EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 15, No. 2, 2022, 726-735
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Existence and Uniqueness of Solutions for Nonlinear Fractional Integro-Differential Equations with Nonlocal Boundary Conditions

M.J. Mardanov[1,2] Y.A. Sharifov[1,2,*], H.N. Aliyev[3]

[1] *Institute of Mathematics and Mechanics, ANAS, Baku, Azerbaijan*
[2] *Baku State University Baku, Azerbaijan*
[3] *Baku Engineering University, Khirdalan City, Azerbaijan*

Abstract. In this paper, the existence and uniqueness of solutions of fractional integro-differential equations with nonlocal boundary conditions is investigated. We establish the existence of solution via Krasnoselskii fixed point theorem; however, the uniqueness results are obtained by applying the contraction mapping principle.

2020 Mathematics Subject Classifications: AMS 34B37, 34B15

Key Words and Phrases: Nonlocal boundary conditions, Caputo fractional derivative, existence, uniqueness, fixed point

---

1. Introduction

Fractional differential equations have been an important tool to describe many problems and processes in different fields of science. In fact, fractional models are more realistic than the classical models. Fractional differential equations appear in many fields such as physics, economics, image processing, blood flow phenomena, aerodynamics, and so on. For more details about fractional differential equations and their applications, we provide the following references [10–12, 14–17, 19–21, 25, 27, 33]. Recently, fractional integro-differential equations were investigated by many researchers in different problems, and a lot of papers were published in this matter (see, for example, [1, 6, 7]). Furthermore, many boundary conditions were considered for the fractional-order integro-differential equations; some of these conditions are the classical, periodic, antiperiodic, nonlocal, multipoint, and the integral boundary conditions. Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two,

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v15i2.4366

Email addresses: sharifov22@rambler.ru (Y.A. Sharifov), misirmardanov@yahoo.com (M.J. Mardanov), hualiyev@beu.edu.az (H.N. Aliyev)

three, multipoint and nonlocal boundary value problems as special cases. Integral boundary value problems occur in the mathematical modeling of variety of physics processes and have recently received considerable attention. For some recent work on boundary value problems with integral boundary conditions we refer to [2, 3, 8, 9, 18, 30] and the references cited therein.

On the other hand, many papers have considered the no separated boundary conditions as they are a very important class of boundary value conditions (we refer the readers to [4, 5, 13, 22–24, 26, 28, 29, 31, 32]). Motivated by the above discussion, in this paper, we establish the existence and uniqueness of solutions for a class of fractional integro-differential equations with no separated boundary value conditions as follows:

In this paper, we study existence and uniqueness of nonlinear fractional integro- differential equations of the type

$$^{c}D_{0+}^{\alpha}x\left(t\right)=f\left(t,x\left(t\right),\varphi x\left(t\right),\psi x\left(t\right)\right),\ for\ t\in[0,T] \tag{1}$$

subject initial-point and integral boundary conditions

$$Ax\left(0\right)+\int_{0}^{T}n\left(t\right)x\left(t\right)dt=C \tag{2}$$

where $0<\alpha<1$, $^{c}D_{0+}^{\alpha}$ is the Caputo fractional derivatives, $A\in R^{n\times n}$ and $n\left(t\right):[0,T]\rightarrow R^{n\times n}$ are given matrices and $\det N\neq0$, $N=\left(A+\int_{0}^{T}n\left(t\right)dt\right)$, $\varphi x\left(t\right)=\int_{0}^{t}\mu\left(t,s\right)x\left(s\right)ds$, $\psi x\left(t\right)=\int_{0}^{T}\gamma\left(t,s\right)x\left(s\right)ds$, where $\mu,\gamma:[0,T]\times[0,T]\rightarrow R^{n\times n}$, with $\mu_{0}=\max\limits_{t,s\in[0,T]}\left\|\mu\left(t,s\right)\right\|$, $\gamma_{0}=\max\limits_{t,s\in[0,T]}\left\|\gamma\left(t,s\right)\right\|$.

The rest of the paper is organized as follows. In Section 2, we give some notations, recall some concepts, and introduce a concept of a continuous solution for our problem. In Section 3, we give two main results: the first result based on the Krasnoselskii's fixed point theorem and the second result based on the Banach contraction principle.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. By $C\left([0,T];R^{n}\right)$ we denote the Banach space of all continuous functions from $[0,T]$ into R^{n} with the norm $\|x\|=\max\left\{|x\left(t\right)|:t\in[0,T]\right\}$, where $|\cdot|$ norm in R^{n}.

Definition 1. *(see [28]). The Riemann–Liouville fractional integral of order $\alpha>0$of a continuous function $y:[0,\infty)\rightarrow R$, is defined by*

$$J^{\alpha}y\left(x\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-s\right)^{\alpha-1}y\left(s\right)ds,$$

provided the right-hand side exists on $(0,\infty)$, where $\Gamma\left(\cdot\right)$ is the Gamma function defined for any complex number z as

$$\Gamma\left(z\right)=\int_{0}^{\infty}t^{z-1}e^{-t}dt.$$

Definition 2. *(see [28]). The (right-sided) Riemann-Liouville fractional derivative is defined by*

$$^{RL}D^{\alpha}y = (D^{\alpha}y)(x) = \frac{d}{dx^n}\left(J^{n-\alpha}y\right)(x), \ x > 0,$$

where $n = [\alpha] + 1$, $[\alpha]$*denotes the integer part of real number* α, *provided the right-hand side is point-wise defined on* $(0, \infty)$.

The Riemann-Liouville fractional derivative is left-inverse (but not right-inverse) of the Riemann-Liouville fractional integral, which is a natural generalization of the Cauchy formula for the n-fold primitive of a function y. As to the initial value problems for fractional differential equations with fractional derivatives in the Riemann-Liouville sense, they should be given as (bounded) initial values of the fractional integral $J^{n-\alpha}$ and of its integer derivatives of order $k = 1, 2, ..., n-1$.

Definition 3. *(see [28]). The Caputo fractional derivative of order* $\alpha > 0$ *of a continuous function* y, *is defined by*

$$^{C}D^{\alpha}_{0+}y = (D^{\alpha}y)(x) = \left(J^{n-\alpha}y^{(n)}\right)(x), \ n-1 < \alpha \leq n, \ x > 0,$$

provided the right-hand side is point-wise defined on (a, ∞*).*

Obviously, this definition allows one to consider the initial-value problems for the fractional differential equations with initial conditions that are expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order.

Remark 1. *Under natural conditions on* $y(x)$, *the Caputo fractional derivative becomes the conventional integer order derivative of the function* $y(x)$ *as* $\alpha \to n$.

Remark 2. *(see [28]). Let* $\alpha, \beta > 0$ *and* $n = [\alpha] + 1$; *then the following relations hold:*

$$\begin{array}{l} ^{c}D^{\alpha}_{0+}t^{\beta} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-\alpha}, \ \beta > n, \\ ^{c}D^{\alpha}_{0+}t^{k} = 0, \ k = 0, 1, ..., n-1. \end{array}$$

Lemma 1. *(see [28]). For* $\alpha > 0$, $y(t) \in C([0,T]) \bigcap L_1([0,T])$, *the homogeneous fractional differential equation*

$$^{c}D^{\alpha}_{0+}y(t) = 0,$$

has a solution

$$y(t) = c_0 + c_1 t + c_2 t^2 ... + c_{n-1}t^{n-1},$$

where $c_i \in R$, $i = 1, 2, ..., n-1$ *and* $n = [\alpha] + 1$.

Lemma 2. *(see [28]). Assume that* $y(t) \in C([0,T]) \bigcap L_1([0,T])$, *with derivative of order* n *that belongs to* $C([0,T]) \bigcap L_1([0,T])$; *then*

$$I^{\alpha}_{0+}\,{}^{c}D^{\alpha}_{0+}y(t) = y(t) + c_0 + c_1 t + c_2 t^2 ... + c_{n-1}t^{n-1},$$

where $c_i \in R$, $i = 1, 2, ..., n-1$ *and* $n = [\alpha] + 1$.

Lemma 3. *(see [28]). Let $p, q \geq 0,\ f \in L_1([0,T])$. Then*

$$I_{0+}^{p} I_{0+}^{q} f(t) = I_{0+}^{p+q} f(t) = I_{0+}^{q} I_{0+}^{p} f(t)$$

is satisfied almost everywhere on $[0,T]$. Moreover, if $f \in C([0,T])$, then (4) is true for all $t \in [0,T]$.

Lemma 4. *(see [28]). If $\alpha > 0,\ f \in C([0,T])$, then ${}^{c}D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t)$ for all $t \in [0,T]$.*

We have the following result which is useful in what follows.

Theorem 1. *Let $y \in C([0,T]; R^n)$. Then the unique solution of the linear boundary value problem*

$$\begin{cases} {}^{c}D_{0+}^{\alpha} x(t) = y(t), \\ \mathrm{Ax}(0) + \int_0^T n(t)\, x(t)\, dt = C \end{cases} \tag{3}$$

is given by

$$x(t) = N^{-1}C + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds -$$

$$- \frac{N^{-1}}{\Gamma(\alpha)} \int_0^T n(t) \int_0^t (t-s)^{\alpha-1} y(s)\, ds dt. \tag{4}$$

Proof. Assume that x is a solution of the boundary value problem (3); then we have

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds,\ \ t \in [0,T],$$

where $x(0)$ is still an arbitrary constant vector.

For determining $x(0)$ we use the boundary value condition $Ax(0) + \int\limits_0^T n(t)\, x(t)\, dt = C$:

$$C = Ax(0) + \int_0^T n(t)\, x(t)\, dt = \left(A + \int_0^T n(t)\, dt\right) x(0) +$$
$$+ \tfrac{1}{\Gamma(\alpha)} \int_0^T n(t) \int_0^t (t-s)^{\alpha-1} y(s)\, ds dt.$$

From here we get

$$x(0) = N^{-1}C - \frac{N^{-1}}{\Gamma(\alpha)} \int_0^T n(t) \int_0^t (t-s)^{\alpha-1} y(s)\, ds dt$$

and consequently for all $t \in [0,T]$ (4) is true.

Lemma 6 (Krasnoselskii's fixed point theorem, [20])

Let M be a closed, bounded, convex and nonempty subset of a Banach space X.Let A, B be the operators such that

(a) $Ax + By \in Mx$ whenever $x, y \in M$;

(b) $A$is compact and continuous;

(c)B is a contraction mapping.

Then there exists $z \in M$ such that $z = Az + Bz$.

3. Main results

In this section, the theorems of uniqueness and existence of a solution for problem (1), (2) will be given. For the forthcoming analysis we impose suitable conditions on the functions involved in the boundary value problem (1), (2). We assume the following conditions are met:

(H1) The function $f : [0,T] \times R^n \to R^n$ is continuous and satisfies the following Lipschitz condition

$$|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_{.3})| \leq L(|x_1-y_1| + |x_2-y_2| + |x_3-y_3|),$$

$$x_i, y_i \in R^n,\ i=1,2,3, \quad t \in [0,T],\ L>0.$$

(H2) $\|f(t,x,\varphi x,\psi x)\| \leq G$, for all $x \in R^n$, $t \in [0,T]$, $G \geq 0$.

Theorem 2. *Assume that $f : [0,T] \times R^n \to R^n$ is jointly continuous and satisfies (H1) and (H2). If*

$$\frac{L\left\|N^{-1}\right\|\|n\|T^{\alpha+1}}{\Gamma(\alpha+2)}\left(1+T(\varphi_0+\psi_0)\right) < 1, \tag{5}$$

then the fractional integro-differential problem (1), (2) *has at least one solution.*

Proof. Consider $B_r = \{x \in C([0,T];R^n) : \|x\| \leq r\}$, where

$$r \geq \frac{GT^\alpha}{\Gamma(\alpha+1)} + \left\|N^{-1}C\right\| + \frac{G\|n\|\left\|N^{-1}\right\|T^{\alpha+1}}{\Gamma(\alpha+2)}.$$

Define two mappings A_1 and A_2 on B_r by

$$(A_1x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),\varphi x(s),\psi x(s))\,ds, \tag{6}$$

$$(A_2x)(t) = N^{-1}\left(C - \frac{1}{\Gamma(\alpha)}\int_0^T n(t)\int_0^t (t-s)^{\alpha-1} f(s,x(s),\varphi x(s),\psi x(s))\,dsdt\right). \tag{7}$$

For $x, y \in B_r$, by (H2), we obtain

$$\|(A_1x)(t) + (A_2y)(t)\| \leq \frac{G}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds +$$

$$+\left\|N^{-1}C\right\| + \frac{G\|n\|\left\|N^{-1}\right\|}{\Gamma(\alpha)}\int_0^T\int_0^t (t-s)^{\alpha-1}\,dsdt \leq$$

$$\leq \frac{GT^\alpha}{\Gamma(\alpha+1)} + \left\|N^{-1}C\right\| + \frac{G\|n\|\left\|N^{-1}\right\|T^{\alpha+1}}{\Gamma(\alpha+2)} \leq r.$$

This shows that $A_1x+A_2y \in B_r$. Therefore, condition (a) of Lemma 6 holds. It is claimed that A_1 is compact and continuous. Continuity of f implies that $(A_1x)(t)$ is continuous. $(A_1x)(t)$ is uniformly bounded on B_r as

$$\|A_1x\| \le \frac{GT^{\alpha}}{\Gamma(\alpha+1)}.$$

Since f is bounded on the compact set $[0,T]\times B_r$, let $\sup\limits_{[0,T]\times B_r} \|f(t,x,y,z)\| = M_f$. Then, for $t_1, t_2 \in [0,T]$, $t_1 < t_2$ we get

$$\|(A_1x)(t_2)-(A_1x)(t_1)\| =$$

$$= \tfrac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) f(s,x(s),\varphi x(s),\psi x(s))\,ds + \right.$$
$$\left. \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s,x(s),\varphi x(s),\psi x(s))\,ds \right\| \le$$

$$\le \frac{M_f}{\Gamma(\alpha)} \left(\frac{t_2^{\alpha}}{\alpha} - \frac{t_1^{\alpha}}{\alpha} \right),$$

which is independent of x and tends to zero as $t_2 \to t_1$. Therefore, A_1 is relatively compact on B_r. By Arzela Ascoli's Theorem, A_1 is compact on B_r.

For x, $y \in B_r$ and $t \in [0,T]$, by (H1), we have

$$\|(A_2x)(t)-(A_2y)(t)\| \le$$

$$\le \frac{1}{\Gamma(\alpha)} \left\| N^{-1} \int_0^T n(t) \int_0^t (t-s)^{\alpha-1} \left(f(s,x(s),\varphi x(s),\psi x(s)) - \right.\right.$$
$$\left.\left. - f(s,y(s),\varphi y(s),\psi y(s)) \right) ds dt \right\| \le$$
$$\le L(1+T(\varphi_0+\psi_0)) \frac{\|N^{-1}\| \|n\| T^{\alpha+1}}{\Gamma(\alpha+1)} \|x-y\|.$$

It follows from (5) that A_2 is a contraction mapping. Thus, by Krasnoselskii's fixed point theorem, boundary value problem (1), (2) has at least one solution.

Theorem 3. *Assume that $f : [0,T]\times R^n \to R^n$ is a continuous function satisfying the assumption (A1). Then the problems* (3) *and* (4) *has a unique solution on* $[0,T]$ *if*

$$L(1+T(\varphi_0+\psi_0))\Lambda < 1,$$

where $\Lambda = \frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\|N^{-1}\|\|n\|T^{\alpha+1}}{\Gamma(\alpha+2)}$.

Proof.

Proof. Define a mapping $F : C([0,T];R^n) \to ([0,T];R^n)$ by

$$(Fx)(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s),\varphi x(s),\psi x(s))\,ds +$$
$$+N^{-1}\left(C - \tfrac{1}{\Gamma(\alpha)} \int_0^T n(t) \int_0^t (t-s)^{\alpha-1} f(s,x(s),\varphi x(s),\psi x(s))\,dsdt \right). \quad (8)$$

Let us first show that $FB_r \subset B_r$, where$F$is the operator defined by (8) and $r \geq \frac{M_f \Lambda + \left\|N^{-1}C\right\|}{1-L(1+T(\varphi_0+\psi_0))\Lambda}$ with $M_f = \sup\limits_{t\in[0,T]} |f(t,0,0,0)|$. Then, in view of the assumptions (H1) and (H2), we have

$$|f(t,x,\varphi x,\psi x)| \leq |f(t,x,\varphi x,\psi x) - f(t,0,0,0)| + |f(t,0,0,0)| \leq$$
$$\leq L(|x| + |\varphi x| + |\psi x|) + M_f \leq L(1 + T(\varphi_0 + \psi_0)) r + M_f.$$

For any $x \in B_r$, we have

$$\|Fx\| = \sup_{t\in[0,T]} |Fx(t)| \leq$$
$$\leq \left\|N^{-1}C\right\| + (L(1+T(\varphi_0+\psi_0)) r + M_f) \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\left\|N^{-1}\right\| \|n\| T^{\alpha+1}}{\Gamma(\alpha+2)} \right\} \leq r,$$

which implies that $FB_r \subset B_r$. Next, for $x, y \in C([0,T]; R^n)$ and for each $t \in [0,T]$, we obtain

$$\|Fx - Fy\| \leq \sup_{[0,T]} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), \varphi x(s), \psi x(s)) -$$
$$-f(s, y(s), \varphi y(s), \psi y(s))| \, ds +$$
$$+ \frac{\left\|N^{-1}\right\| \|n\|}{\Gamma(\alpha)} \int_0^T \int_0^t (t-s)^{\alpha-1} |f(s, x(s), \varphi x(s), \psi x(s)) - f(s, y(s), \varphi y(s), \psi y(s))| \, ds \leq$$
$$\leq L(1+T(\varphi_0+\psi_0)) \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\left\|N^{-1}\right\| \|n\| T^{\alpha+1}}{\Gamma(\alpha+2)} \right\} \|x - y\| =$$
$$= L(1+T(\varphi_0+\psi_0)) \Lambda \|x-y\|.$$

Since $L(1+T(\varphi_0+\psi_0))\Lambda < 1$, the operator F is a contraction. By Banach contraction mapping principle the operator F has a unique fixed point, which means that the problems (1) and (2) has a unique solution for on $[0,T]$.

4. Conclusion

In the paper a system of fractional integro-differential equations with nonlocal conditions is studied. At first the boundary value problem is reduced to the equivalent integral equation. Then, using the theorem on fixed points, the condition on the existence and uniqueness of the solution of the boundary value problem is obtained. The technique used in this research can be applied to the similar problems for fractional differential equations subject to multi-point nonlocal conditions

$$Ex(0) + \sum_{j=1}^{J} B_j x(\lambda_j) = C,$$

where E is a unit matrix, $B_j \in R^{n\times n}$ are given matrices and

$$\sum_{j=1}^{J} \|B_j\| < 1, \; 0 < \lambda_1 < \lambda_2 < ... < \lambda_J < T.$$

References

[1] B. Ahmad, S. K. Ntouyas, R. Agarwal, and A. Alsaedi. Existence results for sequential fractional integro-differential equations with nonlocal multi-point and strip conditions. *Boundary Value Problems*, 2016:205, 2016.

[2] B. Ahmed and S. Sivasundaram. On a four-point nonlocal boundary value problem of nonlinear integro-differential equations of fractional order. *Applied Mathematics and Computation*, 217:480–487, 2010.

[3] B. Ahmed and S. Sivasundaram. Existence of solutions for nonlinear fractional integro-differential equations with three-point nonlocal fractional boundary conditions. *Advances in Difference Equations*, 2010:Article ID 691721, 2020.

[4] H. Alsulami, S. K. Ntouyas, R. P. Agarwal, B. Ahmad, and A. Alsaedi. A study of fractional-order coupled systems with a new concept of coupled non-separated boundary conditions. *Boundary Value Problems*, 68, 2017.

[5] A. Ashyralyev and Y. A. Sharifov. Existence and uniqueness of solutions for the system of nonlinear fractional differential equations with nonlocal and integral boundary conditions. *Advances in Difference Equations*, 2012:Id 594802, 2012.

[6] D. Baleanu, K. Ghafarnezhad, and S. Rezapour. On a three step crisis integro-differential equation. *Advances in Difference Equations*, 1:153, 2019.

[7] D. Baleanu, G. Khadijeh, R. Shahram, and S. Mehdi. On the existence of solutions of a three steps crisis integro-differential, equation. *Advances in Difference Equations*, 1:135, 2018.

[8] D. Baleanu, S. Z. Nazeni, and S. Rezapour. Existence and uniqueness of solutions for multi-term nonlinear fractional integro-differential equations. *Advances in Difference Equations*, 2013:368, 2013.

[9] A. Bragdi, A. Frioui, and A. G. Lakoud. Existence of solutions for nonlinear fractional integro-differential equations. *Advances in Difference Equations*, 2020:418, 2020.

[10] K. Guida, K. Hilal, and L. Ibnelazyz. Existence of mild solutions for a class of impulsive hilfer fractional coupled systems. *Advances in Mathematical Physics*, 2020:Article ID 8406509, 2020.

[11] K. Hilal, K. Guida, L. Ibnelazyz, and M. Oukessou. *Existence Results for an Impulsive Fractional Integro-differential Equations with Non-compact Semigroup.* Springer Nature Switzerland AG, New York, NY, USA, 2019.

[12] K. Hilal, L. Ibnelazyz, K. Guida, and S. Melliani. *Existence of Mild Solutions for an Impulsive Fractional Integro-differential Equations with Non-local Condition.* Springer Nature Switzerland AG, New York, NY, USA, 2019.

[13] K. Hilal, L. Ibnelazyz, K. Guida, and S. Melliani. Fractional langevin equations with non-separated integral boundary conditions. *Advances in Mathematical Physics*, 2020:Article ID 3173764, 2020.

[14] R. Hilfer. *Applications of Fractional Calculs in Physics.* World Scientific, Singapore, 2000.

[15] L. Ibnelazyz, K. Guida, K. Hilal, and S. Melliani. Existence of solution for a fractional langevin system with nonseparated integral boundary conditions. *Journal of Mathematics*, 2021:Article ID 3482153, 2021.

[16] L. Ibnelazyz, K. Guida, K. Hilal, and S. Melliani. Existence results for nonlinear fractional integro-differential equations with integral and antiperiodic boundary conditions. *Computational and Applied Mathematics*, 40:33, 2021.

[17] L. Ibnelazyz, K. Guida, K. Hilal, and S. Melliani. New existence results for nonlinear fractional integro-differential equations. *Advances in Mathematical Physics*, 2021:Article ID 5525591, 2021.

[18] L. Ibnelazyz, K. Guida, S. Melliani, and K. Hilal. On a nonlocal multipoint and integral boundary value problem of nonlinear fractional integro-differential equations. *Journal of Function Spaces*, 2020:Article ID 8891736, 2020.

[19] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations.* North Holland Mathematics Studies, Elsevier, Amsterdam, Netherlands, 2006.

[20] V. Lakshmikantham. Theory of fractional functional differential equations. *Nonlinear Analysis*, 69(10):3337–3343, 2007.

[21] V. Lakshmikantham and A. S. Vatsala. Basic theory of fractional differential equations. *Nonlinear Analysis*, 69(8):2677–2682, 2007.

[22] X. Liu and Y. Liu. Fractional differential equations with fractional non-separated boundary conditions. *Electronic Journal of Differential Equations*, 25(10):1, 2013.

[23] D. Luo, Ak. Zada, Sh. Shaleena, and M. Ahmad. Analysis of a coupled system of fractional differential equations with nonseparated boundary conditions. *Advances in Difference Equation*, 2020:590, 2020.

[24] M.J. Mardanov, Y.A. Sharifov, K.E. Ismayilova, and S.A.Zamanova. Existence and uniqueness of solutions for the system of first-order nonlinear differential equations with three-point and integral boundary conditions. *European journal of pure and applied mathematics*, 12(3):756–770, 2019.

[25] K.S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* Wiley, New York, NY,USA, 1993.

[26] M.J.Mardanov, Y.A. Sharifov, H.N. Aliyev, and R.A. Sardarova. Existence and uniqueness of solutions for the first order non-linear differential equations with multi-point boundary conditions. *European journal of pure and applied mathematics*, 13(3):414–426, 2020.

[27] I. Podlubny. *Fractional Differential Equations.* Academic Press, New York, NY, USA, 1993.

[28] Y.A. Sharifov. Existence and uniqueness of solutions for the system of nonlinear fractional differential equations with nonlocal boundary conditions. *Proceedings of IMM of NAS of Azerbaijan*, XXXVI(XLIV):125–134, 2012.

[29] M. Alesemi S.N. Rao. On a coupled system of fractional differential equations with nonlocal non-separated boundary conditions.

[30] Y. Wang and L. Liu. Uniqueness and existence of positive solutions for the fractional integro-differential equation. *Boundary Value Problems, YEAR = 2017, volume = 12,.*

[31] Y. Xing, F. Jiao, and F. Liu. On the generalization of a solution for a class of integro-differential equations with non-separated integral boundary conditions. *Mathematical Problems in Engineering, YEAR = 2020, volume = 2020, pages = Article ID 8679465,.*

[32] S.M. Zeynally Y.A. Sharifov, F.M. Zeynally. Existence and uniqueness of solutions for nonlinear fractional differential equations with two-point boundary conditions. *Advanced Mathematical Models Applications, YEAR = 2018, volume = 3, number = 1, pages = 54-62,.*

[33] Y. Zhou. *Fractional Differential Equations.* Xiangtan University, Xiangtan, China, 2014.