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Monophonic Eccentric Domination Numbers of Graphs

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Abstract. Let G be a (simple) undirected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A set $S \subseteq V(G)$ is a monophonic eccentric dominating set if every vertex in $V(G) \setminus S$ has a monophonic eccentric vertex in S. The minimum size of a monophonic eccentric dominating set in G is called the monophonic eccentric domination number of G. It shown that the absolute difference of the domination number and monophonic eccentric domination number of a graph can be made arbitrarily large. We characterize the monophonic eccentric dominating sets in graphs resulting from the join, corona, and lexicographic product of two graphs and determine bounds on their monophonic eccentric domination numbers.

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1. Introduction

In a recent study, Santhakumaran and Titus in [3] and [4] defined monophonic distance and obtained some results related to the concept. Using monophonic paths and monophonic distance-related concepts, Titus et al. in [6], and [7] defined and studied a variation of domination called monophonic eccentric domination and the corresponding monophonic eccentric domination number. Titus and Fancy [5] also introduced total monophonic eccentric dominating set. The authors mentioned that the monophonic eccentric domination number and total monophonic eccentric domination number find useful applications in channel assignment problems in radio technologies and in molecular problems in theoretical chemistry.

Recently, Gamorez and Canoy in [1] and [2] also made use of the monophonic distance and related concepts to construct a topology on a vertex set of a given undirected graph. Some characterizations were obtained and subbasic open sets on graphs resulting from some binary operations were determined.

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2. Terminology and Notations

For any two vertices u and v in an undirected connected graph G, the distance $d_G(u,v)$ is the length of a shortest path joining u and v. The *open neighborhood* of a point u is the set $N_G(u)$ consisting of all points v which are adjacent to u. The *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. A chord of a path P in a graph G is an edge joining two non-adjacent vertices of P. A path P in a graph G is called a *monophonic path* if it is chordless. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_G^m(u,v)$ from u to v is defined as the length of a longest u-v monophonic path in G. The *monophonic eccentricity* $e_G^m(v)$ of a vertex v in G is the maximum monophonic distance from v to a vertex of G. The *monophonic radius* $rad_m(G)$ of graph G is given by $rad_m(G) = \min\{e_G^m(v) : v \in V(G)\}$ and the *monophonic diameter* $diam_m(G)$ of G is given by $diam_m(G) = \max\{e_G^m(v) : v \in V(G)\}$. A vertex w in G is a *monophonic eccentric vertex* of a vertex v in G if $e_G^m(v) = d_G^m(w,v)$. In this case, we say that w is a monophonic eccentric neighbor of v. The set consisting of all the monophonic eccentric vertices of $v \in V(G)$ will be denoted by $N_G^m(v)$, i.e., $N_G^m(v) = \{w \in V(G) : e_G^m(v) = d_G^m(w,v)\}$. Here, $N_G^m[v] = N_G^m(v) \cup \{v\}$. A set $S \subseteq V(G)$ is a *monophonic eccentric dominating set* (*total monophonic eccentric dominating set*) of G if each $w \in V(G) \setminus S$ (resp. $w \in V(G)$) has a monophonic eccentric vertex in S. The smallest size of a monophonic eccentric dominating (total monophonic eccentric dominating) set of G, denoted by $\gamma_{me}(G)$ (resp. $\gamma_{tme}(G)$), is called the *monophonic eccentric domination number* (resp. *total monophonic eccentric domination number*) of G. Any monophonic eccentric dominating (total monophonic eccentric dominating) set of G of size $\gamma_{me}(G)$ (resp. $\gamma_{tme}(G)$) is called a minimum monophonic eccentric dominating set or a γ_{me}-set (resp. minimum total monophonic eccentric dominating set or γ_{tme}-set) of G.

Let G be a connected graph with $diam_m(G) \geq 3$. A set $S \subseteq V(G)$ is a d_m^3-*monophonic eccentric set* of G if for each $u \in V(G) \setminus S$ with $e_G^m(u) \geq 3$, there exists $w \in S$ such that $e_G^m(u) = d_G^m(w,u)$. The minimum cardinality of a d_m^3-monophonic eccentric set of G, denoted by $\mu_{me}^3(G)$, is called the d_m^3-*monophonic eccentric number of* G.

The *join* of two graphs G and H, denoted by $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of graphs G and H, denoted by $G \circ H$, is the graph obtained from$G$ by taking a copy H^v of H and forming the join $\langle v \rangle + H^v = v + H^v$ for each $v \in V(G)$. The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and $(v,a)(u,b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$. Note that any non-empty set $C \subseteq V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

3. Results

Theorem 1. *Let G be a connected graph of order $n \geq 1$. Then $\gamma_{me}(G) = 1$ if and only if $G = K_n$ or there exists $v \in V(G)$ satisfying the following properties:*

(*i*) $V(G) \setminus N_G[v] = \{w \in V(G) : d_G^m(v, w) = 2\} \neq \varnothing$;

(*ii*) $N_G(v) \subseteq N_G(w)$ *for each* $w \in V(G) \setminus N_G[v]$; *and*

(*iii*) $d_G^m(x, y) \leq 2$ *for all* $x, y \in V(G) \setminus N_G[v]$.

Proof. Suppose $\gamma_{me}(G) = 1$ and let $S = \{v\}$ be a monophonic eccentric dominating set of G. If $G = K_n$, then we are done. So suppose that $G \neq K_n$. Suppose $N_G[v] = V(G)$. Since $G \neq K_n$, there exist vertices $a, b \in V(G)$ such that $d_G(a, b) = 2 \leq d_G^m(a, b)$. It follows that v is not a monophonic eccentric vertex of a, contrary to our assumption of S. Thus, $V(G) \setminus N_G[v] \neq \varnothing$. Now let $w \in V(G) \setminus N_G[v]$. Since v is a monophonic eccentric vertex of w, we have $e_m(w) = d_G^m(v, w) \geq 2$. Suppose $d_G^m(v, w) \neq 2$, say $[v_1, v_2, ..., v_k]$, where $v_1 = v$, $v_k = w$ and $k \geq 4$, is a v-w monophonic path. Since $d_G(v_2, w) \geq 2$, this would imply that v is not a monophonic eccentric vertex of v_2, a contradiction. Thus, $d_G^m(v, w) = 2$. This shows that (*i*) holds. Next, let $z \in N_G(v)$ and let $w \in V(G) \setminus N_G[v]$. Since v is a monophonic eccentric vertex of z, it follows that $d_G(z, w) = 1$, that is, $z \in N_G(w)$. This shows that (*ii*) holds. Finally, let $x, y \in V(G) \setminus N_G[v]$. If $d_G^m(x, y) \geq 3$, then v is not a monophonic eccentric vertex of x, a contradiction. Thus, $d_G^m(x, y) \leq 2$, showing that (*iii*) holds.

For the converse, suppose first that $G = K_n$. Then, clearly, $\gamma_{me}(G) = 1$. Next, suppose that there exists $v \in V(G)$ satisfying conditions (*i*), (*ii*), and (*iii*). Let $S_0 = \{v\}$. By (*ii*), v is a monophonic eccentric vertex of every element of $N_G(v)$. Let $w \in V(G) \setminus N_G[v]$. Then $d_G^m(v, w) = 2$ by (*i*). Further, by (*iii*), it follows that v is a monophonic eccentric vertex of w. Therefore, $\gamma_{me}(G) = |S_0| = 1$. □

Theorem 2. *Let* $G_1, G_2, ..., G_k$ *be the distinct components of* G *with* $k \geq 2$ *and let* $H = K_1 + G = \langle v \rangle + G$.

(*i*) *If* $diam_m(G_i) \leq 2$ *for each* $i \in \{1, 2, ..., k\}$ *and one of the components is trivial, then* $\gamma_{me}(H) = 1$.

(*ii*) *If* $diam_m(G_i) \leq 2$ *for each* $i \in \{1, 2, ..., k\}$ *and none of the components is trivial, then* $\gamma_{me}(H) = 2$.

Proof. (*i*) Let G_j be a trivial component of G. Set $S = V(G_j) = \{w\}$. Clearly, $N_H(w) = \{v\}$. Since $diam_m(G_i) \leq 2$ for each $i \in \{1, 2, ..., k\}$, the conditions given in Theorem 1 are satisfied. Therefore, $\gamma_{me}(H) = 1$.

(*ii*) Since none of the components is trivial, none of the vertices of H satisfies the conditions in Theorem 1. It follows that $\gamma_{me}(H) \geq 2$. Pick $w_i \in V(G_i)$ for $i = 1, 2$ and let $S = \{w_1, w_2\}$. Clearly, w_1 is a monophonic eccentric vertex of v. Let $z \in V(G) \setminus S$. Suppose $z \notin V(G_1) \cup V(G_2)$. Since $diam_m(G_i) \leq 2$ for each $i \in \{1, 2, ..., k\}$, it follows that w_1 and w_2 are monophonic eccentric vertices of z in H. Suppose $z \in V(G_1)$. By assumption, $d_{G_i}^m(z, w_1) \leq 2$. Hence, $e_H^m(z) = d_H^m(z, w_2) = 2$, that is, w_2 is a monophonic eccentric vertex of z in H. Similarly, w_1 is a monophonic eccentric vertex of z in H if $z \in V(G_2)$. This shows that S is a monophonic eccentric dominating set of H. Therefore, $\gamma_{me}(H) = |S| = 2$. □

Theorem 3. *Let G be a connected non-complete graph and let $K_1 = \langle v\rangle$. Then S is a monophonic eccentric dominating set of $K_1 + G$ if and only if $S \cap V(G)$ is a monophonic eccentric dominating set of G.*

Proof. Suppose S is a monophonic eccentric dominating set of $K_1 + G$. Since G is non-complete, $S_G = S \cap V(G) \neq \varnothing$. Let $w \in V(G) \setminus S_G$. If $d_G(w, x) = 1$ for every $x \in S$, then every element of S_G is a monophonic eccentric vertex of w. Suppose $d_G(w, y) \neq 1$ for some $y \in S$. Then $e^m_{K_1+G}(w) = e^m_G(w) \geq d^m_G(w, y) \geq 2$. Since S is a monophonic eccentric dominating set of $K_1 + G$, there exists a monophonic eccentric vertex $z \in S$ of w. Since $d^m_{K_1+G}(w, v) = 1$, $z \neq v$. Thus, $z \in S_G$ and $e^m_G(w) = d^m_G(z, w)$. Hence, $S \cap V(G)$ is a monophonic eccentric dominating set of G.

For the converse, suppose that $S_G = S \cap V(G)$ is a monophonic eccentric dominating set of G. Let $u \in V(K_1 + G) \setminus S$. If $u = v$, then every element of S is a monophonic eccentric vertex of u in $K_1 + G$. Suppose $u \neq v$. Since S_G is a monophonic eccentric dominating set of G and $u \in V(G) \setminus S_G$, there exists $p \in S_G$ such that $e^m_G(u) = d^m_G(p, u)$. Hence, $e^m_{K_1+G}(u) = d^m_{K_1+G}(p, u)$. This proves that S is a monophonic eccentric dominating set of $K_1 + G$. □

The next result is a consequence of Theorem 3 and the fact that $\gamma_{me}(H) = 1$ for every complete graph H.

Corollary 1. *Let G be a connected graph. Then $\gamma_{me}(K_1 + G) = \gamma_{me}(G)$.*

Theorem 4. *Let $G_1, G_2, ..., G_k$ be the distinct components of G with $k \geq 2$ and let $H = K_1 + G = \langle v\rangle + G$. Suppose $R_G = \{j \in \{1, 2, ..., k\} : diam_m(G_j) \geq 3\} \neq \varnothing$. Then S is a monophonic eccentric dominating set of H if and only if $S_j = S \cap V(G_j)$ is a d^3_m-monophonic eccentric set of G_j for each $j \in R_G$ and, in addition, $S \cap V(G_t) \neq \varnothing$ for some $t \in \{1, 2, ..., k\} \setminus R_G$ whenever $|R_G| = 1$ and there exists $p \in V(G_r) \setminus S_r$ such that $e^m_{G_r}(p) = 1$ or $e^m_{G_r}(p) = 2$ and $d^m_{G_r}(p, w) = 1$ for all $w \in S_r$, where $R_G = \{r\}$.*

Proof. Suppose S is a monophonic eccentric dominating set of H and let $j \in R_G$. Let $u \in V(G_j) \setminus S_j$ with $e^m_{G_j}(u) \geq 3$. Then by assumption, there exists $w \in S$ such that $e^m_H(u) = d^m_H(w, u)$. Since $e^m_H(u) = e^m_{G_j}(u) \geq 3$, it follows that $w \in S_j$ and that $e^m_{G_j}(v) = d^m_{G_j}(w, u)$. This shows that S_j is a a d^3_m-monophonic eccentric set of G_j for each $j \in R_G$. Suppose now that $|R_G| = 1$, say $R_G = \{r\}$. Suppose there exists $p \in V(G_r) \setminus S_r$ such that $e^m_{G_r}(p) = 1$ or $e^m_{G_r}(p) = 2$ and $d^m_{G_r}(p, w) = 1$ for all $w \in S_r$, where $R_G = \{r\}$. Since $e^m_H(p) = 2$ and S is a monophonic eccentric dominating set of H, there exist $t \in \{1, 2, ..., k\} \setminus R_G$ and $x \in S \cap V(G_t)$ such that $e^m_H(p) = d^m_H(p, x) = 2$. This shows that $S \cap V(G_t) \neq \varnothing$ for some $t \in \{1, 2, ..., k\} \setminus R_G$.

For the converse, suppose that $S_j = S \cap V(G_j)$ is a d^3_m-monophonic eccentric set of G_j for each $j \in R_G$ and, in addition, $S \cap V(G_t) \neq \varnothing$ for some $t \in \{1, 2, ..., k\} \setminus R_G$ whenever $|R_G| = 1$ and there exists $p \in V(G_r) \setminus S_r$ such that $e^m_{G_r}(p) = 1$ or $e^m_{G_r}(p) = 2$ and $d^m_{G_r}(p, w) = 1$ for all $w \in S_r$, where $R_G = \{r\}$. Let $z \in V(H) \setminus S$. If $v \notin S$ and $z = v$, then every element of S is a monophonic eccentric vertex of z in H. Suppose that $z \in V(G)$.

If $|R_G| \geq 2$, then by assumption, there exists $q \in S$ ($q \in S_i$ or $q \in S_j$, where $i, j \in R_G$ and $i \neq j$) such that $e^m_H(z) = d^m_H(z, q)$. Suppose $|R_G| = 1$, say $R_G = \{r\}$. Assume first that $z \in V(G_r)$. If $e^m_{G_r}(z) \geq 3$, then $e^m_{G_r}(z) = e^m_H(z)$ and so z has a monophonic eccentric vertex in H (in G_r) by assumption. Suppose $e^m_{G_r}(z) = 2$. If $d^m_{G_r}(p, w) = 2$ for some $w \in S_r$, then w is a monophonic eccentric vertex of z in H. Suppose $d^m_{G_r}(p, w) = 1$ for all $w \in S_r$. By assumption, $S \cap V(G_t) \neq \varnothing$ for some $t \in \{1, 2, ..., k\} \setminus R_G$. Then every element of $S \cap V(G_t)$ is a monophonic eccentric vertex of z in H. If $e^m_{G_r}(z) = 1$, then every element of $S \cap V(G_t)$ is a monophonic eccentric vertex of z in H because $e^m_H(z) = 2$. Next, suppose that $z \in G_i$ for $i \neq r$. Then $e^m_H(z) = 2$. Hence, every element of S_r is a monophonic eccentric vertex of z in H. Therefore, S is a monophonic eccentric dominating set of H. □

Corollary 2. *Let $G_1, G_2, ..., G_k$ be the distinct components of G with $k \geq 2$ and let $H = K_1 + G = \langle v \rangle + G$. Suppose $R_G = \{j \in \{1, 2, ..., k\} : diam_m(G_j) \geq 3\} \neq \varnothing$.*

(i) If $|R_G| \geq 2$, then $\gamma_{me}(H) = \sum_{j \in R_G} \mu^3_m(G_j)$.

(ii) If $R_G = \{r\}$ and $\gamma_{me}(H) \neq \mu^3_m(G_r)$, then $\gamma_{me}(H) = \mu^3_m(G_r) + 1$.

Proof. (i) Suppose $|R_G| \geq 2$. Let S_j be a d^3_m-monophonic eccentric set of G_j such that $\mu^3_m(G_j) = |S_j|$ for each $j \in R_G$ and set $S = \cup_{j \in R_G} S_j$. Then S is a monophonic eccentric dominating set of H by Theorem 4. Hence, $\gamma_{me}(H) \leq |S| = \sum_{j \in R_G} \mu^3_m(G_j)$.

Next, suppose that S_0 is a minimum monophonic eccentric dominating set of H. Let $S'_j = S_0 \cap V(G_j)$ for each $j \in R_G$. By Theorem 4, S'_j is a d^3_m-monophonic eccentric set of G_j for each $j \in R_G$. Since $|S'_j| \geq \mu^3_m(G_j)$ for each $j \in R_G$, $\gamma_{me}(H) = |S_0| \geq \sum_{j \in R_G} \mu^3_m(G_j)$. This proves the assertion.

(ii) Suppose $R_G = \{r\}$ and $\gamma_{me}(H) \neq \mu^3_m(G_r)$. Let S be a minimum monophonic eccentric dominating set of H. Then $S \cap V(G_r)$ is a d^3_m-monophonic eccentric set of G_r by Theorem 4. If $S \cap V(G_r) = S$, then $\mu^3_m(G_r) < |S| = \gamma_{me}(H)$ by assumption. If $S \cap V(G_r) \neq S$, again, $\mu^3_m(G_r) \leq |S \cap V(G_r)| < |S| = \gamma_{me}(H)$ by assumption. Thus, $\mu^3_m(G_r) + 1 \leq \gamma_{me}(H)$. Next, let S_G be a minimum d^3_m-monophonic eccentric set of G_r. Let $t \in \{1, 2, ..., k\} \setminus \{r\}$ and choose any $q \in V(G_t)$. Let $S_0 = S_G \cup \{q\}$. Then S_0 is a monophonic eccentric dominating set of H by Theorem 4. This implies that $\gamma_{me}(H) \leq |S_0| = \mu^3_m(G_r) + 1$. Therefore, $\gamma_{me}(H) = \mu^3_m(G_r) + 1$. □

Theorem 5. *Let G and H be connected non-complete graphs. Then S is a monophonic eccentric dominating set of $G + H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are monophonic eccentric dominating sets of G and H, respectively.*

Proof. Suppose S is a monophonic eccentric dominating set of $G + H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Since G and H are non-complete graphs, $S_G \neq \varnothing$ and $S_H \neq \varnothing$. Let $v \in V(G) \setminus S_G$. If $d^m_G(v, x) = 1$ for all $x \in V(G) \setminus \{v\}$, then every element of S_G is a monophonic eccentic vertex of v in G. Suppose $d^m_G(v, y) \neq 1$ for some $y \in V(G) \setminus \{v\}$. Since S is a monophonic eccentric dominating set of $G + H$, there exists $q \in S$ such that $e^m_{G+H}(v) = d^m_{G+H}(v, q)$. Since $d^m_{G+H}(v, h) = 1$ for all $h \in S_H$, it follows that $q \in S_G$ and $e^m_G(v) = d^m_G(v, q)$. This implies that S_G is a monophonic eccentric dominating set of G. Similarly, S_H is a monophonic eccentric dominating set of H.

For the converse, suppose that $S = S_G \cup S_H$, where S_G and S_H are monophonic eccentric dominating sets of G and H, respectively. Let $x \in V(G + H) \setminus S$. Suppose $x \in V(G)$. Then $x \in V(G) \setminus S_G$. By assumption, there exists $w \in S_G$ such that $e^m_G(x) = d^m_G(x, w)$. It follows that $e^m_{G+H}(x) = d^m_{G+H}(x, w)$. Similarly, if $x \in V(H)$, then there exists $u \in S_H \subseteq S$ such that $e^m_{G+H}(x) = d^m_{G+H}(x, u)$. Therefore, S is a monophonic eccentric dominating set of $G + H$. □

Corollary 3. *Let G and H be connected non-complete graphs. Then*

$$\gamma_{me}(G + H) = \gamma_{me}(G) + \gamma_{me}(H).$$

The next result shows that the absolute difference of the domination number the monophonic eccentric domination number can be made arbitrarily large.

Theorem 6. *Let n be a positive integer. Then the following statements hold:*

(i) *There exists a connected graph G_1 such that $\gamma(G_1) - \gamma_{me}(G_1) = n$.*

(ii) *There exists a connected graph G_2 such that $\gamma_{me}(G_2) - \gamma(G_2) = n$.*

Proof. (i) Consider the corona $G_1 = P_{n+2} \circ K_1$ of $P_{n+2} = [x_1, x_2, ..., x_{n+2}]$ and K_1 in Figure 1. Clearly, $S_1 = \{x_1, x_2, ..., x_{n+1}, x_{n+2}\}$ is a minimum dominating set and $S_2 = \{a, b\}$ is a minimum monophonic eccentric dominating set of G_1. Thus, $\gamma(G_1) - \gamma_{me}(G_1) = n$.

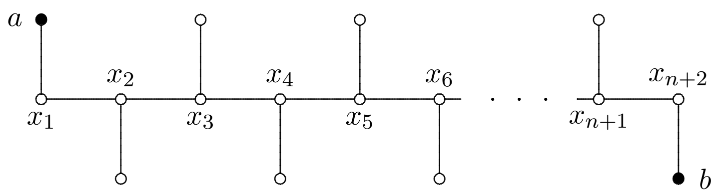

Figure 1: A graph G_1 with $\gamma(G_1) = n + 2$ and $\gamma_{me}(G_1) = 2$

(ii) Consider the graph $G_2 = K_1 + (\cup_{j=1}^{n+2} H_j)$, where $H_j = P_4$ for each $j \in \{1, \ldots, n+1\}$. Clearly, $\gamma(G_2) = 1$. Now $R_{G_2} = \{1, 2, ..., n + 1\}$ (see Theorem 4) and $\mu^3_m(H_j) = \mu^3_m(P_4) = 1$ for each $j \in R_{G_2}$. Hence, $\gamma_{me}(G_2) = \sum_{j \in R_G} \mu^3_m(H_j) = n + 1$ by Corollary 2. Thus, $\gamma_{me}(G_2) - \gamma(G_2) = n$. This proves the assertion. □

Theorem 7. *Let G and H be any connected non-trivial graphs. Then S is a monophonic eccentric dominating set of $G \circ H$ if and only if $S = A \cup (\cup_{v \in V(G)} S_v)$, where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for each $v \in V(G)$, and satisfies the following conditions:*

(i) *If $v \in V(G) \setminus A$, then $S_w \neq \varnothing$ for some $w \in V(G)$ with $e^m_G(v) = d^m_G(v, w)$.*

(*ii*) *If* $x \in V(H) \setminus S_v$ *and* $e^m_{H^v}(x) < e^m_G(v) + 2$*, then* $S_w \neq \varnothing$ *for some* $w \in V(G)$ *with* $e^m_G(v) = d^m_G(v, w)$.

(*iii*) *If* $x \in V(H) \setminus S_v$ *and* $e^m_{H^v}(x) = e^m_G(v) + 2$*, then there exists* $y \in S_v$ *such that* $e^m_{H^v}(x) = d^m_{H^v}(x, y)$ *or* $S_w \neq \varnothing$ *for some* $w \in V(G)$ *with* $e^m_G(v) = d^m_G(v, w)$.

(*iv*) *If* $x \in V(H) \setminus S_v$ *and* $e^m_{H^v}(x) > e^m_G(v) + 2$*, then there exists* $y \in S_v$ *such that* $e^m_{H^v}(x) = d^m_{H^v}(x, y)$.

Proof. Suppose S is a monophonic eccentric dominating set of $G \circ H$. Let $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Let $v \in V(G)$. If $v \in V(G) \setminus A$, then $e^m_{G \circ H}(v) = e^m_G(v) + 1$. Hence, by assumption, there exist $w \in V(G)$ with $e^m_G(v) = d^m_G(v, w)$ and $q \in S_w$ such that $e^m_{G \circ H}(v) = d^m_{G \circ H}(v, q)$. This shows that (*i*) holds. Again, since S is a monophonic eccentric dominating set of $G \circ H$, it is routine to show that (*ii*), (*iii*) and (*iv*) hold.

For the converse, suppose that S is the given set and satisfies the given conditions. Let $x \in V(G \circ H) \setminus S$ and let $v \in V(G)$ such that $x \in v + H^v$. If $x = v$, then $S_w \neq \varnothing$ for some $w \in V(G)$ with $e^m_G(v) = d^m_G(v, w)$ by (*i*). It follows that every element of S_w is a monophonic eccentric vertex of x in $G \circ H$. Suppose $x \in V(H^v) \setminus S_v$. If $e^m_{H^v}(x) > e^m_G(v) + 2$, then S_v contains a monophonic eccentric vertex of x in $G \circ H$ by (*iv*). If $e^m_{H^v}(x) \leq e^m_G(v) + 2$, then x has a monophonic eccentric vertex in $G \circ H$ by (*ii*) and (*iii*). Therefore, S is a monophonic eccentric dominating set of $G \circ H$. □

Theorem 8. *Let* G *and* H *be any connected non-trivial graphs such that* $rad_m(H) > diam_m(G) + 2$*. Then* S *is a monophonic eccentric dominating set of* $G \circ H$ *if and only if* $S_v = S \cap V(H^v)$ *is a monophonic eccentric dominating set of* H^v *for each* $v \in V(G)$*. Moreover,* $\gamma_{me}(G \circ H) = |V(G)|\gamma_{me}(H)$.

Proof. Let $v \in V(G)$ and let $S_v = S \cap V(H^v)$. Let $x \in V(H^v) \setminus S_v$. Since $rad_m(H) > diam_m(G) + 2$, $e^m_{H^v}(x) > e^m_G(v) + 2$. By Theorem 7(*iv*), there exists $y \in S_v$ such that $e^m_{H^v}(x) = d^m_{H^v}(x, y) = d^m_{G \circ H}(x, y)$. This shows that S_v is a monophonic eccentric dominating set of H^v.

For the converse, suppose that S_v is a monophonic eccentric dominating set of H^v for each $v \in V(G)$. Let $z \in V(G \circ H) \setminus S$ and let $w \in V(G)$ such that $z \in w + V(H^w)$. Since $rad_m(H) > diam_m(G) + 2$, the conditions in Theorem 7 are satisfied by S. Therefore, S is a monophonic eccentric dominating set of $G \circ H$.

Next, let D_v be a minimum monophonic eccentric dominating set of H^v for each $v \in V(G)$. Then $S_0 = \cup_{v \in V(G)} D_v$ is a minimum monophonic eccentric dominating set of $G \circ H$. Thus, $\gamma_{me}(G \circ H) = |S_0| = |V(G)|\gamma_{me}(H)$. □

For vertex $v \in V(G)$, we denote by $N^m_G(v)$ the set of all monophonic eccentric vertices of v, i.e., $N^m_G(v) = \{w \in V(G) : e^m_G(v) = d^m_G(v, w)\}$.

Let G be a connected graph. Denote by $V_m(G)$ a smallest set of vertices of G satisfying the properties:

(A) For each $v \in V_m(G)$ there exists $w \in V(G)$ such that $v \in N_G^m(w)$, and

(B) $|V_m(G) \cap N_G^m(u)| = 1$ for each $u \in V(G)$.

As an example, consider the graph G obtained from $C_4 = [a, b, c, d, a]$ by adding the pendant edge ae. The set $\{c, e\}$ is the smallest subset of G satisfying properties (A) and (B). Thus, $V_m(G) = \{c, e\}$.

Theorem 9. *Let G and H be any connected non-trivial graphs such that $diam_m(H) < rad_m(G) + 2$. Then S is a monophonic eccentric dominating set of $G \circ H$ if and only if $S_v \neq \varnothing$ for each $v \in V_m(G)$ having $S_w \neq V(H^w)$ for some $w \in V(G)$ with $v \in N_G^m(w)$, where $S_u = S \cap V(H^u)$ for each $u \in V(G)$. Moreover, $\gamma_{me}(G \circ H) = |V_m(G)|$.*

Proof. Suppose S is a monophonic eccentric dominating set of $G \circ H$. Let $v \in V_m(G)$ and $S_v = S \cap V(H^v)$. Then $Q_v = \{y \in V(G) : v \in N_G^m(y)\} \neq \varnothing$ by property (A) of $V_m(G)$. Suppose that $S_w \neq V(H^w)$ for some $w \in Q_v$, say $z \in V(H^w) \setminus S_w$. By property (B) of $V_m(G)$, it follows that $|V_m(G) \cap N_G^m(w)| = \{v\}$. From the assumption that $diam_m(H) < rad_m(G)+2$, it follows that $e_{H^w}^m(z) < e_G^m(w)+2$. Hence, $e_{G\circ H}^m(z) = e_G^m(w) + 2 = d_G^m(w, v) + 2$. Since S is a monophonic eccentric dominating set of $G \circ H$, Theorem 7(ii) guarantees the existence of $q \in S_v$ such that $e_{G\circ H}^m(z) = d_{G\circ H}^m(z, q)$, showing that $S_v \neq \varnothing$.

For the converse, suppose that the given condition holds. Let $z \in V(G \circ H) \setminus S$ and let $w \in V(G)$ such that $z \in V(w + H^w)$. Let $v \in N_G^m(w) \cap V_m(G)$. Since $diam_m(H) < rad_m(G) + 2$ and $S_v \neq \varnothing$ by assumption, every element of S_v is a monophonic eccentric vertex of z. Since z was arbitrarily chosen, it follows that S is a monophonic eccentric dominating set of $G \circ H$.

Next, choose any point $x_v \in V(H^v)$ for each $v \in V_m(G)$ and let $S_0 = \{x_v : v \in V_m(G)\}$. Then S_0 is a minimum monophonic eccentric dominating set of $G \circ H$. Thus, $\gamma_{me}(G \circ H) = |S_0| = |V_m(G)|$. □

Theorem 10. *Let G and H be non-trivial connected graphs such that $rad_m(G) > diam_m(H)$. Then $C = \bigcup_{x\in S}[\{x\}\times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a monophonic eccentric dominating set of $G[H]$ if and only if the following hold:*

(i) S is a monophonic eccentric dominating set of G.

(ii) For each $x \in S$ such that $T_x \neq V(H)$, $S \cap N_G^m(x) \neq \varnothing$.

Proof. Suppose C is a monophonic eccentric dominating set of $G[H]$ and let $v \in V(G) \setminus S$. Pick any $a \in V(H)$. Then $(v, a) \notin C$ and so by assumption of C, there exists $(w, b) \in C$ such that $e_{G[H]}^m((v, a)) = d_{G[H]}^m((v, a), (w, b))$. It follows that $v \neq w$ and $e_G^m(v) = d_G^m(v, w)$, i.e., $w \in S \cap N_G^m(v)$. This shows that S is a monophonic eccentric dominating set of G.

Next, let $x \in S$ with $T_x \neq V(H)$. Let $p \in V(H) \setminus T_x$. Then $(x, p) \in V(G[H]) \setminus C$. Hence, there exists $(z, q) \in N_{G[H]}^m((x, p)) \cap C$. Since $rad_m(G) > diam_m(H)$, it follows that

$d^m_G(x,z) > d^m_H(p,q)$. Hence, $d^m_{G[H]}((z,q),(x,p)) = d^m_G(x,z)$ and $z \in S \cap N^m_G(x)$, showing that (ii) holds.

For the converse, suppose that (i) and (ii) hold. Let $(v,a) \in V(G[H]) \setminus C$. If $v \notin S$, then there exists $w \in N^m_G(v) \cap S$ by (i). Let $d \in T_w$. Then $(w,d) \in C$. Now, because $rad_m(G) > diam_m(H)$, it follows that

$$e^m_{G[H]}((v,a)) = d^m_{G[H]}((v,a),(w,d)) = d^m_G(w,v) = e^m_G(v).$$

Suppose $v \in S$. Then $a \notin T_v$, i.e., $T_v \neq V(H)$. By (ii), it follows that there exists $z \in S \cap N^m_G(v)$. Pick any $b \in T_z$. Then $(z,b) \in C$ and

$$e^m_{G[H]}((v,a)) = d^m_{G[H]}((v,a),(z,b)) = d^m_G(z,v) = e^m_G(v).$$

Therefore, C is a monophonic eccentric dominating set of $G[H]$. □

Corollary 4. *Let G and H be non-trivial connected graphs such that $rad_m(G) > diam_m(H)$. Then $\gamma_{me}(G[H]) = \gamma_{tme}(G)$.*

Proof. Let S be a γ_{tme}-set of G and let $p \in V(H)$. For each $x \in S$, let $T_x = \{p\}$. Then $C = \bigcup_{x\in S}[\{x\} \times T_x] = S \times \{p\}$ is a monophonic eccentric dominating set of $G[H]$ by Theorem 10. Thus,

$$\gamma_{me}(G[H]) \leq |C| = |S| = \gamma_{tme}(G).$$

Next, let $C_0 = \bigcup_{x\in S_0}[\{x\} \times R_x]$ be a γ_{me}-set of $G[H]$. Then S_0 is a monophonic eccentric dominating set of G by Theorem 10(i). If S_0 is a total monophonic eccentric dominating set, then

$$\gamma_{me}(G[H]) = |C_0| \geq |S_0| \geq \gamma_{tme}(G).$$

Suppose S_0 is not a total monophonic eccentric dominating set. Then there exists $y \in S_0$ such that $N^m_G(y) \cap S_0 = \varnothing$. By Theorem 10$(ii)$, $R_y = V(H)$. Let $S_1 = \{v \in S_0 : N^m_G(v) \cap S_0 = \varnothing\}$. Again, $R_v = V(H)$ for each $v \in S_1$ by Theorem 10(ii). For each $v \in S_1$, choose a vertex $z_v \in N^m_G(v)$ and set $S_2 = \{z_v : v \in S_1\}$. Then $S_2 \cap S_0 = \varnothing$ and $|S_1| \geq |S_2|$. Clearly, $S^* = S_0 \cup S_2$ is a total monophonic eccentric dominating set of G and we have

$$\begin{aligned}
\gamma_{me}(G[H]) &= |C_0| = \sum_{x\in S_0} |R_x| \\
&= \sum_{x\in S_0\setminus S_1} |R_x| + \sum_{x\in S_1} |R_x| \\
&= \sum_{x\in S_0\setminus S_1} |R_x| + |V(H)||S_1| \\
&\geq \sum_{x\in S_0\setminus S_1} |R_x| + 2|S_1| \\
&\geq \sum_{x\in S_0\setminus S_1} |R_x| + (|S_1| + |S_2|) \\
&\geq |S_0 \setminus S_1| + S_1 + S_2 = |S^*| \geq \gamma_{tme}(G).
\end{aligned}$$

Accordingly, $\gamma_{me}(G[H]) = \gamma_{tme}(G)$. □

Theorem 11. *Let G and H be non-trivial connected graphs such that $rad_m(H) > diam_m(G)$. Then $C = \bigcup_{x\in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a monophonic eccentric dominating set of $G[H]$ if and only if*

- *(i) $S = V(G)$ and*
- *(ii) T_x is a monophonic eccentric dominating set of H for each $x \in V(G)$.*

Proof. Suppose C is a monophonic eccentric dominating set of $G[H]$. Suppose $S \neq V(G)$, say $x \in V(G)\setminus S$. Pick any $a \in V(H)$. Then $(v,a) \notin C$. As C is a monophonic eccentric dominating set of $G[H]$, there exists $(w,b) \in C$ such that $e^m_{G[H]}((v,a)) = d^m_{G[H]}((v,a),(w,b))$. However, the assumption that $rad_m(H) > diam_m(G)$ implies that $e^m_{G[H]}((v,a)) = e^m_H(a) = d^m_H(a,b) > e^m_G(v)$. This is impossible because $w \neq x$. Thus, $S = V(G)$, showing that (i) holds.

Let $x \in V(G)$. If $T_x = V(G)$, then it is a monophonic eccentric dominating set of H. Suppose $T_x \neq V(H)$ and let $q \in V(H) \setminus T_x$. Since $(x,q) \in V(G[H]) \setminus C$ and $e^m_{G[H]}((x,q)) = e^m_H(q)$, it follows that there exists $p \in T_x \cap N^m_H(q)$. Hence, T_x is a monophonic eccentric dominating set of H. This shows that (ii) holds.

For the converse, suppose that (i) and (ii) hold. Let $(z,a) \in V(G[H]) \setminus C$. Since $S = V(G)$, it follows that $a \notin T_x$. As T_x is a monophonic eccentric dominating set of H according to (ii), there exists $b \in T_x$ such that $e^m_H(a) = d^m_H(a,b)$. With the assumption that $rad_m(H) > diam_m(G)$, it follows that

$$e^m_{G[H]}((z,a)) = e^m_H(a) = d^m_H(a,b) = d^m_{G[H]}((z,a),(z,b)),$$

where $(z,b) \in C$. Therefore, C is monophonic eccentric dominating set of $G[H]$. □

The next result is an immediate consequence of Theorem 11.

Corollary 5. *Let G and H be non-trivial connected graphs such that* $rad_m(H) > diam_m(G)$. *Then* $\gamma_{me}(G[H]) = |V(G)|\gamma_{me}(H)$.

Conclusion: Monophonic paths and monophonic distance-related concepts had been used to define monophonic eccentric dominating set and monophonic eccentric domination number of a graph. It was shown that the absolute difference of the domination number and the monophonic eccentric domination number can be made arbitrarily large. Monophonic eccentric dominating sets in the join, corona, and lexicographic product of two graphs were characterized and, under some conditions, their monophonic eccentric domination numbers were subsequently determined. Several aspects of the concept (e.g. its complexity) and the corresponding parameter remains to be investigated. Moreover, other variants of the concept may as well be introduced and studied.

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