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# A Triple Integral Containing the Lommel Function $s_{u,v}(z)$: Derivation and Evaluation

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**Abstract.** A three-dimensional integral containing the kernel $g(x, y, z)s_{u,v}(z)$ is derived. The function $g(x, y, z)$ is a generalized function containing the logarithmic and exponential functions and $s_{u,v}(z)$ is the Lommel function and the integral is taken over the cube $0 \leq y \leq \infty, 0 \leq x \leq \infty, 0 \leq z \leq \infty$. A representation in terms of the Lerch function is derived, from which special cases can be evaluated. Almost all Hurwitz-Lerch Zeta functions have an asymmetrical zero distribution. All the results in this work are new.

**2020 Mathematics Subject Classifications**: 30E20, 33-01, 33-03, 33-04, 33-33B

**Key Words and Phrases**: Lommel Function, Triple integral, Catalan's constant, Cauchy integral

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## 1. Significance Statement

Eugen Cornelius Joseph von Lommel (1837-1899) was a German physicist. He is known for the Lommel polynomial, the Lommel function, the Lommel-Weber function, and the Lommel differential equation. The Lommel function given in equation (10.7.10) in [10] is a particular solution to the inhomogeneous Bessel equation given in equation (10.7.4) in [10]. These functions see a myriad of uses in physics and engineering (see [3] for a complete list of references). Definite integrals in the form of Mellin transforms of Lommel functions are tabled in the book of [1]. In this work the authors extend the dimension of the integral and the kernel involving the Lommel function. In this paper the authors derive a triple integral of the product of the Lommel, logarithmic and exponential functions and express this triple integral in terms of the Hurwitz-Lerch Zeta function $\Phi(z, s, v)$.

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## 2. Introduction

In this paper we derive the triple definite integral given by

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{\alpha x^m e^{-b\left(y^2+z^2\right)} y^{-m-v+1} z^{-m+v+1} (\alpha x)^u \log^k \left(\frac{ax}{yz}\right)}{u^2+2u-v^2+1} {}_1F_2\left(1; \frac{u}{2}-\frac{v}{2}+\frac{3}{2}, \frac{u}{2}+\frac{v}{2}+\frac{3}{2}; -\frac{1}{4}x^2\alpha^2\right) dxdydz \quad (1)$$

where the parameters $k, a, \alpha$ are general complex numbers and $-1 < Re(m) < Re(v) < Re(u) < 1, Re(b) > 0, Re(\alpha) > 0$. This definite integral will be used to derive special cases in terms of special functions and fundamental constants. The derivations follow the method used by us in [8]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}} dw. \quad (2)$$

where $C$ is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of $x$, $y$ and $z$, then take a definite triple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (2) by another function of $y$ and take the infinite sum of both sides such that the contour integral of both equations are the same.

## 3. Definite Integral of the Contour Integral

We use the method in [8, 9]. The variable of integration in the contour integral is $r = w + m$. The cut and contour are in the first or second quadrant of the complex $r$-plane. The cut approaches the origin from the interior of the first or second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the triple integral by replacing $y$ by

$$\log\left(\frac{ax}{yz}\right)$$

and multiplying by

$$\frac{\alpha x^m e^{-b\left(y^2+z^2\right)} y^{-m-v+1} z^{-m+v+1} (\alpha x)^u \, {}_1F_2\left(1; \frac{u}{2}-\frac{v}{2}+\frac{3}{2}, \frac{u}{2}+\frac{v}{2}+\frac{3}{2}; -\frac{1}{4}x^2\alpha^2\right)}{u^2+2u-v^2+1}$$

then taking the definite integral with respect to $x \in [0, \infty)$, $y \in [0, \infty)$ and $z \in [0, \infty)$ to obtain

$$\frac{1}{\Gamma(k+1)}\int_0^\infty \int_0^\infty \int_0^\infty \frac{\alpha x^m e^{-b\left(y^2+z^2\right)} y^{-m-v+1} z^{-m+v+1} (\alpha x)^u \log^k \left(\frac{ax}{yz}\right)}{u^2+2u-v^2+1}$$

$$ {}_1F_2\left(1;\frac{u}{2}-\frac{v}{2}+\frac{3}{2},\frac{u}{2}+\frac{v}{2}+\frac{3}{2};-\frac{1}{4}x^2\alpha^2\right)dxdydz $$

$$ =\frac{1}{2\pi i}\int_0^\infty\int_0^\infty\int_0^\infty\int_C\frac{\alpha a^w w^{-k-1}e^{-b\left(y^2+z^2\right)}x^{m+w}(\alpha x)^u y^{-m-v-w+1}z^{-m+v-w+1}}{u^2+2u-v^2+1} {}_1F_2\left(1;\frac{u}{2}-\frac{v}{2}+\frac{3}{2},\frac{u}{2}+\frac{v}{2}+\frac{3}{2};-\frac{1}{4}x^2\alpha^2\right)dwdxdydz $$

$$ =\frac{1}{2\pi i}\int_C\int_0^\infty\int_0^\infty\int_0^\infty\frac{\alpha a^w w^{-k-1}e^{-b\left(y^2+z^2\right)}x^{m+w}(\alpha x)^u y^{-m-v-w+1}z^{-m+v-w+1}}{u^2+2u-v^2+1} {}_1F_2\left(1;\frac{u}{2}-\frac{v}{2}+\frac{3}{2},\frac{u}{2}+\frac{v}{2}+\frac{3}{2};-\frac{1}{4}x^2\alpha^2\right)dxdydzdw $$

$$ =\frac{1}{2\pi i}\int_C\pi a^w w^{-k-1}b^{m+w-2}\left(-2^{m+u+w-4}\right)\alpha^{-m-w} \Gamma\left(\frac{1}{2}(u-v+1)\right)\Gamma\left(\frac{1}{2}(u+v+1)\right)\csc\left(\frac{1}{2}\pi(m+u+w-1)\right)dw \quad (3) $$

from equation (3.37.5.1) in [1] and equations (3.326.3) and (8.574.3) in [4] where $Re(\alpha)>0, |Re\left(\frac{1}{2}\pi(m+u+w-1)\right)|<1, Re(w+m)<3/2$ and using the reflection formula (8.334.3) in [4] for the Gamma function. We are able to switch the order of integration over $x$, $y$, $z$ and $r$ using Fubini's theorem for multiple integrals see (9.112) in [5], since the integrand is of bounded measure over the space $\mathbb{C}\times[0,\infty)\times[0,\infty)\times[0,\infty)$.

## 4. The Hurwitz-Lerch Zeta Function and Infinite Sum of the Contour Integral

In this section we use Equation (2) to derive the contour integral representations for the Hurwitz-Lerch Zeta function.

### 4.1. The Hurwitz-Lerch Zeta Function

The Hurwitz-Lerch Zeta function (25.14) in [2] has a series representation given by

$$ \Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^n \quad (4) $$

where $|z|<1, v\neq 0,-1,..$ and is continued analytically by its integral representation given by

$$ \Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \quad (5) $$

where $Re(v)>0$, and either $|z|\leq 1, z\neq 1, Re(s)>0$, or $z=1, Re(s)>1$.

### 4.2. Infinite sum of the Contour Integral

Using equation (2) and replacing $y$ by

$$\log(a)-\log(\alpha)+\log(b)+\frac{1}{2}i\pi(2y+1)+\log(2)$$

then multiplying both sides by

$$\pi(-1)^{y}b^{m-2}\alpha^{-m}2^{m+u-3}e^{\frac{1}{2}i\pi(2y+1)(m+u)}\Gamma\left(\frac{1}{2}(u-v+1)\right)\Gamma\left(\frac{1}{2}(u+v+1)\right)$$

taking the infinite sum over $y\in[0,\infty)$ and simplifying in terms of the Hurwitz-Lerch Zeta function we obtain

$$\frac{1}{\Gamma(k+1)}\pi^{k+1}b^{m-2}\alpha^{-m}2^{m+u-3}e^{\frac{1}{2}i\pi(k+m+u)}\Gamma\left(\frac{1}{2}(u-v+1)\right)\Gamma\left(\frac{1}{2}(u+v+1)\right)$$
$$\Phi\left(-e^{i\pi(m+u)},-k,\frac{-2i\log(2a)-2i\log(b)+2i\log(\alpha)+\pi}{2\pi}\right)$$

$$=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_{C}\pi a^{w}w^{-k-1}b^{m+w-2}2^{m+u+w-3}\alpha^{-m-w}$$
$$\Gamma\left(\frac{1}{2}(u-v+1)\right)\Gamma\left(\frac{1}{2}(u+v+1)\right)e^{\frac{1}{2}i\pi(2y(m+u+w+1)+m+u+w)}dw$$

$$=\frac{1}{2\pi i}\int_{C}\sum_{y=0}^{\infty}\pi a^{w}w^{-k-1}b^{m+w-2}2^{m+u+w-3}\alpha^{-m-w}$$
$$\Gamma\left(\frac{1}{2}(u-v+1)\right)\Gamma\left(\frac{1}{2}(u+v+1)\right)e^{\frac{1}{2}i\pi(2y(m+u+w+1)+m+u+w)}dw$$

$$=\frac{1}{2\pi i}\int_{C}\pi a^{w}w^{-k-1}b^{m+w-2}2^{m+u+w-4}\alpha^{-m-w}$$
$$\Gamma\left(\frac{1}{2}(u-v+1)\right)\Gamma\left(\frac{1}{2}(u+v+1)\right)\sec\left(\frac{1}{2}\pi(m+u+w)\right)dw \quad (6)$$

from equation (1.232.2) in [4] where $Im\left(\frac{1}{2}\pi(m+u+w)\right)>0$ in order for the sum to converge.

## 5. Definite Integral in terms of the Hurwitz-Lerch Zeta Function

**Theorem 1.** *For all* $k,a\in\mathbb{C},-1<Re(m)<Re(v)<Re(u)<1,Re(b)>0,Re(\alpha)>0$ *then,*

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{\alpha x^m e^{-b\left(y^2+z^2\right)} y^{-m-v+1} z^{-m+v+1} (\alpha x)^u \log^k\left(\frac{ax}{yz}\right)}{u^2+2u-v^2+1} {}_1F_2\left(1; \frac{u}{2}-\frac{v}{2}+\frac{3}{2}, \frac{u}{2}+\frac{v}{2}+\frac{3}{2}; -\frac{1}{4}x^2\alpha^2\right) dxdydz$$

$$= \pi^{k+1} b^{m-2} \alpha^{-m} 2^{m+u-3} e^{\frac{1}{2}i\pi(k+m+u)} \Gamma\left(\frac{1}{2}(u-v+1)\right) \Gamma\left(\frac{1}{2}(u+v+1)\right) \Phi\left(-e^{i\pi(m+u)}, -k, \frac{-2i\log(2a)-2i\log(b)+2i\log(\alpha)+\pi}{2\pi}\right) \quad (7)$$

*Proof.* The right-hand sides of relations (3) and (6) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion.

**Example 1.** *The degenerate case.*

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{\alpha x^m e^{-b\left(y^2+z^2\right)} y^{-m-v+1} z^{-m+v+1} (\alpha x)^u}{u^2+2u-v^2+1} {}_1F_2\left(1; \frac{u}{2}-\frac{v}{2}+\frac{3}{2}, \frac{u}{2}+\frac{v}{2}+\frac{3}{2}; -\frac{1}{4}x^2\alpha^2\right) dxdydz$$
$$= \pi b^{m-2} \alpha^{-m} 2^{m+u-4} \sec\left(\frac{1}{2}\pi(m+u)\right) \Gamma\left(\frac{1}{2}(u-v+1)\right) \Gamma\left(\frac{1}{2}(u+v+1)\right) \quad (8)$$

*Proof.* Use equation (7) and set $k = 0$ and simplify using entry (2) in Table below (64:12:7) in [7].

**Example 2.** *The inverse tangent function* $\tan^{-1}(x)$,

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-y^2-z^2} x^{m+u} y^{-m-v+1} z^{-m+v+1}}{\left(u^2+2u-v^2+1\right)\log\left(-\frac{x}{2yz}\right)} {}_1F_2\left(1; \frac{u}{2}-\frac{v}{2}+\frac{3}{2}, \frac{u}{2}+\frac{v}{2}+\frac{3}{2}; -\frac{x^2}{4}\right) dxdydz$$

$$= 2^{m+u-2} e^{-\frac{1}{2}i\pi(2m+2u+1)} \left(e^{\frac{1}{2}i\pi(m+u)} - \tan^{-1}\left(e^{\frac{1}{2}i\pi(m+u)}\right)\right) \Gamma\left(\frac{1}{2}(u-v+1)\right) \Gamma\left(\frac{1}{2}(u+v+1)\right) \quad (9)$$

*Proof.* Use equation (7) and set $k = -1, a = -1/2, b = 1, \alpha = 1$ and simplify using entry (3) in Table below (64:12:7) in [7].

**Example 3.** *The inverse hyperbolic tangent function* $\tanh^{-1}(x)$*,*

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{10/21} y^{17/28} z^{31/28} e^{-y^2-z^2} {}_1F_2\left(1; \frac{37}{24}, \frac{43}{24}; -\frac{x^2}{4}\right)}{\log\left(-\frac{x}{2yz}\right)} dxdydz$$
$$= -\sqrt[42]{-1}2^{10/21}\left((-1)^{5/21} + i\tanh^{-1}\left((-1)^{31/42}\right)\right)\Gamma\left(\frac{37}{24}\right)\Gamma\left(\frac{43}{24}\right) \quad (10)$$

*Proof.* Use equation (9) and set $u = 1/3, v = 1/4, m = 1/7$ and simplify.

**Example 4.** *The Polylogarithm function* $Li_k(z)$*,*

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-y^2-z^2} x^{m+u} y^{-m-v+1} z^{-m+v+1} \log^k\left(\frac{ix}{2yz}\right)}{u^2+2u-v^2+1}$$
$${}_1F_2\left(1; \frac{u}{2} - \frac{v}{2} + \frac{3}{2}, \frac{u}{2} + \frac{v}{2} + \frac{3}{2}; -\frac{x^2}{4}\right) dxdydz$$
$$= \pi^{k+1}\left(-2^{m+u-3}\right) e^{\frac{1}{2}i\pi(k+m+u)-i\pi(m+u)}$$
$$\Gamma\left(\frac{1}{2}(u-v+1)\right)\Gamma\left(\frac{1}{2}(u+v+1)\right) Li_{-k}\left(-e^{i\pi(m+u)}\right) \quad (11)$$

*Proof.* Use equation (7) and set $a = i/2, b = 1, \alpha = 1$ and simplify using equation (64:12:2) in [7].

**Example 5.** *The constant* $\pi$*,*

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{y^{7/6} z^{11/6} e^{-y^2-z^2} {}_1F_2\left(1; \frac{19}{12}, \frac{23}{12}; -\frac{x^2}{4}\right)}{\log^2\left(\frac{ix}{2yz}\right)} dxdydz = -\frac{77\pi\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{3456} \quad (12)$$

*Proof.* Use equation (11) and set $k = -2, u = 1/2, v = 1/3, m = -1/2$ and simplify.

**Example 6.** *Catalan's constant* $K$*,*

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{\sqrt{x} y^{2/3} z^{4/3} e^{-y^2-z^2} {}_1F_2\left(1; \frac{19}{12}, \frac{23}{12}; -\frac{x^2}{4}\right)}{\log^2\left(\frac{ix}{2yz}\right)} dxdydz$$
$$= -\frac{\left(\frac{77}{13824} - \frac{77i}{13824}\right)\left(\pi^2 + 48iK\right)\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{\pi} \quad (13)$$

*Proof.* Use equation (11) and set $k = -2, u = 1/2, v = 1/3, m = 0$ and simplify using equation (2.2.1.2.7) in [6].

## 6. Discussion

In this paper, we have presented a novel method for deriving a new triple integral containing the Lommel function $s_{u,v}(z)$ along with some interesting definite integrals, using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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