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Triple Integral involving the Bessel-Integral Function $Ji_v(z)$: Derivation and Evaluation

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Abstract. A triple integral involving the Bessel-integral function $Ji_v(z)$ is derived and evaluated for certain real numbers of the parameters. The derived integral allows a representation in terms of the product of the Hurwitz-Lerch zeta and Gamma functions with seven parameters. All the results in this work are new.

2020 Mathematics Subject Classifications: 30E20, 33-01, 33-03, 33-04, 33-33B

Key Words and Phrases: Triple integral, Bessel-integral function, Hurwitz-Lerch zeta function, Catalan's constant

1. Significance Statement

The Bessel integral function $Ji_v(z)$ has been studied in many works namely; the operational solution of linear differential equations and properties of their solutions [11], and the expansion of the works of by van der Pol were published in [7] and [5]. In the book of Prudnikov et al. [9] section (3.3.2) some very interesting triple integrals containing the Bessel function of order zero $J_0(z)$ are tabled without derivation. In this current work we aim to expand on the Table of Prudnikov et al. by deriving and evaluating a triple integral involving the Bessel integral function given in Table (3.37.4) in [1] and provide a formal derivation by expressing the triple integral in terms of the Hurwitz-Lerch zeta and Gamma functions.

2. Introduction

In this paper we derive the triple definite integral given by

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{v^2\Gamma(v)} x^{m-1} y^{-m+v+1} z^{-m-v+1} (\alpha x)^v e^{-by^2-cz^2} \log^{k-1}$$

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$$\left(\frac{ax}{yz}\right)\left(m\log\left(\frac{ax}{yz}\right)+k\right)\,_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{1}{4}x^2\alpha^2\right)dxdydz \quad (1)$$

where the parameters k, a, b, c, v, m are general complex numbers and $\alpha \in \mathbb{R}_+, Re(b), Re(c) > 0, Re(m) < 0 < Re(v) < 2$. This definite integral will be used to derive special cases in terms of special functions and fundamental constants. The derivations follow the method used by us in [10]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \quad (2)$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x, y and z, then take a definite triple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (2) by another function of x y and z and take the infinite sums of both sides such that the contour integral of both equations are the same.

3. Definite Integral of the Contour Integral

We use the method in [10]. The variable of integration in the contour integral is $s = w+m$. The cut and contour are in the first quadrant of the complex s-plane. The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the triple integral by replacing y by $\log\left(\frac{ax}{yz}\right)$ and multiplying by

$$-\frac{m2^{-v}x^{m-1}y^{-m+v+1}z^{-m-v+1}(\alpha x)^v e^{-by^2-cz^2}\,_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{1}{4}x^2\alpha^2\right)}{v^2\Gamma(v)} \quad (3)$$

for the first equation and replacing y by $\log\left(\frac{ax}{yz}\right)$ and replacing $k \to k-1$ and multiplying by

$$\frac{2^{-v}x^{m-1}y^{-m+v+1}z^{-m-v+1}(\alpha x)^v e^{-by^2-cz^2}\,_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{1}{4}x^2\alpha^2\right)}{v^2\Gamma(v)} \quad (4)$$

to form the second equation. Next we add both equations then take the definite triple integral with respect to $x \in [0,\infty)$, $y \in [0,\infty)$ and $z \in [0,\infty)$ to obtain

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{v^2\Gamma(v)\Gamma(k+1)}x^{m-1}y^{-m+v+1}z^{-m-v+1}(\alpha x)^v e^{-by^2-cz^2}$$
$$\log^{k-1}\left(\frac{ax}{yz}\right)\left(m\log\left(\frac{ax}{yz}\right)+k\right)\,_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{1}{4}x^2\alpha^2\right)dxdydz$$

$$= -\frac{1}{2\pi i}\int_0^\infty\int_0^\infty\int_0^\infty\int_C \frac{1}{v^2\Gamma(v)}2^{-v}a^w w^{-k-1}(m+w)x^{m+w-1}(\alpha x)^v e^{-by^2-cz^2}$$
$$y^{-m+v-w+1}z^{-m-v-w+1}{}_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{1}{4}x^2\alpha^2\right)dwdxdydz$$

$$= -\frac{1}{2\pi i}\int_C\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{v^2\Gamma(v)}2^{-v}a^w w^{-k-1}(m+w)x^{m+w-1}(\alpha x)^v e^{-by^2-cz^2}$$
$$y^{-m+v-w+1}z^{-m-v-w+1}{}_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{1}{4}x^2\alpha^2\right)dxdydzdw$$

$$= \frac{1}{2\pi i}\int_C \pi a^w w^{-k-1}2^{m+w-3}\alpha^{-m-w}b^{\frac{1}{2}(m-v+w-2)}c^{\frac{1}{2}(m+v+w-2)}$$
$$\csc\left(\frac{1}{2}\pi(m+v+w)\right)dw \quad (5)$$

from equation (4) in [2], equation (3.37.4.1) in [1] and equation (3.326.2) in [4] where $0 < Re(w+m) < 2, Re(m) < Re(v) < 2, \alpha \in \mathbb{R}_+$ and using the reflection formula (8.334.3) in [4] for the Gamma function. We are able to switch the order of integration over x, y and z using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C}\times[0,\infty)\times[0,\infty)\times[0,\infty)$

4. The Hurwitz-Lerch zeta Function and Infinite Sum of the Contour Integral

In this section we use Equation (2) to derive the contour integral representations for the Hurwitz-Lerch zeta function.

4.1. The Hurwitz-Lerch zeta Function

The Hurwitz-Lerch zeta function (25.14) in [3] has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \quad (6)$$

where $|z|< 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \quad (7)$$

where $Re(v) > 0$, and either $|z|\leq 1, z\neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

4.2. Infinite sum of the Contour Integral

Using equation (2) and replacing y by $\log(a)-\log(\alpha)+\frac{\log(b)}{2}+\frac{\log(c)}{2}+\frac{1}{2}i\pi(2y+1)+\log(2)$ then multiplying both sides by $-i\pi 2^{m-2}\alpha^{-m}b^{\frac{1}{2}(m-v-2)}c^{\frac{1}{2}(m+v-2)}e^{\frac{1}{2}i\pi(2y+1)(m+v)}$ taking the infinite sum over $y\in[0,\infty)$ and simplifying in terms of the Hurwitz-Lerch zeta function we obtain

$$\begin{aligned}
&-\frac{1}{\Gamma(k+1)}i\pi^{k+1}2^{m-2}\alpha^{-m}b^{\frac{1}{2}(m-v-2)}c^{\frac{1}{2}(m+v-2)}e^{\frac{1}{2}i\pi(k+m+v)}\\
&\quad\Phi\left(e^{i\pi(m+v)},-k,\frac{-2i\log(2a)-i\log(b)-i\log(c)+2i\log(\alpha)+\pi}{2\pi}\right)\\
&\quad=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C i\pi 2^{m-2}a^w w^{-k-1}\alpha^{-m}b^{\frac{1}{2}(m-v-2)}c^{\frac{1}{2}(m+v-2)}\\
&\quad\exp\left(\frac{1}{2}(w(-2\log(\alpha)+\log(b)+\log(c)+\log(4))+i\pi(2y+1)(m+v+w))\right)dw\\
&\quad=-\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}i\pi 2^{m-2}a^w w^{-k-1}\alpha^{-m}b^{\frac{1}{2}(m-v-2)}c^{\frac{1}{2}(m+v-2)}\\
&\quad\exp\left(\frac{1}{2}(w(-2\log(\alpha)+\log(b)+\log(c)+\log(4))+i\pi(2y+1)(m+v+w))\right)dw\\
&\quad=\frac{1}{2\pi i}\int_C\pi a^w w^{-k-1}2^{m+w-3}\alpha^{-m-w}b^{\frac{1}{2}(m-v+w-2)}c^{\frac{1}{2}(m+v+w-2)}\\
&\quad\csc\left(\frac{1}{2}\pi(m+v+w)\right)dw
\end{aligned}\tag{8}$$

from equation (1.232.3) in [4] where $Im\left(\frac{1}{2}\pi(m+v+w)\right)>0$ in order for the sum to converge.

5. Definite Integral in terms of the Hurwitz-Lerch zeta Function

Theorem 1. *For all* $k,a\in\mathbb{C},Re(b)>0,Re(c)>0,Re(m)<0<Re(v)<2,\alpha\in\mathbb{R}_+,$

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty\frac{1}{v^2\Gamma(v)}x^{m-1}y^{-m+v+1}z^{-m-v+1}(\alpha x)^v e^{-by^2-cz^2}\\
&\quad\log^{k-1}\left(\frac{ax}{yz}\right)\left(m\log\left(\frac{ax}{yz}\right)+k\right){}_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{1}{4}x^2\alpha^2\right)dxdydz\\
&\quad=i\pi^{k+1}\alpha^{-m}2^{m+v-2}b^{\frac{1}{2}(m-v-2)}c^{\frac{1}{2}(m+v-2)}e^{\frac{1}{2}i\pi(k+m+v)}\\
&\quad\Phi\left(e^{i\pi(m+v)},-k,\frac{-2i\log(2a)-i\log(b)-i\log(c)+2i\log(\alpha)+\pi}{2\pi}\right)
\end{aligned}\tag{9}$$

Proof. The right-hand sides of relations (5) and (8) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion.

Example 1. *The degenerate case.*

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{v^2\Gamma(v)} m x^{m-1} y^{-m+v+1} z^{-m-v+1} (\alpha x)^v e^{-by^2-cz^2} {}_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{1}{4}x^2\alpha^2\right) dxdydz$$
$$= \pi\alpha^{-m}\left(-2^{m+v-3}\right) b^{\frac{1}{2}(m-v-2)} c^{\frac{1}{2}(m+v-2)} \csc\left(\frac{1}{2}\pi(m+v)\right) \quad (10)$$

Proof. Use equation (9) and set $k=0$ and simplify using entry (2) in Table below (64:12:7) in [8].

Example 2. *The Hurwitz zeta function $\zeta(s,v)$*

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{25x^{5/3}\Gamma\left(\frac{5}{3}\right)} 9y^{10/3}(\alpha x)^{5/3} e^{-by^2-cz^2}\left(k-\frac{2}{3}\log\left(\frac{ax}{yz}\right)\right) \log^{k-1}\left(\frac{ax}{yz}\right) {}_1F_2\left(\frac{5}{6};\frac{11}{6},\frac{8}{3};-\frac{1}{4}x^2\alpha^2\right) dxdydz$$

$$= \frac{i\alpha^{2/3}2^{k-1}e^{\frac{1}{2}i\pi(k+1)}\pi^{k+1}\zeta\left(-k,\frac{-2i\log(2a)-i\log(b)-i\log(c)+2i\log(\alpha)+\pi}{4\pi}\right)}{b^{13/6}\sqrt{c}}$$
$$- \frac{i\alpha^{2/3}2^{k-1}e^{\frac{1}{2}i\pi(k+1)}\pi^{k+1}\zeta\left(-k,\frac{1}{2}\left(\frac{-2i\log(2a)-i\log(b)-i\log(c)+2i\log(\alpha)+\pi}{2\pi}+1\right)\right)}{b^{13/6}\sqrt{c}} \quad (11)$$

Proof. Use equation (9) and set $m=-2/3, v=5/3$ and simplify using entry (4) in Table below (64:12:7) in [8].

Example 3.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{y^{10/3}e^{-y^2-z^2}\left(2\log\left(-\frac{x}{yz}\right)+3\right) {}_1F_2\left(\frac{5}{6};\frac{11}{6},\frac{8}{3};-x^2\right)}{\log^2\left(-\frac{x}{yz}\right)} dxdydz \quad (12)$$

$$= \frac{25}{24}i(\pi-4)\Gamma\left(\frac{5}{3}\right)$$

Proof. Use equation (11) and set $a=-1, b=c=\alpha=1$ and simplify.

Example 4. *The zeta function of Riemann $\zeta(s)$.*

$$\int_0^\infty\int_0^\infty\int_0^\infty y^{10/3}e^{-y^2-z^2}\left(3k-2\log\left(\frac{ix}{2yz}\right)\right)\log^{k-1}\left(\frac{ix}{2yz}\right) {}_1F_2\left(\frac{5}{6};\frac{11}{6},\frac{8}{3};-\frac{x^2}{4}\right)dxdydz$$

$$=\frac{25}{6}\left(2^{k+1}-1\right)e^{\frac{i\pi k}{2}}\pi^{k+1}\Gamma\left(\frac{5}{3}\right)\zeta(-k) \quad (13)$$

Proof. Use equation (11) and set $a=i/2, b=c=\alpha=1$ and simplify using entry (2) in Table below (64:7) in [8].

Example 5.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{\log^2\left(\frac{x}{yz}\right)}x^{v-1}e^{-y^2-z^2}y^{-m-p+v+1}z^{-m-p-v+1} \left(y^mz^mx^p\left(p\log\left(\frac{x}{yz}\right)-1\right)+x^my^pz^p\left(1-m\log\left(\frac{x}{yz}\right)\right)\right) {}_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-x^2\right)dxdydz$$

$$=\frac{1}{2}v\Gamma(v+1)\left(\tanh^{-1}\left(e^{\frac{1}{2}i\pi(p+v)}\right)-\tanh^{-1}\left(e^{\frac{1}{2}i\pi(m+v)}\right)\right) \quad (14)$$

Proof. Use equation (9) and form a second equation by replacing $m\to p$ and take their difference. Next set $k=-1, a=b=c=1, \alpha=2$ and simplify using entry (3) in Table below (64:12:7) in [8].

Example 6.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{\sqrt[3]{x}\log^2\left(\frac{x}{yz}\right)}y^{44/15}z^{4/15}e^{-y^2-z^2} \left(\left(10\sqrt[15]{y}\sqrt[15]{z}-9\sqrt[15]{x}\right)\log\left(\frac{x}{yz}\right)-15\left(\sqrt[15]{x}-\sqrt[15]{y}\sqrt[15]{z}\right)\right) {}_1F_2\left(\frac{2}{3};\frac{5}{3},\frac{7}{3};-x^2\right)dxdydz$$

$$=-5\Gamma\left(\frac{7}{3}\right)\tanh^{-1}\left(2+\frac{3}{\sin\left(\frac{2\pi}{15}\right)-2}\right) \quad (15)$$

Proof. Use equation (14) and set $m=-2/3, p=-3/5, v=4/3$ and simplify.

Example 7. *The Polylogarithm function* $Li_n(z)$.

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-y^2-z^2}x^{m+v-1}y^{-m+v+1}z^{-m-v+1}\log^{k-1}\left(\frac{ix}{2yz}\right)\left(k+m\log\left(\frac{ix}{2yz}\right)\right){}_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{x^2}{4}\right)dxdydz$$

$$= i\pi^{k+1}v2^{m+v-2}\Gamma(v+1)e^{-\frac{1}{2}i\pi(-k+m+v)}Li_{-k}\left(e^{i\pi(m+v)}\right) \quad (16)$$

Proof. Use equation (9) and set $a = i/2, b = c = \alpha = 1$ and simplify using equation (64:12:2) in [8].

Example 8. *Catalan's constant* C.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{\sqrt{x}\log^3\left(\frac{ix}{2yz}\right)}\sqrt{z}y^{2v+\frac{1}{2}}e^{-y^2-z^2}\left(-2+\left(\frac{1}{2}-v\right)\log\left(\frac{ix}{2yz}\right)\right){}_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{x^2}{4}\right)dxdydz$$

$$= \frac{ie^{\frac{3i\pi}{4}}\left(-\frac{\pi^2}{48}+iC\right)v\Gamma(v+1)}{2\sqrt{2}\pi} \quad (17)$$

Proof. Use equation (16) and set $k = -2, m = 1/2 - v$ and simplify using equation (2.2.1.2.7) in [6].

Example 9. *The Hypergeometric function* $s^{-1}\,{}_2F_1\,(1, s, 1+s; z)$.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{\sqrt{x}\log^2\left(-\frac{x}{2yz}\right)}\sqrt{z}y^{2v+\frac{1}{2}}e^{-y^2-z^2}\left(\left(\frac{1}{2}-v\right)\log\left(-\frac{x}{2yz}\right)-1\right){}_1F_2\left(\frac{v}{2};\frac{v}{2}+1,v+1;-\frac{x^2}{4}\right)dxdydz$$

$$= \left(\frac{1}{2}-\frac{i}{2}\right)\left(-1+{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};i\right)\right)v\Gamma(v+1) \quad (18)$$

Proof. Use equation (9) and set $k = -1, a = -1/2, b = c = \alpha = 1, m = 1/2 - v$ and simplify using equation (9.559) in [4].

6. Discussion

In this paper, we have presented a novel method for deriving a new integral transform involving the Bessel integral function $Ji_v(z)$ along with some interesting definite integrals using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

References

[1] Yu. A. Brychkov, O. I. Marichev, and N. V. Savischenko. *Handbook of Mellin tranforms*. CRC Press., 2019.

[2] Yong-Kum Cho, Seok-Young Chung, and Hera Yun. An extension of positivity for integrals of bessel functions and buhmann's radial basis functions. *Proceedings of the American Mathematical Society, Series B*, 5:25–39, 2018.

[3] Nist digital library of mathematical functions. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[4] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007.

[5] P. Humbert. Bessel-integral functions. *Proceedings of the Edinburgh Mathematical Society*, 4:276–285, 1933.

[6] Leonard Lewin. *Polylogarithms and Associated Functions*. North Holland, 1981.

[7] F. Oberhettinger. On some expansions for bessel integral functions. *Journal of Research of the National Bureau of Standards*, 59, 1957.

[8] Keith B. Oldham, Jan Myland, and Jerome Spanier. *An Atlas of Functions: with Equator, the Atlas Function Calculator*. Springer Science & Business Media, 07 2010.

[9] Anatoliĭ Platonovich Prudnikov, ÎUriĭ Aleksandrovich Brychkov, and Oleg Igorevich Marichev. *Integrals and Series: Special functions Volume 2*. CRC Press, 1986.

[10] Robert Reynolds and Allan Stauffer. A method for evaluating definite integrals in terms of special functions with examples. *International Mathematical Forum*, 15:235–244, 2020.

[11] B. van der Pol. On the operational solution of linear differential equations and an investigation of the properties of their solutions. *Phil. Mag.*, 8:861–898, 1929.