J_{\frac{5}{4}}(t) e^{-x^2-y}}{t^{3/4} y^{3/4} \log^2\left(\frac{it}{2x\sqrt{y}}\right)} dxdydt = \frac{\left(-\frac{1}{2}\right)^{3/4}\left(\pi^2 + 48iG\right)}{48\pi} \tag{13}$$

Proof. Use equation (12) and set $k = -2, m = -3/4, v = 5/4$ and simplify using equation (2.2.1.2.7) in [7].

Example 6.

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{3/2} J_{\frac{3}{4}}(t) e^{-x^2-y} \log^k\left(\frac{it}{2x\sqrt{y}}\right)}{t^{3/4}\sqrt{y}} dxdydt = -\frac{\left(2^{k+1}-1\right) e^{\frac{i\pi k}{2}} \pi^{k+1} \zeta(-k)}{2^{3/4}} \quad (14)$$

Proof. Use equation (12) and set $m = -3/4, v = 3/4$ and simplify using entry (2) in Table below (64:7) in [8].

Example 7. *The fundamental constant* $\log(2)$,

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{3/2} J_{\frac{3}{4}}(t) e^{-x^2-y}}{t^{3/4}\sqrt{y} \log\left(\frac{it}{2x\sqrt{y}}\right)} dxdydt = -\frac{i\log(2)}{2^{3/4}} \quad (15)$$

Proof. Use equation (14) apply l'Hopital's rule as $k \to -1$ and simplify.

Example 8. *Apéry's constant* $\zeta(3)$

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{3/2} J_{\frac{3}{4}}(t) e^{-x^2-y}}{t^{3/4}\sqrt{y} \log^3\left(\frac{it}{2x\sqrt{y}}\right)} dxdydt = \frac{3i\zeta(3)}{4\ 2^{3/4}\pi^2} \quad (16)$$

Proof. Use equation (14) set $k = -3$ and simplify.

Example 9. *The fundamental constant* $\zeta(5)$,

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{3/2} J_{\frac{3}{4}}(t) e^{-x^2-y}}{t^{3/4}\sqrt{y} \log^5\left(\frac{it}{2x\sqrt{y}}\right)} dxdydt = -\frac{15i\zeta(5)}{16\ 2^{3/4}\pi^4} \quad (17)$$

Proof. Use equation (14) set $k = -5$ and simplify.

6. Discussion

In this paper, we have presented a novel method for deriving a new Bessel function integral transform along with some interesting definite integrals similar to those published by Prudnikov et al. [9], using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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