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The fractional differential equations with uncertainty by conformable derivative

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Abstract. We provide a fractional order fuzzy fractional differential equation $q \in (0, 1]$. A fuzzy fractional integral and a fuzzy conformable derivative are shown and proved. To prove fuzzy solutions for fractional differential equations with fuzzy beginning values and deterministic or fuzzy functions, two alternative techniques are used. The application has been submitted.

2020 Mathematics Subject Classifications: 34K36, 34K37, 46S40

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1. Introduction

In this paper we will consider the fractional differential equation

$$\begin{aligned} y^{(q)}(t) &= F(t, y, k), \quad q \in (0, 1] \\ y(0) &= c \end{aligned} \tag{1}$$

where $k = (k_1, \ldots, k_n)$ is a vector of constants, $t \in (0, a)$ and $y^{(q)}$ is the conformable derivative of y of order $q \in (0, 1]$ see [12]. We suppose the existence of imprecise parameters k_j and c in Eq (1). Because fuzzy sets theory is a valuable tool for representing imprecision and processing vagueness in mathematical models [10, 12], the goal of this paper is to solve Eq (1) with fuzzy parameters using fuzzy conformable derivative using the same technique as Buckley and Feuring [10]. As a result, the first part of this work proposes a new solution to the $q \in (0, 1]$ order fuzzy fractional initial value problem. This new solution's basic attributes are listed. [8] developed the concept of the fuzzy conformable derivative, which is the most natural and efficient definition of the conformable derivative of order $q \in (0, 1]$. The following are the key advantages of this derivative:

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Many applications and phenomena can be modeled using conformable derivatives and need to be solved (physical applications see[2, 4, 5, 13–17].)
It can be extended to solve exactly and numerically fractional differential equations and systems easily and efficiently. It creates new comparisons of conformable derivatives and other previous fractional definitions in many applications. This paper initially proposes a new solution to the $q \in (0,1]$ order fuzzy fractional initial value problem. This new solution's basic qualities are listed below.
The following is a breakdown of the paper's sections: section 2 covers the fundamental ideas of fuzzy numbers. In section 3, we prove certain results on a fuzzy fractional integral and a fuzzy conformable derivative. The extension principle of Zadeh and the concept of fuzzy conformable derivatives are applied in section 4 using two different ways. The fuzzy fractional differential equation is demonstrated. The applications are in section 5.

2. Preliminaries

We place a bar over a letter to denote a fuzzy number of $\mathbb{R}$. So, $\bar{u}$, all represent fuzzy numbers of $\mathbb{R}$. We write $\mu_{\bar{u}}(t)$, a number in $[0,1]$, for the membership function of $\bar{u}$ evaluated at $t \in \mathbb{R}$.
Let us denote by $\mathbb{R}_{\mathcal{F}} = \{\mu_{\bar{u}} : \mathbb{R} \to [0,1]\}$ the class of fuzzy subsets of the real axis satisfying the following properties :

(i) $\bar{u}$ is normal i.e, there exists an $x_0 \in \mathbb{R}$ such that $\mu_{\bar{u}}(x_0) = 1$,

(ii) $\bar{u}$ is fuzzy convex i.e for $x, y \in \mathbb{R}$ and $0 < \lambda \leq 1$,

$$\mu_{\bar{u}}(\lambda x + (1-\lambda)y) \geq \min\left[\mu_{\bar{u}}(x), \mu_{\bar{u}}(y)\right]$$

(iii) $\bar{u}$ is upper semicontinuous,

(iv) $[\bar{u}]^0 = cl\left\{x \in \mathbb{R} \mid \mu_{\bar{u}}(x) > 0\right\}$ is compact.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. For $0 < \alpha \leq 1$ denote $[\bar{u}]^\alpha = \left\{x \in \mathbb{R} \mid \mu_{\bar{u}}(x) \geq \alpha\right\}$, then from (i) to (iv) it follows that the α-level sets $[\bar{u}]^\alpha \in P_K(\mathbb{R})$ for all $0 \leq \alpha \leq 1$ is a closed bounded interval which is denoted by $[\bar{u}]^\alpha = [u_1^\alpha, u_2^\alpha]$.
By $P_K(\mathbb{R})$ we denote the family of all nonempty compact convex subsets of $\mathbb{R}$, and define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual.

Theorem 1. *see [1] If $\bar{u} \in \mathbb{R}_{\mathcal{F}}$, then*

(i) $[\bar{u}]^\alpha \in P_K(\mathbb{R})$ *for all* $0 \leq \alpha \leq 1$

(ii) $[\bar{u}]^{\alpha_2} \subset [\bar{u}]^{\alpha_1}$ *for all* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$

(iii) $\{\alpha_k\} \subset [0,1]$ *is a nondecreasing sequence which converges to* α *then*

$$[\bar{u}]^\alpha = \bigcap_{k \geq 1} [\bar{u}]^{\alpha_k}$$

Conversely, if $A_\alpha = \{[u_1^\alpha, u_2^\alpha]\,; \alpha \in (0,1]\}$ *is a family of closed real intervals verifying* (i) *and* (ii)*, then* $\{A_\alpha\}$ *defined a fuzzy number* $\bar{u} \in \mathbb{R}_{\mathcal{F}}$ *such that* $[\bar{u}]^\alpha = A_\alpha$ *for* $0 < \alpha \leq 1$ *and* $[\bar{u}]^0 = \overline{\cup_{0<\alpha\leq 1} A_\alpha} \subset A_0$.

Definition 1. *[7, 9, 11] We represent an arbitrary fuzzy number by an ordered pair of functions* $[\bar{u}]^\alpha = [u_1^\alpha, u_2^\alpha]\,, \alpha \in [0,1]$, *which satisfy the following requirements :*

1. u_1^α *is an increasing function over* $[0,1]$;
2. u_2^α *is a decreasing function on* $[0,1]$;
3. u_1^α *and* u_2^α *are bounded left continuous on* $(0,1]$, *and right continuous at* $\alpha = 0$;
4. $u_1^\alpha \leq u_2^\alpha$, *for* $0 \leq \alpha \leq 1$.

Definition 2. *For a fuzzy set* $\bar{u} = (u_1, u_2, u_3)\,, (u_1 < u_2 < u_3)\,, \bar{u}$ *is called triangular fuzzy number with peak* u_2, *left width* $u_2 - u_1 > 0$ *and right width* $u_3 - u_2 > 0$, *if its membership function has the following form:*

$$\mu_{\bar{u}}(t) = \begin{cases} 1 - \frac{(u_2 - t)}{u_2 - u_1}, & u_1 \leq t \leq u_2, \\ 1 - \frac{(t - u_2)}{u_3 - u_2}, & u_2 \leq t \leq u_3, \\ 0, & \textit{otherwise.} \end{cases}$$

Lemma 1. *see [3] Let* $\bar{u}, \bar{v} : \mathbb{R} \to [0,1]$ *be the fuzzy sets. Then* $\bar{u} = \bar{v}$ *if and only if* $[\bar{u}]^\alpha = [\bar{v}]^\alpha$ *for all* $\alpha \in [0,1]$.

The following arithmetic operations on fuzzy numbers are well known and frequently used below. If $\bar{u}, \bar{v} \in \mathbb{R}_{\mathcal{F}}$ then

$$[\bar{u} + \bar{v}]^\alpha = [u_1^\alpha + v_1^\alpha, u_2^\alpha + v_2^\alpha]$$

$$[\lambda \bar{u}]^\alpha = \lambda [\bar{u}]^\alpha = \begin{cases} [\lambda u_1^\alpha, \lambda u_2^\alpha] & \textit{if } \lambda \geq 0 \\ [\lambda u_2^\alpha, \lambda u_1^\alpha] & \textit{if } \lambda < 0, \end{cases}$$

Definition 3. *Let* $\bar{u}, \bar{v} \in \mathbb{R}_{\mathcal{F}}$. *If there exists* $\bar{w} \in \mathbb{R}_{\mathcal{F}}$ *such as* $\bar{u} = \bar{v} + \bar{w}$ *then* $\bar{w}$ *is called the* H*-difference of* $\bar{u}, \bar{v}$ *and it is denoted* $\bar{u} \ominus \bar{v}$

3. Fuzzy conformable differentiability and fuzzy fractional integral

Now, we present our new definition, which is the simplest and most natural and efficient definition of conformable derivative of order $q \in (0,1]$.

Definition 4. *[8] Let* $\bar{F} : [0,a) \to \mathbb{R}_{\mathcal{F}}$ *be a fuzzy function.* q^{th} *order fuzzy conformable derivative of* $\bar{F}$ *is defined by*

$$T_q(\bar{F})(t) = \lim_{\varepsilon \to 0^+} \frac{\bar{F}\left(t + \varepsilon t^{1-q}\right) \ominus \bar{F}(t)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{\bar{F}(t) \ominus \bar{F}\left(t - \varepsilon t^{1-q}\right)}{\varepsilon}$$

for all $t > 0, q \in (0,1)$. Let $\bar{F}^{(q)}(t)$ stands for $T_q(\bar{F})(t)$. Hence

$$\bar{F}^{(q)}(t) = \lim_{\varepsilon\to 0^+} \frac{\bar{F}\left(t+\varepsilon t^{1-q}\right) \ominus \bar{F}(t)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\bar{F}(t) \ominus \bar{F}\left(t-\varepsilon t^{1-q}\right)}{\varepsilon}$$

If $\bar{F}$ is q-differentiable in some $(0,a)$, and $\lim_{t\to 0^+} \bar{F}^{(q)}(t)$ exists, then

$$\bar{F}^{(q)}(0) = \lim_{t\to 0^+} \bar{F}^{(q)}(t)$$

and the limits (in the metric d.)

Remark 1. *From the definition, it directly follows that if $\bar{F}$ is q-differentiable then the multi valued mapping $\bar{F}_\alpha$ is q-differentiable for all $\alpha \in [0,1]$ and*

$$T_q\bar{F}_\alpha = \left[\bar{F}^{(q)}(t)\right]^\alpha \tag{2}$$

Here $T_q\bar{F}_\alpha$ is denoted the conformable fractional derivative of $\bar{F}_\alpha$ of order q.

Theorem 2. *[8] Let $\bar{F} : [0,a) \to \mathbb{R}_{\mathcal{F}}$ be q-differentiable. Denote $\bar{F}_\alpha(t) = [f_1^\alpha(t), f_2^\alpha(t)]$, $\alpha \in [0,1]$. Then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are q-differentiable and*

$$\left[\bar{F}^{(q)}(t)\right]^\alpha = \left[(f_1^\alpha)^{(q)}(t), (f_2^\alpha)^{(q)}(t)\right].$$

Theorem 3. *If a function $\bar{F} : [0,a) \to \mathbb{R}_{\mathcal{F}}$ is q-differentiable at $t_0 > 0$, $q \in (0,1]$ Denote $\bar{F}_\alpha(t) = [f_1^\alpha(t), f_2^\alpha(t)], \alpha \in [0,1]$. Then $f_1^\alpha(t)$ and $f_2^\alpha(t)$ are continuous at t_0 so $\bar{F}$ is continuous at t_0.*

Proof. If $\varepsilon > 0$ and $\alpha \in [0,1]$, we have :

$$\left[\bar{F}\left(t_0+\varepsilon t_0^{1-q}\right) \ominus \bar{F}(t_0)\right]^\alpha = \left[f_1^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_1^\alpha(t_0), f_2^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_2^\alpha(t_0)\right]$$

Dividing and multiplying by ε, we have :

$$\left[\bar{F}\left(t_0+\varepsilon t_0^{1-q}\right) \ominus \bar{F}(t_0)\right]^\alpha = \left[\frac{f_1^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_1^\alpha(t_0)}{\varepsilon}\cdot\varepsilon, \frac{f_2^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_2^\alpha(t_0)}{\varepsilon}\cdot\varepsilon\right]$$

Similarly, we obtain:

$$\left[\bar{F}(t_0) \ominus \bar{F}\left(t_0-\varepsilon t_0^{1-q}\right)\right]^\alpha = \left[\frac{f_1^\alpha(t_0) - f_1^\alpha\left(t_0-\varepsilon t_0^{1-q}\right)}{\varepsilon}\cdot\varepsilon, \frac{f_2^\alpha(t_0) - f_2^\alpha\left(t_0-\varepsilon t_0^{1-q}\right)}{\varepsilon}\cdot\varepsilon\right]$$

Then

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\left[\bar{F}\left(t_0+\varepsilon t_0^{1-q}\right) \ominus \bar{F}(t_0)\right]^\alpha = \Bigg[&\lim_{\varepsilon\to 0^+}\frac{f_1^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_1^\alpha(t_0)}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\varepsilon \\ &, \lim_{\varepsilon\to 0^+}\frac{f_2^\alpha\left(t_0+\varepsilon t_0^{1-q}\right) - f_2^\alpha(t_0)}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\varepsilon\Bigg]\end{aligned}$$

Similarly, we obtain :

$$\lim_{\varepsilon\to 0^+}\left[\bar{F}\left(t_0\right)\ominus \bar{F}\left(t_0-\varepsilon t_0^{1-q}\right)\right]^{\alpha}=\left[\lim_{\varepsilon\to 0^+}\frac{f_1^{\alpha}\left(t_0\right)-f_1^{\alpha}\left(t_0-\varepsilon t_0^{1-q}\right)}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\varepsilon\right.$$
$$\left.,\lim_{\varepsilon\to 0^+}\frac{f_2^{\alpha}\left(t_0\right)-f_2^{\alpha}\left(t_0-\varepsilon t_0^{1-q}\right)}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\varepsilon\right]$$

Let $h=\varepsilon t_0^{1-q}$. Then

$$\lim_{h\to 0^+}\left[\bar{F}\left(t_0+h\right)\ominus \bar{F}\left(t_0\right)\right]^{\alpha}=\left[\left(f_1^{\alpha}\right)^{(q)}\left(t_0\right).0,\left(f_2^{\alpha}\right)^{(q)}\left(t_0\right).0\right]$$

Similarly, we obtain :

$$\lim_{h\to 0^+}\left[\bar{F}\left(t_0\right)\ominus \bar{F}\left(t_0-h\right)\right]^{\alpha}=\left[\left(f_1^{\alpha}\right)^{(q)}\left(t_0\right).0,\left(f_2^{\alpha}\right)^{(q)}\left(t_0\right).0\right]$$

which implies that

$$\lim_{h\to 0^+}\left[\bar{F}\left(t_0+h\right)\right]^{\alpha}=\left[\bar{F}\left(t_0\right)\right]^{\alpha}$$

Similarly, we obtain :

$$\lim_{h\to 0^+}\left[\bar{F}\left(t_0-h\right)\right]^{\alpha}=\left[\bar{F}\left(t_0\right)\right]^{\alpha}$$

Hence, $\bar{F}$ is continuous at t_0.

Theorem 4. *Let $q\in(0,1]$*

- *If $\bar{F}$ is differentiable and $\bar{F}$ is q-differentiable then* $T_q\bar{F}(t)=t^{1-q}\bar{F}'(t)$

Proof. Let $h=\varepsilon t^{1-q}$ in Definition 4 , and then $\varepsilon=t^{q-1}h$. Therefore, if $\varepsilon>0$ and $\alpha\in[0,1]$, we have

$$\left[\bar{F}\left(t+\varepsilon t^{1-q}\right)\ominus \bar{F}(t)\right]^{\alpha}=\left[f_1^{\alpha}\left(t+\varepsilon t^{1-q}\right)-f_1^{\alpha}(t),f_2^{\alpha}\left(t+\varepsilon t^{1-q}\right)-f_2^{\alpha}(t)\right].$$

Dividing by ε, we have

$$\frac{\left[\bar{F}\left(t+\varepsilon t^{1-q}\right)\ominus \bar{F}(t)\right]^{\alpha}}{\varepsilon}=\left[\frac{f_1^{\alpha}\left(t+\varepsilon t^{1-q}\right)-f_1^{\alpha}(t)}{\varepsilon},\frac{f_2^{\alpha}\left(t+\varepsilon t^{1-q}\right)-f_2^{\alpha}(t)}{\varepsilon}\right],$$

and passing to the limit,

$$\begin{aligned}\lim_{\varepsilon\longrightarrow 0^+}\frac{\left[\bar{F}\left(t+\varepsilon t^{1-q}\right)\ominus \bar{F}(t)\right]^{\alpha}}{\varepsilon}&=\lim_{\varepsilon\longrightarrow 0^+}\left[\frac{f_1^{\alpha}\left(t+\varepsilon t^{1-q}\right)-f_1^{\alpha}(t)}{\varepsilon},\frac{f_2^{\alpha}\left(t+\varepsilon t^{1-q}\right)-f_2^{\alpha}(t)}{\varepsilon}\right]\\&=\lim_{h\longrightarrow 0^+}\left[\frac{f_1^{\alpha}(t+h)-f_1^{\alpha}(t)}{t^{q-1}h},\frac{f_2^{\alpha}(t+h)-f_2^{\alpha}(t)}{t^{q-1}h}\right]\\&=t^{1-q}\lim_{h\longrightarrow 0^+}\left[\frac{f_1^{\alpha}(t+h)-f_1^{\alpha}(t)}{h},\frac{f_2^{\alpha}(t+h)-f_2^{\alpha}(t)}{h}\right]\\&=t^{1-q}\left[\left(f_1^{\alpha}\right)'(t),\left(f_2^{\alpha}\right)'(t)\right].\end{aligned}$$

Similarly, we obtain

$$\frac{\left[\bar{F}(t) \ominus \bar{F}\left(t-\varepsilon t^{1-q}\right)\right]^{\alpha}}{\varepsilon}=\left[\frac{f_1^{\alpha}(t)-f_1^{\alpha}\left(t-\varepsilon t^{1-q}\right)}{\varepsilon}, \frac{f_2^{\alpha}(t)-f_2^{\alpha}\left(t-\varepsilon t^{1-q}\right)}{\varepsilon}\right],$$

and passing to the limit and $\varepsilon=t^{q-1}h$ gives $T_q\bar{F}(t)=t^{1-q}\left[\left(f_1^{\alpha}\right)'(t),\left(f_2^{\alpha}\right)'(t)\right]$.

Let $q \in (0,1]$ and $\bar{F}:[0,a) \rightarrow \mathbb{R}_{\mathcal{F}}$ be such that $[\bar{F}(t)]^{\alpha}=[f_1^{\alpha}(t), f_2^{\alpha}(t)]$ for all $\alpha \in [0,1]$. Suppose that $f_1^{\alpha}, f_2^{\alpha} \in C([0,a),\mathbb{R}) \cap L^1([0,a),\mathbb{R})$ for all $\alpha \in [0,1]$ and let

$$A_{\alpha}=:\left[\int_0^t \frac{f_1^{\alpha}(x)}{x^{1-q}}dx, \int_0^t \frac{f_2^{\alpha}(x)}{x^{1-q}}dx\right], \quad t \in (0,a) \tag{3}$$

Lemma 2. *[5] The family $\{A_{\alpha};\alpha \in [0,1]\}$, given by Eq(3), defined a fuzzy number $\bar{F} \in \mathbb{R}_F$ such that $\left[\bar{F}\right]^{\alpha}=A_{\alpha}$.*

Proof. For $\alpha<\beta$ we have that $f_1^{\alpha}(x) \leq f_1^{\beta}(x)$ and $f_2^{\alpha}(x) \geq f_2^{\beta}(x)$. It follows that

$$\begin{gathered} A_{\alpha} \supseteq A_{\beta}. \text{ Since } f_1^0(x) \leq f_1^{\alpha_n}(x) \leq f_1^1(x) \text{ we have} \\ \left|x^{q-1} f_i^{\alpha_n}(x)\right| \leq \max\left\{a^{q-1}\left|f_i^0(x)\right|, a^{q-1}\left|f_i^1(x)\right|\right\}=: g_i(x) \end{gathered}$$

for $\alpha_n \in [0,1]$ and $i=1,2$. Obviously, g_i is integrable on $[0,a)$. Therefore, if $\alpha_n \uparrow \alpha$ then by the Lebesque's Dominated convergence Theorem, we have

$$\lim_{n\to\infty} \int_0^t \frac{f_i^{\alpha_n}}{x^{1-q}}(x)dx=\int_0^t \frac{f_i^{\alpha}}{x^{1-q}}(x)dx,\ i=1,2$$

From Theorem (1), the proof is complete.

Remark 2. *[12] By using the Weierstrass theorem, it is enough to define the fractional integral on polynomials. This suggests the following. Let $q \in (0,1]$. Define $I_q\left(t^p\right)=\frac{t^{p+q}}{p+q}$ for any $p \in \mathbb{R}$, and $q \neq -p$*

- *If $f(t)=\sum_{k=0}^n b_k t^k$, then we define $I_q(f)=\sum_{k=0}^n b_k I_q\left(t^k\right)=\sum_{k=0}^n b_k \frac{t^{k+q}}{k+q}$*
- *If $f(t)=\sum_{k=0}^{\infty} b_k t^k$, where the series is uniformly convergent, then we define $I_q(f)=\sum_{k=0}^{\infty} b_k \frac{t^{k+q}}{k+q}$*

Clearly, I_q is linear on its domain. Further, if $q=1$, then I_q is the usual integral.

Definition 5. *Let $\bar{F} \in C([0,a),\mathbb{R}_F) \cap L^1([0,a),\mathbb{R}_F)$, Define the fuzzy fractional integral for $a \geq 0$ and $q \in (0,1)$*

$$I_q(\bar{F})(t)=I_1\left(t^{q-1}\bar{F}\right)(t)=\int_0^t \frac{\bar{F}(x)}{x^{1-q}}dx$$

by

$$\begin{aligned}\left[I_q(\bar{F})(t)\right]^\alpha &= \left[I_1\left(t^{q-1}\bar{F}\right)(t)\right]^\alpha = \left[\int_0^t \frac{\bar{F}(x)}{x^{1-q}}dx\right]^\alpha \\ &= \left[\int_0^t \frac{f_1^\alpha(x)}{x^{1-q}}dx, \int_0^t \frac{f_2^\alpha(x)}{x^{1-q}}dx\right]\end{aligned}$$

where the integral $\int_0^t \frac{f_i^\alpha}{x^{1-q}}(x)dx$, *for* $i = 1, 2$ *is the usual Riemann improper integral.*

Also, the following properties are obvious.

(*i*) $I_q\lambda\bar{F}(t) = \lambda I_q\bar{F}(t)$ *for each* $\lambda \in \mathbb{R}$

(*ii*) $I_q(\bar{F}+\bar{G})(t) = I_q\bar{F}(t) + I_q\bar{G}(t)$

Theorem 5. $T_qI_q(\bar{F})(t) = \bar{F}(t)$, *for* $t \geq 0$, *where* $\bar{F}$ *is any continuous function in the domain of* I_q.

Proof. Since $\bar{F}$ is continuous, then $I_q(\bar{F})(t)$ is clearly differentiable. Hence,

$$\begin{aligned}\left[T_qI_q(\bar{F})(t)\right]^\alpha &= \left[t^{1-q}\frac{d}{dt}I_q(\bar{F})(t)\right]^\alpha \\ &= \left[t^{1-q}\frac{d}{dt}\int_0^t \frac{f_1^\alpha(x)}{x^{1-q}}dx, t^{1-q}\frac{d}{dt}\int_0^t \frac{f_2^\alpha(x)}{x^{1-q}}dx\right] \\ &= \left[t^{1-q}\frac{f_1^\alpha(t)}{t^{1-q}}, t^{1-q}\frac{f_2^\alpha(t)}{t^{1-q}}\right] \\ &= [\bar{F}(t)]^\alpha\end{aligned}$$

4. Fuzzy fractional differential equation

In this section, we consider Eq (1) has an unique solution $y(t) = G(t,k,c)$, for $t \in (0,a)$, $k \in \mathbb{R}^n, c \in \mathbb{R}$ i.e are given

- $(0,c)$ is in $(0,a) \times I$ where I be an interval for the y-values.
- F is continuous in $(0,a) \times I$(k is fixed) and
- $\frac{\partial F}{\partial y}$ is continuous in $(0,a) \times I$

Let $\bar{K} = \left(\bar{K}_1, \ldots, \bar{K}_n\right)$ be a vector of triangular fuzzy numbers and let $\bar{C}$ be another triangular fuzzy number. Substitute $\bar{K}$ for k and $\bar{C}$ for c in Eq (1) and we get the fuzzy fractional differential equation

$$\begin{aligned}\overline{Y^q}(t) &= \bar{F}(t,\bar{Y},\bar{K}), \quad q \in (0,1] \\ \bar{Y}(0) &= \bar{C}\end{aligned} \tag{4}$$

4.1. Buckley-Feuring solution

The Buckley-Feuring solution, written BF-solution, to the fuzzy fractional dif- ferential equation [**?**], we first fuzzify the crisp solution $y(t) = g(t, k, c)$ to obtain $\bar{Y}(t) = \bar{G}(t, \bar{K}, \bar{C})$ using the extension principle. Alternatively, we get α-cuts as follows :

$$[\bar{Y}(t)]^\alpha = [y_1^\alpha(t), y_2^\alpha(t)]\,, \tag{5}$$

$$[\bar{F}(t, \bar{Y}, \bar{K})]^\alpha = [f_1^\alpha(t), f_2^\alpha(t)]\,, \tag{6}$$

Let $W = [\bar{K}]^\alpha \times [\bar{C}]^\alpha$. By definition

$$y_1^\alpha(t) = \min\{G(t, k, c) : (k, c) \in W\}, \tag{7}$$

$$y_2^\alpha(t) = \max\{G(t, k, c) : (k, c) \in W\}, \tag{8}$$

$$f_1^\alpha(t) = \min\left\{F(t, y, k) : y \in [\bar{Y}(t)]^\alpha, k \in [\bar{K}]^\alpha\right\}, \tag{9}$$

$$f_2^\alpha(t) = \max\left\{F(t, y, k) : y \in [\bar{Y}(t)]^\alpha, k \in [\bar{K}]^\alpha\right\}, \tag{10}$$

for $t \in (0, a)$ and $\alpha \in [0, 1]$.
Let for $\bar{Y}(t)$ to be a solution to the fuzzy fractional differential equation we need that $\bar{Y}^{(q)}(t)$ exist but also (4) must hold. Assume that $(y)_i^\alpha(t)$ for all $i = \{1, 2\}$, is q-differentiable (conformable derivative) with respect to $t \in (0, a)$ for each $\alpha \in [0, 1]$ and $q \in (0, 1]$.

$$\left(y^{(q)}\right)_i^\alpha(t) = f_i^\alpha(t) \tag{11}$$

Or

$$\left(y^{(q)}\right)_1^\alpha(t) = f_1^\alpha(t) \tag{12}$$

$$\left(y^{(q)}\right)_2^\alpha(t) = f_2^\alpha(t) \tag{13}$$

$$y_1^\alpha(0) = c_1^\alpha \tag{14}$$

$$y_2^\alpha(0) = c_2^\alpha \tag{15}$$

where $[\bar{C}]^\alpha = [c_1^\alpha, c_2^\alpha]$.
We write the partial of $y_i^\alpha(t), \quad i = 1, 2$ with respect to t as $\left(y^{(q)}\right)_i^\alpha,\ i = 1, 2 \textit{ and } q \in (0, 1]$.
Let

$$\Gamma(t, \alpha) = \left[\left(y^{(q)}\right)_1^\alpha(t), \left(y^{(q)}\right)_2^\alpha(t)\right] \tag{16}$$

for $t \in (0, a),\ \alpha \in [0, 1]$ and for $q \in (0, 1]$.
If $\Gamma(t, \alpha)$ defines the α-cuts of a fuzzy number for each $t \in (0, a)$ we will say that $\bar{Y}$, is q-differentiable and write

$$\left[\bar{Y}^{(q)}(t)\right]^\alpha = \Gamma(t, \alpha) = \left[\left(y^{(q)}\right)_1^\alpha(t), \left(y^{(q)}\right)_2^\alpha(t)\right] \tag{17}$$

for all $t \in (0, a),\ \alpha \in [0, 1]$ and for $q \in (0, 1]$. Notice, that Eq (17) is just the conformable derivative (with respect to t) of Eq (6) . So, Eq (17) could be written $\left[\bar{Y}^{(q)}(t)\right]^\alpha$.
Sufficient conditions for $\Gamma(t, \alpha)$ to define the α-cuts of a fuzzy number are: (see [6, 9, 11])

(i) $\left(y^{(q)}\right)_1^\alpha$ and $\left(y^{(q)}\right)_2^\alpha$ are continuous on $(0,a)\times[0,1]$ and for $q\in(0,1]$

(ii) $\left(y^{(q)}\right)_1^\alpha$ is an increasing function of α for each $t\in(0,a)$ and for $q\in(0,1]$

(iii) $\left(y^{(q)}\right)_2^\alpha$ is a decreasing function of α for each $t\in(0,a)$ and for $q\in(0,1]$

(iv) $\left(y^{(q)}\right)_1^\alpha \leq \left(y^{(q)}\right)_2^\alpha$ all $t\in(0,a)$ and $q\in(0,1]$

Hence, if conditions $(i)-(iv)$ above hold, $\bar{Y}(t)$ is q-differentiable.
$\bar{Y}^{(q)}(t), \forall q\in(0,1]$ will be a solution to Eq (4) if,

(a) $\bar{Y}^{(q)}(t),\ \forall q\in(0,1]$ is q-differentiable;

(b) (4) holds for $\bar{Y}(t)=\bar{G}(t,\bar{K},\bar{C})$;

(c) $\bar{Y}^{(q)}(t),\ \forall q\in(0,1]$ satisfies the initial and boundary conditions.

Since there is no specified particular initial and boundary conditions, then only is checked if (4) holds.

We will say that $\bar{Y}(t)$ is a solution (without the initial and boundary conditions) if $\bar{Y}^{(q)}(t)$ exists and

$$\bar{Y}^{(q)}(t)=\bar{F}(t), \quad \forall q\in(0,1]$$

or the following equations must hold

$$\left(y^{(q)}\right)_1^\alpha(t)=f_1^\alpha(t), \tag{18}$$

$$\left(y^{(q)}\right)_2^\alpha(t)=f_2^\alpha(t), \tag{19}$$

for all $t\in(0,a), q\in(0,1]$ and all $\alpha\in[0,1]$. We have the following results regarding BF-solution $=\bar{Y}(t)$.

Theorem 6. *Assume $\bar{Y}^{(q)}(t)$, for all $q\in(0,1]$ is q-differentiable for $t\in(0,a)$. Then if*

(a)

$$\frac{\partial F}{\partial y}>0, \quad \frac{\partial G}{\partial c}>0 \tag{20}$$

and

$$\left(\frac{\partial F}{\partial k_j}\right)\left(\frac{\partial G}{\partial k_j}\right)>0, \quad j=1,\ldots,n \tag{21}$$

Then $\bar{Y}(t)$ is a BF-solution.

(b) *If Eq (20) does not hold or Eq (21) does not hold for some j, then $\bar{Y}(t)$ does not a BF-solution.*

Proof. Let us assume there is only one $k_i = k$ and that $\frac{\partial G}{\partial k} > 0$, $\frac{\partial F}{\partial k} > 0$, the proof for $\frac{\partial G}{\partial k} < 0$, $\frac{\partial F}{\partial k} < 0$ is similar and omitted. Since $\frac{\partial G}{\partial k} > 0$ and $\frac{\partial G}{\partial c} > 0$ we have

$$\begin{aligned} y_1^\alpha(t) &= G\left(t, k_1^\alpha, c_1^\alpha\right), \quad (22) \\ y_2^\alpha(t) &= G\left(t, k_2^\alpha, c_2^\alpha\right). \quad (23) \end{aligned}$$

Also, because $\frac{\partial G}{\partial y} > 0$ and $\frac{\partial F}{\partial k} > 0$ we see that

$$\begin{aligned} f_1^\alpha(t) &= G\left(t, y_1^\alpha(t), k_1^\alpha\right), \quad (24) \\ f_2^\alpha(t) &= G\left(t, y_2^\alpha(t), k_2^\alpha\right). \quad (25) \end{aligned}$$

Now, $y(t) = G(t, k, c)$ is unique solution to

$$\begin{aligned} y^{(q)}(t) &= F(t, y, k), \text{ for all } q \in (0,1] \\ y(0) &= c \end{aligned}$$

which implies that

$$G^{(q)}(t) = F(t, G(t, k, c), k), \text{ } \textit{for all } \text{ } q \in (0,1] \quad (26)$$

and

$$g(0, k, c) = c \quad (27)$$

Assuming is q-differentiable we see that

$$\begin{aligned} \left(y^{(q)}\right)_1^\alpha(t) &= G^{(q)}\left(t, k_1^\alpha, c_1^\alpha\right), \quad q \in (0,1] \\ &= F\left(t, G\left(t, k_1^\alpha, c_1^\alpha\right), k_1^\alpha\right) \\ &= F\left(t, y_1^\alpha(t), k_1^\alpha\right) \\ &= f_1^\alpha(t) \end{aligned}$$

and

$$\begin{aligned} y_1^\alpha(0) &= G\left(0, k_1^\alpha, c_1^\alpha\right) \\ &= c_1^\alpha \end{aligned}$$

and also

$$\begin{aligned} \left(y^{(q)}\right)_2^\alpha(t) &= G^{(q)}\left(t, k_2^\alpha, c_2^\alpha\right), \quad q \in (0,1] \\ &= F\left(t, G\left(t, k_2^\alpha, c_2^\alpha\right), k_1^\alpha\right) \\ &= F\left(t, y_2^\alpha(t), k_2^\alpha\right) \\ &= f_2^\alpha(t) \end{aligned}$$

and

$$\begin{aligned} y_2^\alpha(0) &= G\left(0, k_2^\alpha, c_2^\alpha\right) \\ &= c_2^\alpha \end{aligned}$$

for all $\alpha \in [0,1]$ and $t \in (0,a)$. Hence Eqs (12)-(15) hold.
Now consider the situation where Eq (20) or (21) does not hold. Let us only look at one case where $\frac{\partial G}{\partial c} < 0$ (assume $\frac{\partial G}{\partial k} < 0, \quad \frac{\partial F}{\partial y} > 0 \quad$ and $\quad \frac{\partial F}{\partial k} < 0$). Then we have

$$f_1^\alpha(t) = F\left(t, y_1^\alpha(t), k_2^\alpha\right),$$

$$\begin{gathered} f_2^\alpha(t) = F\left(t, y_2^\alpha(t), k_1^\alpha\right), \\ y_1^\alpha(t) = G\left(t, k_2^\alpha, c_2^\alpha\right) \end{gathered}$$

and

$$y_2^\alpha(t) = G\left(t, k_1^\alpha, c_1^\alpha\right)$$

Eqs (12)-(15) becames

$$\begin{aligned} \left(y^{(q)}\right)_1^\alpha(t) &= G^{(q)}\left(t, k_2^\alpha, c_2^\alpha\right), \quad q \in (0,1] \\ &= F\left(t, G\left(t, k_2^\alpha, c_2^\alpha\right), k_2^\alpha\right) \\ &= F\left(t, y_1^\alpha(t), k_2^\alpha\right) \\ &= f_1^\alpha(t) \end{aligned}$$

and

$$\begin{aligned} y_1^\alpha(0) &= G\left(0, k_2^\alpha, c_2^\alpha\right) \\ &= c_2^\alpha \end{aligned}$$

which is not true.

4.2. Seikkala solution

In this section, we present situations where the BF-solution can, and cannot exist, with these fuzzy fractional differential equations the main problem is determining where the Seikkala solution exists (when the BF-solution does not exist).
The Seikkala solution, written SS, to the fuzzy fractional differential equations

$$\begin{aligned} \overline{X^q}(t) &= \bar{F}(t, \bar{X}, \bar{K}), \quad q \in (0,1] \\ \bar{X}(0) &= \bar{C} \end{aligned} \tag{28}$$

Eq (28) is equivalent to Eq (4) substituting x_i for y_i, for $\bar{X} \in \mathbb{R}_{\mathcal{F}}$ with α-cut $[\bar{X}]^\alpha(t) = [x_1^\alpha(t), x_2^\alpha(t)]$, $\alpha \in [0,1]$. Since the fuzzy conformable fractional derivative $\bar{X}^{(q)}$, $q \in (0,1]$ of fuzzy process $\bar{X} : (0,a) \to \mathbb{R}_{\mathcal{F}}$ is defined by

$$\left[\bar{X}^{(q)}(t)\right]^\alpha = \left[\left(x^{(q)}\right)_1^\alpha(t), \left(x^{(q)}\right)_2^\alpha(t)\right], \quad \alpha \in [0,1] \quad \textit{and} \quad q \in (0,1] \tag{29}$$

We call $\bar{X} : (0,a) \to \mathbb{R}_{\mathcal{F}}$ a fuzzy solution of (28) on $(0,a)$, if

$$\begin{aligned} \left(x^{(q)}\right)_1^\alpha(t) &= \min\left\{F(t,y,k) : y \in [\bar{X}(t)]^\alpha, k \in [\bar{K}]^\alpha\right\} && (30)\\ x_1^\alpha(0) &= c_1^\alpha && (31) \end{aligned}$$

$$\begin{aligned} \left(x^{(q)}\right)_2^\alpha(t) &= \max\left\{F(t,y,k) : y \in [\bar{X}(t)]^\alpha, k \in [\bar{K}]^\alpha\right\} && (32)\\ x_2^\alpha(0) &= c_2^\alpha && (33) \end{aligned}$$

for $t \in (0,a)$, for all $q \in (0,1]$ and $\alpha \in [0,1]$. Thus for fixed α, we have an initial value problem in $\mathbb{R}^2$.

If we can solve it (uniquely), we have only to verify that the intervals $[x_1^\alpha(t), x_2^\alpha(t)], \alpha \in [0,1]$, define a fuzzy number $\bar{X} \in \mathbb{R}_{\mathcal{F}}$

Theorem 7. *Let F satisfy*

$$|F(t,x) - F(t,\tilde{x})| \le h(t, |x-\tilde{x}|), t \ge 0, x, \tilde{x} \in \mathbb{R} \tag{34}$$

where $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous mapping such that $r \to h(t,r)$ is nondecreasing, the initial value problem

$$y^{(q)}(t) = h(t, y(t)), \quad y(0) = y_0 \tag{35}$$

has a solution on $\mathbb{R}_+$ for $y_0 > 0$ and that $y(t) = 0$ is the only solution of (35) *for $y_0 = 0$. Then the initial value problem* (28) *has a unique fuzzy solution.*

Proof. Denote $\widetilde{F} = (F_1, F_2)$, $F_1(t,x) = \min\{F(t,y) : y \in [x_1, x_2]\}$ and $F_2(t,x) = \max\{F(t,y) : y \in [x_1,x_2]\}$ where $x = (x_1, x_2) \in \mathbb{R}^2$. It can be shown that (26) implies

$$\|\widetilde{F}(t,x) - \widetilde{F}(t,\widetilde{x})\| \le h(t, \|x - \widetilde{x}\|), t \ge 0, \ x, \widetilde{x} \in \mathbb{R}^2 \tag{36}$$

where the norm $\|.\|$ is defined by $\|x\| = \max\{|x_1|, |x_2|\}$. It is well known that (36) and the assumptions on h gurantee tha existence, uniqueness and continuous dependence on initial value of a solution to

$$x^{(q)}(t) = \widetilde{F}(t, x(t)), \quad x(0) = x_0 \in \mathbb{R}^2 \tag{37}$$

and that for any continuous function $x_1 : \mathbb{R}_+ \longrightarrow \mathbb{R}^2$ the successive approximations $t \ge 0, n = 1, 2, \ldots$

$$\begin{aligned} x_{n+1}(t) &= x_0 + I_q(\tilde{F})\,(t, x_n(t)) \\ &= x_0 + I\left(t^{1-q}\widetilde{F}\right)(t, x_n(t)) \\ &= x_0 + \int_0^t \frac{\widetilde{F}}{x^{1-q}}\,(s, x_n(s))\,ds && (38) \end{aligned}$$

converge uniformly on closed subintervals of $\mathbb{R}_+$ to the solution of (37) . By choosing $x_0 = (c_1^\alpha, c_2^\alpha)$ in (37) we get a unique solution $x^\alpha(t) = (x_1^\alpha(t), x_2^\alpha(t))$ to (30),(32) for each $\alpha \in [0,1]$.
Next we will show that the intervals $[x_1^\alpha(t), x_2^\alpha(t)]\,, \alpha \in [0,1]$, define a fuzzy number $\bar{X} \in \mathbb{R}_{\mathcal{F}}$ for eatch $t \geq 0$, i.e that $\bar{X}$ is a fuzzy solution to (28).
The successive approximations $x_1(t) = x_0 \in \mathbb{R}_{\mathcal{F}}$

$$x_{n+1}(t) = x_0 + \int_0^t \frac{F}{x^{1-q}}\,(s, x_n(s))\,ds, \quad t \geq 0, n = 1, 2, \dots$$

where the integral is the fuzzy integral, define a sequence of fuzzy numbers $x_n(t) \in \mathbb{R}_{\mathcal{F}}, \forall t \geq 0$ Hence

$$[x_n(t)]^\alpha \supset [x_n(t)]^\beta \quad \text{if} \quad 0 < \alpha \leq \beta \leq 1$$

which implies that

$$[x_1^\alpha(t), x_2^\alpha(t)] \supset \left[x_1^\beta(t), x_2^\beta(t)\right], \quad 0 < \alpha \leq \beta \leq 1$$

since, by the convergence of sequence (38), the end points of $[x_n(t)]^\alpha$ converge to $x_1^\alpha(t)$ and $x_2^\alpha(t)$ respectively. Thus the inclusion (ii) of Theorem (1) holds for the intervals $[x_1^\alpha(t), x_2^\alpha(t)]\,, \alpha \in [0,1]$. For the proof of the continuity Theorem (1) by (iii), let (α_k) be a increasing sequence in $[0,1]$ converging to α. Then $c_1^{\alpha_k} \to c_1^\alpha$ and $c_2^{\alpha_k} \to c_2^\alpha$ because $\bar{X}(0) \in \mathbb{R}_{\mathcal{F}}$. But then, by the continous dependence on the initial value of the solution of (37), $x_1^{\alpha_k} \to x_1^\alpha$ and $x_2^{\alpha_k} \to x_2^\alpha$ i.e (iii) holds for the intervals $[x_1^\alpha(t), x_2^\alpha(t)]\,, \alpha \in [0,1]$. By Theorem (1), $\bar{X} \in \mathbb{R}_{\mathcal{F}}$, so $\bar{X}$ is a fuzzy solution of (28) The uniqueness follows from the uniqueness of the solution of (37).

5. Applications

Now we will solve fuzzy fractional differential equations according to our theorems and definitions.
Let

$$\begin{aligned} y^{(q)}(t) &= ky(t),\ q \in (0,1] \\ y(0) &= c \end{aligned}$$

so that the solution is given by $y(t) = c\exp\left(\frac{k}{q}t^q\right)$. Now we fuzzify $F(t, y, k) = ky(t)$ and $G(t, k, c) = c\exp\left(\frac{k}{q}t^q\right)$. Clearly let

$$\begin{aligned} \bar{F}(t, \bar{Y}, \bar{K}) &= \overline{KY}(t), \quad q \in (0,1] \\ \bar{Y}(0) &= \bar{C} \end{aligned}$$

so that $f_1^\alpha(t) = k_1^\alpha y_1^\alpha(t), \quad f_2^\alpha(t) = k_2^\alpha y_2^\alpha(t)$.
Also $\bar{G}(t, \bar{Y}, \bar{K}) = \overline{\text{Cexp}}\left(\frac{\bar{K}}{q}t^q\right)$, therefore

$$y_i^\alpha(t) = c_i^\alpha \exp\left(\frac{k_i^\alpha}{q}t^q\right)$$

for $i = 1, 2$ and $q \in (0, 1]$, $[\bar{K}]^\alpha = [k_1^\alpha, k_2^\alpha]$ and $[\bar{C}]^\alpha = [c_1^\alpha, c_2^\alpha]$, $\bar{Y}$ is q-differentiable because $(y_i^\alpha)^{(q)}(t), \forall q \in (0, 1]$, for $i = 1, 2$ are α-cuts of $\overline{KY}(t)$ i.e α-cuts of a fuzzy number. Due to $\frac{\partial G}{\partial k} > 0, \frac{\partial G}{\partial c} > 0, \frac{\partial F}{\partial k} = y > 0$ for all t, $\frac{\partial F}{\partial y} = k > 0$. So Theorem (6) implies the result that $\bar{Y}(t)$ is a BF-solution. we easily see that $y_i^\alpha(0) = c_i^\alpha$ for $i = 1, 2$, so $\bar{Y}(t)$ also satisfies the initial condition. The BF-solution may be written as

$$\bar{Y}(t) = \overline{\text{Cexp}}\left(\frac{\bar{K}}{q}t^q\right)$$

for all $t \in (0, a)$.
So if $\frac{\partial F}{\partial y} = k < 0$, we look for a SS. The function $F(t, y, k) = ky$ and $k < 0$ satisfies the assumptions of Theorem (7) with $h(t, y) = y$ and hance the problem

$$\begin{aligned} \bar{X}^{(q)}(t) &= k\bar{X}(t),\ q \in (0, 1] \\ \bar{X}(0) &= \bar{C} \end{aligned}$$

i.e

$$\begin{aligned} \left(x^{(q)}\right)_1^\alpha(t) &= k_1^\alpha x_2^\alpha(t) \\ \left(x^{(q)}\right)_2^\alpha(t) &= k_2^\alpha x_1^\alpha(t) \\ x_1^\alpha(0) &= c_1^\alpha \\ x_2^\alpha(0) &= c_2^\alpha \end{aligned}$$

has a unique fuzzy solution x on $\mathbb{R}_+$. It is given by the α-cuts $[x_1^\alpha(t), x_2^\alpha(t)]$, $\alpha \in [0, 1]$, $t \in (0, a)$, where

$$\begin{aligned} x_1^\alpha(t) &= \tfrac{1}{2}\left(c_1^\alpha + \sqrt{\tfrac{k_1^\alpha}{k_2^\alpha}}c_2^\alpha\right)\exp\left(\sqrt{k_1^\alpha k_2^\alpha}\tfrac{t^q}{q}\right) + \tfrac{1}{2}\left(c_1^\alpha - \sqrt{\tfrac{k_1^\alpha}{k_2^\alpha}}c_2^\alpha\right)\exp\left(-\sqrt{k_1^\alpha k_2^\alpha}\tfrac{t^q}{q}\right) \\ x_2^\alpha(t) &= \tfrac{1}{2}\left(c_1^\alpha\sqrt{\tfrac{k_2^\alpha}{k_1^\alpha}} + c_2^\alpha\right)\exp\left(\sqrt{k_1^\alpha k_2^\alpha}\tfrac{t^q}{q}\right) - \tfrac{1}{2}\left(c_1^\alpha\sqrt{\tfrac{k_2^\alpha}{k_1^\alpha}} - c_2^\alpha\right)\exp\left(-\sqrt{k_1^\alpha k_2^\alpha}\tfrac{t^q}{q}\right) \end{aligned}$$

Then SS exist for $t \in (0, a)$.

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